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ANALYSIS QUALIFYING EXAM

Janurary 1999

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Evaluate

$$\int_{|z|=1} (e^{2\pi z} + 1)^{-2}\, dz$$

where the integral is taken in the counterclockwise direction.

2. Give an example of a subset of $\mathbb{R}$ having uncountably many connected components. Can such a subset be open? Closed?

3. Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of 0. Suppose its Taylor series at 0

$$\sum_{k=0}^{\infty} a_k z^k$$

has all $a_k \geq 0$. Let $r = \min\{|z_0| : f \textit{ has a pole at } z_0\} < \infty$. Prove that f must have a pole at the real point $z = r$.

4. Let $f(x)$ be a real-valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that

$$\lim_{x \to \infty} f(x)$$

exists and is less than $1 + \frac{\pi}{4}$.

5. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous.
a) Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0$.
b) Prove that $\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx = f(1)$.

6. Let a be a complex number and ϵ a positive number. Prove that the function $f(z) = \sin z + \frac{1}{z-a}$ has infinitely many zeros in the strip $|Imz| < \epsilon$