Analysis Exam, May, 2003

1. Suppose that $x_1 > 0$ and $x_{n+1} = (2 + x_n)^{-1}$ for $n = 1, 2, \ldots$. Prove that the sequence x_n converges and find its limit.

2. Evaluate

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx$$

where $a > 0$.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

(A) Show that the Lebesgue measure of $\{x \ : \ |f(x)| > \lambda\}$ approaches 0 as $\lambda \uparrow \infty$.

(B) Show that the additional assumption $\int_{-\infty}^{\infty} |f'(x)|\, dx < \infty$ implies that $f(x) \to 0$ as $x \to \infty$.

4. Find all entire functions f which satisfy $\mathcal{R}e\big(f(z)\big) \leq \frac{2}{|z|}$ for $|z| > 1$. Prove your answer.

5. Suppose that f is a bounded measurable function on $\mathbf{R}$ and $\int_{-\infty}^{\infty} |g(x)|\, dx < \infty$. Prove that

$$\lim_{t \to 0} \int f(x)\big[g(x+t) - g(x)\big]\, dx \ = \ 0 \ .$$

6. A. State Rouche's Theorem.

B. State Schwarz's Lemma.

C. Suppose f is holomorphic in the unit disk $|z| < 1$ with $|f(z)| \leq 1$ and $f(0) = 0$. Prove that for any integer $n \geq 1$,

$$f(z) \ - \ 2^n z^n$$

has precisely n zeros (counting multiplicity) in the disk $|z| < \frac{1}{2}$.