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Constant mean curvature surfaces, harmonic maps and integrable systems, by Frédéric Hélein, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel–Boston–Berlin, 2000, xii+227 pp., $29.95, ISBN 3-7643-6576-5

A remarkable feature of many systems of partial differential equations in two variables that arise in geometry and mechanics is their explicit solvability based on some symmetries or a nonlinear transformation of variables. Such integrable systems include the Korteweg-de-Vries equation, harmonic maps from a sphere to a Lie group, minimal surfaces and certain other conformal constant mean curvature (CMC) surfaces, the two dimensional Toda lattice, and the chiral model from physics. The notion of an *integrable system* is frequently not defined precisely but rather is characterized generally by various inter-related common features including the action of a large Lie group of symmetries to give some algebraic solvability, space-localized solutions (solitons), a nonlinear change of variables transforming to linear equations, solutions generated by a Hamiltonian flow, and a transformation of a solution to a conjugate solution. The present monograph by Frédéric Hélein is a very nice introduction to such equations with emphasis on differential geometry, in particular harmonic maps and CMC surfaces. As outlined briefly in these notes, the foundations for integrable systems began in the sixties and seventies with studies of the Korteweg-de-Vries equation, inverse scattering solutions, and loop group methods. Differential, complex, and algebraic geometry all come in with twistor theory, Lie group-valued harmonic maps, and special tori. Most especially the recent developments in the last 15 years concerning the richness of CMC surfaces provide good motivation for the timeliness and unique emphasis of the present lectures. These latter developments begin with the striking construction of H. Wente [W] of an immersed CMC torus in R^3. This was followed by papers of U. Abresch [A], U. Pinkhall and I. Sterling [PS], N. Kapouleas [K], and D. Ferus, F. Pedit, U. Pinkhall, and I. Sterling [FPPS], giving further construction and important classification results of CMC surfaces. Generalizing the case of classical minimal surfaces, J. Dorfmeister, F. Pedit and H. Wu [DPW] obtained a Weierstrass-type representation formula for symmetric-space-valued harmonic maps.

Introducing these results, their connections, and some of their proofs is a background theme of this monograph. Hélein begins and ends with CMC surfaces. An immersed minimal surface has constant mean curvature 0, and, when conformally parameterized as $X : \Omega \to R^3$, has a Gauss map

$$u = \frac{\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}}{|\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}|} : \Omega \to S^2$$

that is anti-holomorphic. Generalizing this, Ruh-Vilms [RV] showed that a conformally immersed surface in R^3 is CMC if and only if its Gauss map is a *harmonic map*. The derivation of a weakly conformal CMC surface with prescribed Gauss map being a given harmonic map to S^2 leads to the important independent question

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of constructing harmonic maps. The middle third of this monograph is devoted to many ideas and relations involved doing this.

First, Noether's theorem shows how an infinitesimal symmetry for the energy of a regular harmonic map (or, more generally, action of a Hamiltonian) leads to an associated PDE. This is of interest with global symmetries of the target, as when it is the quotient of a Lie group. It also applies with arbitrary infinitesimal smooth deformations of the domain, giving rise to the *stationarity* equations. As an application of the stationarity condition for a smooth harmonic map u of a two dimensional domain Ω, the expression

$$4(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})2\,(dx + idy) \otimes (dx + idy)$$

defines a *holomorphic* quadratic differential. Globally for $\Omega = S^2$, this must vanish identically so that harmonic maps of the 2 sphere are necessarily conformal. In case the target is also S^2, they are simply rational functions of z or $\overline{z}$. One also obtains H. Hopf's theorem [H] that a conformal CMC immersion of S^2 in R^3 is necessarily a round sphere by consideration of the associated Gauss map.

Hélein also interprets the CMC immersibility condition as a zero curvature condition for a connection form defined using a moving frame of the immersion. The nice computations here facilitate, via rotation of the above quadratic differential, construction of a congugate family of CMC surfaces. For harmonic maps from S^2 to higher dimensional S^n, there is next given a brief introduction to twistor theory [BR], treating the construction, from some holomorphic data, of the special class of *isotropic* harmonic maps. Spheres may be viewed as quotients of Lie groups, and one may insert a parameter into the target with an S^1 action. The very general story of working with *loop groups* and more general integrable systems produces a surprisingly large, rich family of solutions. What is crucial in the construction is the finite dimensionality of a vector space associated with the loop Lie algebra being used. The constructed harmonic maps are then said to be of *finite type*. Hélein gives a full proof of the striking Pinkhall-Sterling [PS] classification result that the Gauss map of a conformally immersed CMC torus is of finite type. The book is completed by a revisit to the geometry of the Wente tori, a brief discussion of the Dorfmeister-Pedit-Wu [DPW] Weierstrass-type representation formulas, and a very brief mention of Willmore and Lagrangian surfaces.

This monograph is the compilation of the author's 1999 lectures at ETH, and they retain the same spirit, structure, and vitality. Some of the longer proofs are completed in the appendices of chapters. The remarks and references are short, but very chosen. For the casual reader, it is often necessary to consult earlier chapters to recall notations, context, and definitions. The more serious reader, going from cover to cover, will be rewarded by the enjoyable directness and continuity in the development and bundling of these important and beautiful topics.

For a deeper and very systematic treatment of integrable systems, in one and two dimensions, and their relation to harmonic maps, one should next read the elegant book [G] by Martin Guest. Finally there is a collection of excellent research-expository essays on harmonic maps and integrable systems edited by Fordy and J. Wood, now out-of-print but still available by internet [FW]. All of these sources give many references to the original papers.

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