ANALYSIS QUALIFYING EXAM

May 2002

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable,

$$C_f \;=\; \{a \in \mathbb{R} \;:\; f'(a) = 0\} \textit{ and } I_f \;=\; \{a \in \mathbb{R} \;:\; f''(a) = 0\}\ .$$

(a) Show that the number of points in C_f and I_f satisfy: $\#C_f \;\leq\; 1 + \#I_f$.

(b) Give a specific example of a polynomial f with $\#C_f = 1$, $\#I_f = 1$, and $\#(C_f \cap I_f) = 0$.

2. Suppose that Γ is the counter-clockwise oriented circle of radius 2 about the origin.

(a) Find $\int_\Gamma \frac{e^z}{1+z^2}\,dz$.

(b) Find $\int_\Gamma e^{-\frac{1}{z^2}}\,dz$.

3. Suppose that f is a differentiable function on $\mathbb{R}^2$, $|\frac{\partial f}{\partial y}|$ is bounded, and $\{(x,y) \;:\; f(x,y) \neq 0\}$ is a bounded set.

(a) Prove that $\frac{d}{dt}\int_{-\infty}^{t} f(x,y)\,dx = f(t,y)$.

(b) Prove that $\frac{d}{dy}\int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\,dx$.

4. (a) Describe all entire functions f having $\inf_{z\neq 0} \frac{|f(z)|}{|z|} > 0$.

(b) Describe all holomorphic functions g on $\mathbb{C} \setminus \{0\}$ having $\inf_{z\neq 0} \frac{|g(z)|}{|z|} > 0$.

5. Suppose that f is meromorphic on $\mathbb{C}$ and holomorphic and bounded on the unit disk $\{z \;:\; |z| < 1\}$. Prove that f is holomorphic on a larger disk $\{z \;:\; |z| < 1+\epsilon\}$ for some $\epsilon > 0$.

6. Suppose that f_n is a sequence of positive continuous functions on $[0,1]$ and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Let $M_n = \max_{x\in[0,1]} f_n(x)$.

(a) Give an example (either by sketching a careful graph or by writing a formula) of such a sequence where $\lim_{n\to\infty} M_n = 1$.

(b) Show that if the sequence f_n is *also decreasing* (i.e. $f_{n+1}(x) \leq f_n(x)$ for all n, x), then $\lim_{n\to\infty} M_n = 0$.