SOLUTIONS OF ANALYSIS QUALIFYING EXAM

August 2004

1. (a) The Cauchy-Riemann equations imply that

$$u_x = v_y = -2y(3x+1)\ , \quad u_y = -v_x = -3x^2 - 2x + 3y^2\ .$$

Integrating we find that $u = -3x^2y - 2xy + y^3$.

(b) The general solution is $u = -3x^2y - 2xy + y^3 + c$ for some constant c, because the difference of any two solutions has by the Cauchy-Riemann equations, gradient zero. So the difference must be a constant.

2. Suppose that g is twice continuously differentiable and real-valued on $\mathbf{R}^2$. You are to prove that

$$\frac{\partial^2 g}{\partial x \partial y}(0,0) \;=\; \frac{\partial^2 g}{\partial y \partial x}(0,0)\ , \tag{$*$}$$

using the following steps:

(a)

$$\begin{aligned}
\int_0^b \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,y)\, dx\, dy \;&=\; \int_0^b \big[\frac{\partial g}{\partial y}(a,y) - \frac{\partial g}{\partial y}(0,y)\big]\, dy \\
&=\; g(a,b) \;-\; g(a,0) \;-\; g(0,b) \;+\; g(0,0)\ .
\end{aligned}$$

(b) Using Fubini's Theorem, we also find that

$$\begin{aligned}
\int_0^b \int_0^a \frac{\partial^2 g}{\partial y \partial x}(x,y)\, dx\, dy \;&=\; \int_0^a \int_0^b \frac{\partial^2 g}{\partial y \partial x}(x,y)\, dy\, dx \\
&=\; \int_0^a \big[\frac{\partial g}{\partial x}(x,b) - \frac{\partial g}{\partial x}(x,0)\big]\, dx \\
&=\; g(a,b) \;-\; g(0,b) \;-\; g(a,0) \;+\; g(0,0)\ .
\end{aligned}$$

(c) (a) and (b) are clearly the same.

(d) Using the Fundamental Theorem of Calculus, we conclude

$$\begin{aligned}
\frac{\partial^2 g}{\partial x \partial y}(0,0) \;&=\; \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,0)\, dx \\
&=\; \lim_{b \to 0} \frac{1}{b} \int_0^b \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,y)\, dx\, dy \\
&=\; \lim_{b \to 0} \frac{1}{b} \int_0^b \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial y \partial x}(x,y)\, dx\, dy \\
&=\; \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,0)\, dx \\
&=\; \frac{\partial^2 g}{\partial y \partial x}(0,0)\ .
\end{aligned}$$

3. Suppose that $D = \{z \in \mathbf{C} : \ |z| < 1\}$, $f : D \to D$ is holomorphic, and $z_0 \in D$. Let $w_0 = f(z_0)$. Let

$$F(z) = \frac{z - z_0}{1 - \bar{z}_0 z} \ , \quad G(w) \ = \ \frac{w - w_0}{1 - \bar{w}_0 w} \ , \quad g(\zeta) = (G \circ f \circ F^{-1})(\zeta) \ .$$

Then, $g : D \to D$ is holomorphic with $g(0) = (G \circ f)(z_0) = G(w_0) = 0$. Applying the Schwarz Lemma to g, we conclude that $|g(\zeta)| \le |\zeta|$ for all $\zeta \in D$. So, with $\zeta = F(z)$,

$$\Big|\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}\Big| \ = \ |G \circ f(z)| \ = \ |g(\zeta)| \ \le \ |\zeta| \ = \ \Big|\frac{z - z_0}{1 - \bar{z}_0 z}\Big| \ .$$

(b) From (a) we have that

$$\frac{\big|\frac{f(z) - w_0}{z - z_0}\big|}{|1 - \bar{w}_0 f(z)|} \le \frac{1}{|1 - \bar{z}_0 z|} \ .$$

Taking the limit as $z \to z_0$, and noting that $w_0 = f(z_0)$ and that both $1 - |f(z_0)|^2$ and $1 - |z_0|^2$ are positive, we conclude that

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \le \frac{1}{1 - |z_0|^2} \ ,$$

and then replace z_0 by z.

4.(a) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function such that, for almost all $t \in \mathbf{R}$, $f'(t)$ exists and $|f'(t)| \le 1$. Is it true that, $f(b) - f(a) \ = \ \int_a^b f'(t)\, dt$ for $-\infty < a < b < \infty$?

No. If $f(t)$ is the Cantor function for $0 \le t \le 1$, $f|(-\infty, 0] \equiv 0$, and $f|[0, +\infty) \equiv 0$, the f is continuous with $f'(t) = 0$ for a.e. t, but $f(1) \ne f(0)$.

(b) Suppose $g : \mathbf{R} \to \mathbf{R}$ is differentiable at *every* point $t \in \mathbf{R}$. Is g necessarily of *bounded variation* on every closed interval $[a, b] \subset \mathbf{R}$?

No, we can define $g(0) = 0$ and $g(t) = t^2 \cos(2\pi/t^2)$ for $t \ne 0$. Here $g'(0) = 0$ because $|g(t)| \le t^2$ and, for $t \ne 0$, $g'(t)$ exists by the product and chain rules. Taking $t_n = n^{-1/2}$, we find that the variation of g is infinite on any interval containing 0 because $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

5. Suppose that f is a holomorphic function on the punctured plane $\mathbf{C} \setminus \{0\}$.

(a) For each positive numbers $\varepsilon < R < \infty$, find a formula for $f(z)$ on the annulus $\{z \in \mathbf{C} \ : \ \varepsilon < |z| < R\}$ in terms of the values of f on the inner boundary circle $\{z \ : \ |z| = \varepsilon\}$ and on the outer boundary circle $\{z \ : \ |z| = R\}$. T get a suitable contour of integration, we may remove any thin radial strip from the annular region $\{z \in \mathbf{C} \ : \ \varepsilon < |z| < R\}$. Applying the Cauchy integral formula on the boundary of this region and then letting the width of the thin strip approach 0, we conclude that

$$f(z) \ = \ \frac{1}{2\pi \mathbf{i}} \Big[\int_{\partial \mathbf{B}_R} \frac{f(\zeta}{z - \zeta} d\zeta \ - \ \int_{\partial \mathbf{B}_\varepsilon} \frac{f(\zeta}{z - \zeta} d\zeta \ \Big] \ .$$

(b) If f is meromorphic and

$$\int_{\{z\,:\,0<|z|<1\}} |f(z)|\,dx\,dy \;<\infty\;, \tag{**}$$

then, at 0, f either has a removable singularity or is meromorphic with a pole of order 1. Since f is meromorphic, one has, on a punctured neighborhood of the origin, $z^k f(z) = g(z)$, for some nonnegative integer k and nonvanishing holomorphic function g. If $k \geq 2$, then, for $|z| < \epsilon$ with ϵ sufficiently small,

$$\frac{1}{2}\frac{|g(0)|}{|z|^k} \;<\; |f(z)| \;<\; 2\frac{|g(0)|}{|z|^k}\;.$$

Also

$$\int_{\mathbf{B}_\epsilon} |z|^{-k}\,dx\,dy \;=\; \int_0^{2\pi}\int_0^{\epsilon} r^{1-k} dr\,d\theta \;<\; \infty$$

if and only if $k < 2$. For $k = 0$ the singularity is removable. For $k = 1$, it is a pole of order 1.

(c) Does the integrability assumption (**) alone imply that f is automatically meromorphic at 0. Yes, for any $r > 0$ we can choose, by Fubini's Theorem, a number $\varepsilon(r) \in [\frac{r}{2}, r]$ so that

$$\int_{\partial\mathbf{B}_{\varepsilon(r)}} |f| \;\leq\; \frac{2}{r}\int_{\mathbf{B}_r} |f|\,dx\,dy\;.$$

It follows that for fixed z the line integral

$$\left|\int_{\partial\mathbf{B}_{\varepsilon(r)}} \frac{\zeta f(\zeta)}{z-\zeta}\,d\zeta\right| \;\leq\; \frac{2}{\mathrm{dist}\,(z,\partial\mathbf{B}_{\varepsilon(r)})}\int_{\mathbf{B}_r} |f|\,dx\,dy \;\to\; 0 \;\text{ as }\; r\to 0\;.$$

Applying the formula from (a) with $f(z)$ replaced by $zf(z)$, we conclude that $zf(z)$ has a removable singularity at 0, so that f is meromorphic with a pole of order ≤ 1 at 0.

6. Suppose that $E_1, E_2, E_3, \ldots$ is a sequence of Lebesgue measurable subsets of the unit ball $\mathbf{B}$ in $\mathbf{R}^n$, and that each E_k has positive Lebesgue measure $\mu(E_k) > \varepsilon$ for a fixed $\varepsilon > 0$. For each $x \in \mathbf{B}$, let $n(x)$ denote the number of integers k so that $x \in E_k$.

(a) Show that $n(x) \geq 2$ for some $x \in \mathbf{B}$. Otherwise, the E_k are disjoint and

$$\infty \;>\; \mu(\mathbf{B}) \;\geq\; \mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty}\mu(E_k) = \infty\;.$$

(b) Show that $\sup_{x\in\mathbf{B}} n(x) = \infty$.

$$\infty \;=\; \sum_{k=1}^{\infty}\int \chi_{E_k} \;\leq\; \int\sum_{k=1}^{\infty}\chi_{E_k} \;\leq\; [\sup_{x\in\mathbf{B}} n(x)]\mu(B)\;.$$

(c) Show that $n(x) = \infty$ for some $x \in \mathbf{B}$.

The sets $F_j = \cup_{k=j}^{\infty} E_k$, form a decreasing sequence of measurable subsets of the finite measure set $\mathbf{B}$. So $F = \cap_{j=1}^{\infty} F_j$ is measurable with

$$\mu(F) \;=\; \lim_{j\to\infty} \mu(F_j) \;\geq\; \lim_{j\to\infty} \mu(E_j) \;=\; \varepsilon \;> 0\ .$$

So F contains a point x. Since $x \in F_1$, $x \in E_{n(1)}$ for some positive integer $n(1)$. Since $x \in F_{n(1)+1}$, $x \in E_{n(2)}$ for some integer $n(2) > n(1)$. Continuing, we inductively find a sequence of integers $n(1) < n(2) < n(3) < \ldots$ so that $x \in \cap_{i=1}^{\infty} E_{n(i)}$.