ANALYSIS QUALIFYING EXAM

May 2004

1. (a) For what real numbers p is

$$\int_1^\infty t^p (\sin^2 t)\, dt \;<\; \infty\ ?$$

(b) For what real numbers q is

$$\int_0^1 t^q (\sin^2 t)\, dt \;<\; \infty\ ?$$

(c) For what real numbers s is

$$\int_{\mathbf{R}^3 \setminus \mathbf{B}_1} |x|^s (\sin^2 |x|\,)\, dx_1 dx_2 dx_3 \;<\; \infty\ ,$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\mathbf{B}_1 = \{x \in \mathbf{R}^3 \;:\; |x| < 1\}$?

2. For $\varepsilon > 0$, let $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(t/\varepsilon)$ where $\phi(t) = 1 - |t|$ for $|t| \le 1$ and $\phi(t) = 0$ for $|t| > 1$. Then, for $f \in L^1(\mathbf{R})$, let

$$f_\varepsilon(x) \;=\; \int_{-\infty}^{\infty} f(y)\phi_\varepsilon(x - y)\, dy\ .$$

(a) Show that each f_ϵ is continuous, and even satisfies the estimate

$$|f_\varepsilon(w) - f_\varepsilon(x)| \le\; \varepsilon^{-1}\Big(\int |f(t)|\, dt\Big)|w - x|\ .$$

(b) Show that if f itself is uniformly continuous, then f_ε approaches f uniformly as $\varepsilon \to 0$.

3. (a) Does there exist, for every $\varepsilon > 0$, an open dense subset U of the plane $\mathbf{R}^2$ with 2-dimensional Lebesgue measure less than ε? If so, construct one. If not, explain why it can't exist.

(b) Suppose, for $\theta \in [0, 2\pi)$, ℓ_θ is the ray $\{(t\cos\theta, t\sin\theta) \in \mathbf{R}^2 \;:\; 0 \le t < +\infty\}$. If E is a measurable subset of $\mathbf{R}^2$ with positive 2-dimensional Lebesgue measure, then

$$\{\theta \in [0, 2\pi) :\; E \cap \ell_\theta \text{ has positive } 1 - \text{dimensional Lebesgue measure in } \ell_\theta\}$$

has positive 1-dimensional Lebesgue measure.

4. Compute
$$\int_0^\infty \frac{dx}{x^2 + \mathbf{i}} \quad \big(\text{where } \mathbf{i} \text{ is the usual complex } \sqrt{-1}\big)\ .$$

5. Suppose that D is the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ and f is a nonconstant holomorphic function on some connected open neighborhood of $\overline{D}$ and that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f(D) = D$.

6. (a) Give a necessary and sufficient (topological) condition on an open set Ω in the complex plane so that for *every* holomorphic function f on Ω there will exist a holomorphic function F on Ω with $F' = f$. Justify your answer.

(b) Suppose A is a finite subset of the unit disk D and $U = D \backslash A$. Give a necessary and sufficient condition on a holomorphic function f on U so that there will exist a holomorphic function F with $F' = f$ on U. Justify your answer.