Solutions to ANALYSIS QUALIFYING EXAM

January 2005

1. Suppose $f : \mathbf{C} \to \mathbf{C}$ is continuous and the complex derivative $f'(z)$ exists for all $z \in \mathbf{C}$.

(a) What is the Cauchy integral formula for f on the disk $|z| < R$?

$$f(z) \;=\; \frac{1}{2\pi \mathbf{i}} \int_{\partial \mathbf{B}_R(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \;.$$

(b) Since $\frac{d^n}{dz^n}\left(\frac{1}{\zeta - z}\right) = \frac{n!}{(\zeta - z)^{n+1}}$, which is bounded for $|z| \leq r < R$, we may differentiate under the integral in the Cauchy Integral formula to find that all the complex derivatives exist and satisfy

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi \mathbf{i}} \int_{\partial \mathbf{B}_R(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \;.$$

(c) Taking $z = 0$ in this formula, we readily find

$$|f^{(n)}(0)| \;\leq\; M_R \frac{n!}{R^n} \;.$$

2. For $0 < \alpha \leq 1$, a function $f : [0,1] \to [0,1]$ is α-Hölder continuous if there is a positive constant C so that

$$|f(x) - f(y)| \;\leq\; C\,|x - y|^{\alpha} \quad \text{for } 0 \leq x < y \leq 1 \;.$$

(a) Since $x = \sqrt{x}\sqrt{x} < \sqrt{y}\sqrt{x}$, $\quad y + 2x < y + 2\sqrt{y}\sqrt{x}$, $\quad$ and

$$\left(\sqrt{y} \;-\; \sqrt{x}\right)^2 \;=\; y - 2\sqrt{y}\sqrt{x} + x \;<\; y - x \;=\; \left(\sqrt{y - x}\right)^2 \;.$$

Taking square roots gives the desired Hölder estimate $\sqrt{y} - \sqrt{x} \;<\; \sqrt{y - x}$.

(b) If $g(x) = \sqrt{x}$ were 1-Hölder continuous at $x = 0$, then $\sqrt{y} = g(y) - g(0) \leq C(y - 0)$. But this inequality is false for $y \leq C^{-2}$.

3. (a) Show that if f is meromorphic (but not holomorphic) at 0, then, for some $n \in \{1, 2, \cdots\}$,

$$\lim_{r \to 0} r^n \int_0^{2\pi} |f(re^{i\theta})|\, d\theta \quad \text{exists and is nonzero}\,.$$

Choose n to be the order of the pole of f at 0 so that the Laurent expansion for f begins $\frac{b_n}{z^n} + \frac{b_n}{z^{n-1}} + \cdots$ with $b_n \neq 0$. For $0 < r < 1$ this series is absolutely and uniformly convergent on $|z| = r$. Taking the absolute value, multiplying by r^n, integrating, using the triangular inequality, and taking the limit at $r \to 0$, we find that $\lim_{r \to 0} r^n \int_0^{2\pi} |f(re^{i\theta})|\, d\theta = |b_n|$.

(b) Show that if g is an entire holomorphic function, and

$$\lim_{r\to\infty} r^{-1/2} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta \ < \ \infty \ , \quad \text{then } g \text{ is a constant} \ .$$

We may repeat the proof of 1(c) with f replaced by g and M_R replaced by

$$N_R \ = \ \int_0^{2\pi} |g(Re^{\mathbf{i}\theta}|\, d\theta \ \le \ CR^{1/2} \ .$$

So for $n \ge 1$, $\frac{|g^{(n)}(0)|}{n!} \le N_R\, R^{-n} \ \le C\, R^{\frac{1}{2}-n} \ \to \ 0$ an $R \to \infty$. Using the power series expansion for g at 0 we find that $g \equiv g(0)$, a constant.

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable with $\int_0^\infty |f(t)|\, dt \ < \ \infty$.

(a) Since $|f(t)| \ \ge \ |f(t)e^{-\varepsilon t^2}|$ and $f(t)e^{-\varepsilon t^2} \to f(t)$ as $\varepsilon \to 0$, Lebesgue's dominated convergence gives

$$\lim_{\varepsilon\to 0} \int_0^\infty f(t)e^{-\varepsilon t^2}\, dt \ = \ \int_0^\infty f(t)\, dt \ .$$

(b) Since $|f(t)| \ \ge \ |f(t)e^{-t^2/\varepsilon}|$ and $f(t)e^{-t^2/\varepsilon} \to 0$ as $\varepsilon \to 0$, dominated convergence this time gives

$$\lim_{\varepsilon\to 0} \int_0^\infty f(t)e^{-t^2/\varepsilon}\, dt \ = \ 0 \ .$$

(c) Write $f = f_1 + f_2$ with f_1 and f_2 both integrable and continuously differentiable, $f_1 \mid [3,\infty) \equiv 0$, and $f_2 \mid [0,1] \equiv 0$. We readily check (for example by differentiating) that, for each $\varepsilon < 1$, the function $te^{-t^2/\varepsilon}$ is decreasing on $[1,\infty)$. So

$$|\frac{1}{\varepsilon}\int_0^\infty f_2(t)te^{-t^2/\varepsilon}\, dt| \ \le \ \frac{1}{\varepsilon}\, e^{-1/\varepsilon} \int_1^\infty |f_2(t)|\, dt \ \to \ 0 \ \text{ as } \varepsilon \to 0 \ .$$

For the f_1 term, we use integration by parts with $g(t) = -\frac{1}{2}e^{-t^2/\varepsilon}$ to get that

$$\begin{aligned} \frac{1}{\varepsilon}\int_0^\infty f_1(t)\, t\, e^{-t^2/\varepsilon}\, dt \ &= \ \int_0^3 f_1(t)g'(t)\, dt \\ &= \ f_1(3)g(3) \ - \ f_1(0)g(0) \ - \ \int_0^3 f_1'(t)g(t)\, dt \\ &= \ 0 \ + \ \frac{1}{2}f(0) \ + \ \frac{1}{2}\int_0^3 f'(t)e^{-t^2/\varepsilon}\, dt \\ &\to \ \frac{1}{2}f(0) \ + \ 0 \ \text{ as } \varepsilon \to 0 \ . \end{aligned}$$

So $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\infty f_1(t)te^{-t^2/\varepsilon}\, dt = \frac{1}{2}f(0)$.

5. (a) Suppose that A is a (possibly uncountable) set. Prove that if $f_a : \mathbf{R} \to [0,1]$ is a continuous function for each $a \in A$, then $f(x) = \sup_{a \in A} f_a(x)$ is Lebesgue measurable. For each $t \in \mathbf{R}$, let $E_t = \{x \ : \ f(x) > t\}$ and $E_{a,t} = \{x \ : \ f_a(x) > t\}$. So $x \in E_t$ iff $f(x) > t$ iff $f_a(x) > t$ for some $a \in A$ iff $x \in \cup_{a \in A} E_{a,t}$. Since f is continuous, each $E_{a,t}$ is open so $E_t = \cup_{a \in A} E_{a,t}$ is also open, hence measurable. So f is measurable.

(b) Show that there exists a set A and a family $\{g_a \ : \ a \in A\}$ of Lebesgue measurable functions $g_a : \mathbf{R} \to [0,1]$ so that $g(x) = \sup_{a \in A} g_a(x)$ is <u>not</u> Lebesgue measurable. For an unmeasurable subset A of $\mathbf{R}$, $g = \chi_A$ is not measurable. But $g = \sup_{a \in A} g_a$ where each $g_a = \chi_{\{a\}}$ is measurable.

6. (a) For what complex numbers z is the series $\sum_{k=0}^{\infty} 2^{-k} e^{kz}$ absolutely convergent?

Since a geometric series $\sum_{k=0}^{\infty} a^k$ converges absolutely iff $|a| < 1$ and $|2^{-k} e^{kz}| = (\frac{1}{2}e^x)^k$ we see that the series from 6(a) converges iff $\frac{1}{2}e^x < 1$ or $x < \log 2$.

(b) For these z, the sum of this series is given by the formula for the sum of a geometric series, namely,

$$\frac{1}{1 - \frac{1}{2}e^z} = \frac{2}{2 - e^z} .$$

(c) The series $\sum_{k=0}^{\infty} 2^{-k} \cos kz$ is absolutely convergent iff $|y| < \log 2$. In fact, note that $\cos kz = \frac{1}{2}(e^{\mathbf{i}kz} + e^{-\mathbf{i}ks})$. Also

$$|\frac{1}{2}e^{\mathbf{i}z}| = \frac{1}{2}e^{-y} < 1 \Leftrightarrow e^y > \frac{1}{2}$$

and

$$|\frac{1}{2}e^{-\mathbf{i}z}| = \frac{1}{2}e^{y} < 1 \Leftrightarrow e^y < 2 .$$

Thus for $|y| < \log 2$ both geometric series of exponentials $\sum_{k=0}^{\infty} (\frac{1}{2}e^{\mathbf{i}z})^k$ and $\sum_{k=0}^{\infty} (\frac{1}{2}e^{-\mathbf{i}z})^k$ converge absolutely and hence so does $\sum_{k=0}^{\infty} 2^{-k} \cos kz$.

For $|y| \geq \log 2$ one of the two geometric series of exponentials does converge absolutely while the other definitely does not. Thus it is impossible that their average converges absolutely.