Solutions to ANALYSIS QUALIFYING EXAM

January 2004

1. (a) Classify all entire functions $f : \mathbf{C} \to \mathbf{C}$ such that

$$\sup_{z\in\mathbf{C}} \frac{|f(z)|}{1+|z|^4} \ < \ \infty \ .$$

The function $f(\frac{1}{z})$ has an isolated singularity at 0. If this singularity is removable, then f is bounded and so constant by Louiville's theorem, which is one possibility. If it had a transcendental singularity at 0, then $z^4 f(\frac{1}{z})$ would also have a transcendental singularity at 0 and be unbounded, contradicting the growth assumption on f at ∞. We see that $f(\frac{1}{z})$ must have a pole at 0 so that f is necessarily a polynomial. Also we see that the degree of f is at most 4, and any such polynomial satisfies the hypothesis. Thus $f(z) = a_0 + a_1 z + a - 2z^2 + a_3 z^3 + a_4 z^4$ for some complex numbers $a_0, \ldots, a_4$

(b) Classify all entire functions $g : \mathbf{C} \to \mathbf{C}$ such that

$$\inf_{z\in\mathbf{C}} \frac{|g(z)|}{|z|^4} \ > \ 0 \ .$$

Again $g(\frac{1}{z})$ cannot have a transcendental singularity at 0 because then $z^4 g(\frac{1}{z})$ would be arbitrarily close to zero for some points z near 0. So again g is a polynomial. But now the condition implies that g can vanish only at the origin. So, by the fundamental theorem of algebra, $g(z) = az^m$. The condition $\inf_{z\in\mathbf{C}} |a||z|^{m-4} > 0$ requires that $m - 4 \geq 0$ (for z near 0) and $m - 4 \leq 0$ (for z near ∞). So $g(z) = az^4$ with $a \neq 0$.

2. Suppose that $f_n : \mathbf{R} \to \mathbf{R}$ is a differentiable function for every positive integer n, $M = \sup_{n,x} |f_n'(x)| < \infty$ and that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbf{R}$ exists for all $x \in \mathbf{R}$.

(a) Show that the functions f_n are *uniformly bounded* on each fixed interval $[a, b] \subset \mathbf{R}$.

Since $f(a) = \lim_{n\to\infty} f_n(a)$, $N = \sup_n |f_n(a)| < \infty$. Then for any $x \in [a, b]$ the fundamental theorem of calculus gives the uniform bound

$$|f_n(x)| \ \leq \ |f_n(a)| + |\int_a^x f_n'(t)\, dt| \ \leq \ N + M|b - a| \ .$$

(b) Is f *continuous* on $\mathbf{R}$? Prove or find a counterexample. Yes, as in (a) the fundamental theorem of calculus implies that for $-\infty < x < y < \infty$,

$$|f(y) - f(x)| \ = \ \lim_{n\to\infty} |f_n(y) - f_n(x)| \ \leq \ \limsup_{n\to\infty} \int_x^y |f_n'(t)|\, dt \ \leq \ M(y - x) \ .$$

(c) Is f *differentiable* on $\mathbf{R}$? Prove or find a counterexample. Not necessarily. One easily obtains an example with $f(x) = |x|$ and the graph of $f_n(x)$ being obtained by slightly rounding the graph of $|x|$.

3. Compute the (improper) integral

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx \ .$$

This improper integral exists as $\lim_{R\to\infty} I_R$ where

$$I_R \ = \ \int_{1/R}^{R} \frac{\sin x}{x(x^2+1)}\, dx \ = \ \frac{1}{2}\Big[\int_{-R}^{-1/R} + \int_{1/R}^{R}\Big] \frac{\sin x}{x(x^2+1)}\, dx \ .$$

because $|\frac{\sin x}{x}| \le 1$ and $\frac{1}{x^2+1}$ is integrable on $[0,\infty)$. We want to use the Cauchy integral formula, but we need to choose the $f(z)$ so that the integral on the extra outer boundary curve will approach 0 as the domain gets larger. [Warning: The estimate $|\sin z| \le 1$ is not always true for z complex.] One thing that works is to note that $\frac{\sin x}{x(x^2+1)} = \mathcal{I}m \frac{e^{ix}}{x(x^2+1)}$ for x real and take

$$f(z) \ = \ \frac{e^{iz}}{z(z^2+1)}$$

on the domain Ω_R in the upper halfplane bounded by the 4 curves

$$[-R, -\frac{1}{R}], \ \ \gamma_R = \{\frac{1}{R}e^{i\theta} \ : \ \pi \ge \theta \ge 0\}, \ \ [\frac{1}{R}, R], \ \ \Gamma_R = \{Re^{i\theta} \ : \ 0 \le \theta \le \pi\} \ .$$

Inside Ω_R, $f(z)$ has a single pole at $z = i$ with residue $\frac{e^{i^2}}{i(i+i)} = -\frac{1}{2e}$. Thus, Cauchy's residue formula gives

$$-\frac{\pi}{e} \ = \ \mathcal{I}m\big(2\pi i(-\frac{1}{2e})\big) \ = \ \mathcal{I}m \int_{\partial\Omega_R} f(z)\, dz \ = \ 2I_R + \mathcal{I}m \int_{\gamma_R} f(z)\, dz + \mathcal{I}m \int_{\Gamma_R} f(z)\, dz \ .$$

On Γ_R, $|e^{iRe^{i\theta}}| = |e^{-R\sin\theta}| \le 1$ because $\sin\theta \in [0,1]$. So we see that

$$\Big|\int_{\Gamma_R} f(z)\, dz\Big| \ \le \ \frac{1}{R^3}\pi R \ \to \ 0 \text{ as } R \to \infty \ .$$

Finally

$$\int_{\gamma_R} f(z)\, dz \ = \ -\frac{1}{2}\int_{\partial \mathbf{B}_{1/R}} f(z)\, dz \ = \ -\frac{1}{2}(2\pi i)\mathrm{Res}_{\,0} f \ = \ -\pi i(1) \ .$$

So, taking imaginary parts,

$$\lim_{R\to\infty} I_R \ = \ \frac{1}{2}\big[-\frac{\pi}{e} + \pi\big] \ = \ \frac{\pi}{2}(1 - e^{-1}) \ .$$

4. (a) In the unit disk $\{z \in \mathbf{C} \ : \ |z| < 1\}$ how many solutions are there to the equation $z^8 - 5z^3 + z \ = \ 2$? We apply Rouché's Theorem with $f(z) = z^8 - 5z^3 + z_2$ and $g(z) = -5z^3$ on the unit disk noting that for $|z| = 1$,

$$|f(z) - g(z)| \ = \ |z^8 + z - 2| \ \le \ |z|^8 + |z| + 2 \ = \ 1 + 1 + 2 = 4 \ < \ 5(1)^3 \ = \ |g(z)|.$$

Thus, in the unit disk, $f(z)$ has the same number of zeros as $g(z)$ (counting multiplicities), namely 3. So the equation $z^8 - 5z^3 + z \ = \ 2$ has 3 solutions in the unit disk.

(b) In the radius-2 disk $\{z \in \mathbf{C} : |z| < 2\}$ how many solutions are there to the same equation $z^8 - 5z^3 + z = 2$? Here we use the same f but now take $g(z) = z^8$ and note that for $|z| = 2$ one has

$$|f(z) - g(z)| = |-5z^3 + z - 2| \leq 5(2)^3 + 2 + 2 = 44 < (2)^8 = |g(z)| .$$

So the equation $z^8 - 5z^3 + z = 2$ has 8 solutions in the radius-2 disk.

5. (a) Suppose that f is integrable on $[0,1]$. Show that there exists a sequence of positive numbers $a_n \downarrow 0$ so that $\lim_{n\to\infty} a_n|f(a_n)| = 0$.

If this were false, then $\epsilon = \liminf_{x\to 0} x|f(x)| > 0$, and there there would exist a positive δ so that $x|f(x)| \geq \frac{1}{2}\epsilon$ whenever $0 < x \leq \delta$. But then

$$\int_0^1 |f(x)|\,dx \geq \int_0^\delta |f(x)|\,dx \geq \int_0^\delta \frac{\epsilon}{2x}\,dx = \infty ,$$

contradicting the integrability of f.

(b) Let f_n be a sequence of functions integrable on $[0,1]$ with $\sup_n \int_0^1 |f_n(x)|\,dx < \infty$. Does there exist a subsequence f_{n_k} of f_n and sequence of positive numbers $b_k \downarrow 0$ and so that $\lim_{k\to\infty} b_k|f_{n_k}(b_k)| = 0$. If so, prove it. If not, find a counterexample.

As Frank pointed out, a stronger result is true. One need only assume that each f_n is integrable and one doesn't need to pass to a subsequence f_{n_k} for the conclusion. Here we first choose $\alpha_k \downarrow 0$ so that $\sum_{k=1}^\infty \alpha_k \int_0^1 |f_k(x)|\,dx < \infty$, and apply (a) to the integrable function $f(x) = \sum_{k=1}^\infty \alpha_k |f_k(x)|$ to find points $a_m \downarrow 0$ so that $\lim_{m\to\infty} a_m f(a_m) = 0$. Passing to a subsequence we can make this sequence converge as fast as we want. In particular we can choose inductively $a_{m_k} \downarrow 0$ so that $a_{m_k} f(a_{m_k}) \leq \alpha_k^2$. Letting $b_k = a_{m_k}$, we conclude that

$$b_k f_k(b_k) \leq b_k \alpha_k^{-1} f(b_k) \leq \alpha_k^{-1}\alpha_k^2 = \alpha_k \to 0 \text{ as } k \to \infty .$$

6. Suppose $1 \leq p \leq \infty$, $f \in L^p\big([0,1]\big)$, and $h(t)$ is the Lebesgue measure of the set $\{x \in [0,1] : |f(x)| > t\}$ for $0 \leq t < \infty$.

(a) Show that $\int_0^\infty h(t)\,dt < \infty$ if $1 < p \leq \infty$.

(b) Is this still true for $p = 1$? Prove or find a counterexample.

Here this is true for $p = 1$. Since Hölder's inequality implies that $L^p\big([0,1]\big) \subset L^1\big([0,1]\big)$, we only need do the case $p = 1$ and part (a) follows.

For this, one uses Fubini's theorem with the characteristic function of the subgraph

$$A = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y < |f(x)|\} .$$

Let λ denote 1 dimensional Lebesgue measure. By Fubini's theorem, A is 2 dimensional Lebesgue measurable with 2 dimensional measure

$$|A| \;=\; \int_0^1 \lambda\{y \,:\, (x,y) \in A\}\, dx \;=\; \int_0^1 |f(x)|\, dx \;<\; \infty \;.$$

But slicing the other way shows that

$$\int_0^\infty h(y)\, dy \;=\; \int_0^\infty \lambda\{x \,:\, |f(x)| > y\}\, dy \;=\; \int_0^\infty \lambda\{x \,:\, (x,y) \in A\}\, dy \;=\; |A| \;<\; \infty \;.$$

One can get an alternate proof of (a) (but not (b)) by using Chebychev's inequality to see that

$$h(t) \;=\; \lambda\{x \in [0,1] \,:\, |f(x)|^p > t^p\} \;\leq\; \frac{1}{t^p} \int_0^1 |f(x)|^p\, dx \;.$$

So

$$\int_0^\infty h(t)\, dt \;\leq\; 1 + \int_1^\infty h(t)\, dt \;\leq\; 1 + \Big(\int_0^1 |f(x)|^p\, dx \Big) \int_1^\infty t^{-p}\, dt \;<\; \infty \;.$$