

Analysis Exam, Fall 2005

1. (a) Suppose $f : [0, 1] \rightarrow \mathbf{R}$ is Lebesgue integrable. Find

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nx^2}{2 + nx} f(x) dx ,$$

and justify each of your steps.

(b) Give an example of a Lebesgue integrable $f : [0, 1] \rightarrow \mathbf{R}$ for which

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nx}{2 + nx^2} f(x) dx = +\infty .$$

2. Suppose g is holomorphic on $\{z \in \mathbf{C} : |z| < 2\}$ and $|g(z)| < 1$ whenever $|z| = 1$. Show that there exists a unique point $w \in \mathbf{C}$ with $|w| < 1$ and $g(w) = w$.

3. Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is infinitely differentiable.

(a) Define carefully, for each point a with $f(a) = 0$, the *order of vanishing* $N(f, a)$ of f at a . Hint: $N(f, a) \in \{1, 2, \dots\} \cup \{+\infty\}$.

(b) Let $N(f) = \sum_{\{a: f(a)=0\}} N(f, a)$. (Here $N(f) = +\infty$ in case f has infinitely many zeros.) Prove the inequality:

$$N(f') \geq N(f) - 1 .$$

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x - \mathbf{i}} dx .$$

5. Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is infinitely differentiable.

(a) Derive a formula for

$$\int_0^x f''(t) \frac{(x-t)}{2} dt$$

in terms of $f(x)$, $f(0)$, $f'(0)$.

(b) Give an upper bound for $|f(x)|$ in terms of

$$|f(0)|, |f'(0)|, |f''(0)|, \text{ and } \sup_{|t| \leq |x|} |f'''(t)| .$$

6. Suppose g is holomorphic on $A = \{z \in \mathbf{C} : 0 < |z| < 1\}$ and

$$\limsup_{|z| \rightarrow 0} |g(z) - \lambda| > 0$$

for every $\lambda \in \mathbf{C}$. Show that either

(I) $\lim_{|z| \rightarrow 0} |z|^{1/2} |g(z)| = \infty$ or

(II) $g(A)$ is dense in \mathbf{C} .

Give a specific example of a g satisfying (II).