Application of Scans and Fractional Power Integrands

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In this note we describe the notion of a rectifiable scan and consider some applications [DH1], [DH2] to Plateau-type minimization problems. "Scans" were first introduced in the work [HR1] of Tristan Rivière and the second author to adequately describe certain bubbling phenomena. There, the behavior of certain $W^{1,3}$ weakly convergent sequences of smooth maps from 4 dimensional domains into $\mathbf{S}^2$ led to the consideration of a necessarily infinite mass generalization of a rectifiable current. The definition of a scan is motivated by the fact that a rectifiable current can be expressed in terms of its lower dimensional slices by oriented affine subspaces. By integral geometry, the slicing function for the rectifiable current is a mass integrable function of the subspaces. With a scan one considers more general such functions that are not necessarily mass integrable.

§1. RECTIFIABLE CURRENTS AND THE PLATEAU PROBLEM

An m dimensional *rectifiable set* R in $\mathbf{R}^n$ is a subset of some countable union $\cup_{i=0}^{\infty} M_i$ where $M_1, M_2, \ldots$ are m dimensional $\mathcal{C}^1$ submanifolds and M_0 has m dimensional Hausdorff measure $\mathcal{H}^m(M_0) = 0$. At $\mathcal{H}^m$ almost every point $x \in R$, R has an approximate tangent space. [S],§3. An m dimensional (integer-multiplicity) *rectifiable current* T in $\mathbf{R}^n$ is given by a bounded m dimensional Borel measurable rectifiable *concentration* set R_T together with an $\mathcal{H}^m$ integrable density function $\theta_T : R_T \to \{1, 2, \ldots\}$ and an $\mathcal{H}^m$ measurable orientation $\vec{T} : R_T \to \bigwedge_m \mathbf{R}^n$ so that at $\mathcal{H}^m$ almost every $x \in R_T$, $\vec{T}(x)$ is the wedge product of vectors from an orthonormal basis of the approximate tangent space of R_T at x. See [F1],4.1.24 or [S],§27. Thus the action of the current T on a differential m form $\phi \in \mathcal{D}^m(\mathbf{R}^n)$ is given by the integration

$$T(\phi) \ = \ \int_{R_T} \langle \vec{T}(x), \phi(x) \rangle \, \theta_T(x) \, d\mathcal{H}^m x \ .$$

The *mass* of T is then simply $\mathbf{M}(T) = \int_{R_T} \theta_T(x) \, d\mathcal{H}^m x$. For $m \geq 1$, *boundary* of T is the $m-1$ dimensional current defined by the formula $\partial T(\psi) = T(d\psi)$ for $\psi \in \mathcal{D}^{m-1}(\mathbf{R}^n)$. A rectifiable current generalizes an oriented submanifold M. We sometimes use the abbreviated notation $[\![M]\!]$, in case the orientation is known, for the corresponding multiplicity one rectifiable current. So if M is an oriented manifold with boundary, Stokes Theorem becomes $\partial [\![M]\!] = [\![\partial M]\!]$. Even though $\mathcal{H}^m(R_T) < \infty$, one should be aware that

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the *support* of the current, $\operatorname{spt} T$, which may be much larger than R_T, might even be n dimensional for $m \geq 1$. On the other hand a *zero* dimensional rectifiable current is simply a finite integral combination of point masses. Let $\mathcal{R}_m$ denote the group of m dimensional rectifiable currents in $\mathbf{R}^n$.

In 1960, H. Federer and W. Fleming obtained the following fundamental existence theorem:

1.1 Theorem. [FF] *Given any $T_0 \in \mathcal{R}_m$ with $\partial T_0 \in \mathcal{R}_{m-1}$, the family of currents $\{T \in \mathcal{R}_m \ : \ \partial T = \partial T_0\}$ contains a rectifiable current of least mass.*

This theorem is valid for all $m \geq 1$ in any $\mathbf{R}^n$ as well as in any compact Riemannian manifold (provided the admissible family is nonempty). There one also has, in any homology class, a rectifiable current that minimizes mass. Among *general* currents the existence of a mass minimizers is an easy consequence of the Banach-Alaoglu theorem, but what is important in [FF] is the rectifiability, which should be understood as an initial regularity for minimizers.

The complete interior regularity, of such rectifiable mass-minimizers, i.e. that $\operatorname{spt} T \setminus \operatorname{spt} \partial T$ is an embedded real analytic submanifold, was established in the sixties for $1 \leq m = n-1 \leq 6$ by works of Fleming [Fl], De Giorgi [D], Almgren [A1], Triscari [Tr], and Simons [Ss]. De Giorgi's work showed that a 7 dimensional mass-minimizer in $\mathbf{R}^8$ would have at most isolated interior singularities, and, in fact in 1970, Bombieri, De Giorgi [D], and Giusti [BDG] established the mass-minimality of the specific example

$$Q \ = \ \partial[\![\{(x,y) \in \mathbf{R}^4 \times \mathbf{R}^4 \ : \ |x| < |y|\}]\!] \llcorner \mathbf{B}^8_1 \ ,$$

which has an isolated singularity at $(0,0)$. Then the cartesian product with a cube $Q \times [\![-1,1]\!]^j$ is mass minimizing in $\mathbf{R}^{8+j}$, and so the following result of H. Federer gives the optimal estimate of the Hausdorff dimension of the interior singular set of a codimension one minimizer:

1.2 Theorem. [F2] *For any m dimensional mass-minimizing rectifiable current T in $\mathbf{R}^{m+1}$ and $\epsilon > 0$*

$$\mathcal{H}^{m-7+\epsilon}\big(\operatorname{Sing}(\operatorname{spt} T \setminus \operatorname{spt} \partial T)\big) \ = \ 0 \ .$$

The complete boundary regularity of T was established by Hardt and Simon [HS] near any point where the given ∂T_0 is a smooth $m-1$ dimensional oriented embedded submanifold of $\mathbf{R}^{m+1}$. In 1984, F.J. Almgren completed a massive work treating the higher codimension interior partial regularity:

1.3 Theorem. [A4] *For any m dimensional mass-minimizing rectifiable current T in $\mathbf{R}^n$ and $\epsilon > 0$*

$$\mathcal{H}^{m-2+\epsilon}\big(\operatorname{Sing}(\operatorname{spt} T \setminus \operatorname{spt} \partial T)\big) \ = \ 0 \ ,$$

and S. Chang [C] showed that interior singularities are at most isolated points for $m = 2$. Note that [FF] already contained the singular minimizing example of the sum of two oriented totally orthogonal disks in $\mathbf{R}^2 \times \mathbf{R}^2$.

$$[\![\mathbf{B}_1^2 \times \{0\}]\!] \ + \ [\![\{0\} \times \mathbf{B}_1^2]\!] \ .$$

§2. SIZE AND FRACTIONAL POWERS OF THE DENSITY

In $\mathbf{R}^3$, two dimensional mass-minimizing rectifiable currents have no interior singularities and provide a nice model for some but not all "soap films". General soap films may have interior singular curves which simultaneously border three surfaces meeting at equal angles. To use currents in a better model for soap films, Almgren [A3] introduced the notion of *size* for a rectifiable current:

$$\mathrm{Size}\,(T) \ = \ \mathcal{H}^m(R_T) \ .$$

Thus one ignores the density function in computing size. To understand size versus mass minimization, consider the one dimensional rectifiable current in the plane consisting of 2 parallel similarly oriented intervals

$$T_0 \ = \ [\![(-\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{7}{2}, \frac{\sqrt{3}}{2})]\!] \ + \ [\![(-\frac{1}{2}, -\frac{\sqrt{3}}{2}), (\frac{7}{2}, -\frac{\sqrt{3}}{2})]\!] \ .$$

Then T_0 is mass-minimizing among all rectifiable currents having boundary equal to ∂T_0. However, the size minimizer is sum of five oriented intervals,

$$\begin{aligned} T_1 \ &= [\![(-\frac{1}{2}, \frac{\sqrt{3}}{2}), (0,0)]\!] \ + \ [\![(-\frac{1}{2}, -\frac{\sqrt{3}}{2}), (0,0)]\!] \\ &\quad + \ 2[\![(0,0), (3,0)]\!] \ + \ [\![(3,0), (\frac{7}{2}, \frac{\sqrt{3}}{2})]\!] \ + \ [\![(3,0), (\frac{7}{2}, \frac{\sqrt{3}}{2})]\!] \ , \end{aligned}$$

one of which has multiplicity 2. Note that

$$\mathbf{M}(T_0) \ = \ 2 \cdot 4 \ < \ 4 \cdot 1 + 2 \cdot 3 \ = \ \mathbf{M}(T_1) \text{ and } \mathrm{Size}\,(T_0) \ = \ 8 \ > \ 4 \cdot 1 + 3 \ = \ \mathrm{Size}\,(T_1).$$

Similarly in dimension 2 one may consider the sum T_0 of two close coaxial, parallel and similarly oriented, disks. Then T_0 is mass-minimizing, but the size-minimizer T_1 with boundary ∂T_0 contains a single multiplicity 2 disk in the middle and the set $\mathrm{spt}\, T_1$ models a soap film with an interior singular curve.

A fundamental problem with size-minimization is the lack of a general existence theorem. One is faced with the possibility of a size-minimizing sequence of rectifiable currents having unbounded masses and failing to have subsequences convergent as currents. This is what happens in an example of F. Morgan.

2.1 Example [M]. For a fixed $1 < \beta < 2$ consider the following countable sum of vertical oriented intervals in the plane:

$$T_0 \;=\; \sum_{j=1}^{\infty} [\![(\frac{1}{j}, -\frac{1}{j^\beta}), (\frac{1}{j}, \frac{1}{j^\beta})]\!] \;.$$

Then $T_0 \in \mathcal{R}_1$ because $\mathbf{M}(T_0) = \sum_{j=1}^{\infty} \frac{2}{j^\beta} < \infty$. One may check that T_0 is mass-minimizing by an easy callibration argument. However, one sees, as in the previous section, that one may decrease the size, at the expense of increasing the mass, by replacing the oriented interval $[\![(1,-1),(1,1)]\!]$ by the sum of three intervals

$$[\![(1,-1), (\frac{1}{2}, 1)]\!] \;+\; [\![(\frac{1}{2}, -\frac{1}{2^\beta}), (\frac{1}{2}, \frac{1}{2^\beta})]\!] \;+\; [\![(\frac{1}{2}, \frac{1}{2^\beta}), (1,1)]\!] \;.$$

The new current has the multiplicity 2 interval $2[\![(\frac{1}{2}, -\frac{1}{2^\beta}), (\frac{1}{2}, \frac{1}{2^\beta})]\!]$ which we may then replace by the sum

$$2[\![(\frac{1}{2}, -\frac{1}{2^\beta}), (\frac{1}{3}, -\frac{1}{3^\beta})]\!] \;+\; 2[\![(\frac{1}{3}, -\frac{1}{3^\beta}), (\frac{1}{3}, \frac{1}{3^\beta})]\!] \;+\; 2[\![(\frac{1}{3}, \frac{1}{3^\beta}), (\frac{1}{2}, -\frac{1}{2^\beta})]\!] \;.$$

Continuing we obtain a size-minimizing sequence whose mass approaches infinity. These do not converge as currents and the resulting formal countable sum of oriented intervals with integer multiplicities is *not* a current. We will see that this can be understood as a "scan".

There are some positive results concerning size-minimization.

2.2 Theorem. [M] *If* $\mathrm{spt}\,(\partial T_0)$ *is a smooth* $m-1$ *dimensional submanifold of* $\mathbf{R}^{m+1}$ *that lies on the boundary of a smooth compact convex body, then there exists a size-minimizing rectifiable current* T *with* $\partial T = \partial T_0$.

The idea here is that one can modify a size-minimizing sequence using decompositions into oriented boundaries of sets which extend all the way to the boundary Γ of the convex body. Since $\Gamma \setminus \mathrm{spt}\,\partial T_0$ contains only finitely many components, one thus obtains a bound $\mathcal{H}^m$ almost everywhere on the densities in the sequence. So the masses are bounded, and one has convergence as currents. The lower semicontinuity of size under this condition was established in [A3].

In [DH1] we obtain, for a general codimension one rectifiable current T_0, (without the convex hull property of $\mathrm{spt}\,(\partial T_0)$) a weaker result concerning the existence of a *minimizing set*. An m dimensional set $S \subset \mathbf{R}^n$ is *minimizing* [A2] relative to a compact set K if

$$\mathcal{H}^m[f(S)] \;\geq\; \mathcal{H}^m(S)$$

for any Lipschitz map $f : \mathbf{R}^n \to \mathbf{R}^n$ with $\overline{\{x \,:\, f(x) \neq x\}} \subset \mathbf{R}^n \setminus K$.

2.3 Theorem. [DH1] *If, in* $\mathbf{R}^{m+1}$, $T_0 \in \mathcal{R}_m$ *and* $\mathcal{H}^m(\operatorname{spt}\partial T_0)) = 0$, *then there exists a minimizing set* S *with respect to* $\operatorname{spt}\partial T_0$ *having* $\operatorname{spt}\partial T_0 \subset \overline{S}$.

Here is a brief outline of our construction.

First we penalize the lack of compactness by choosing, for any $0 < \epsilon < 1$, a rectifiable current T_ϵ minimizing $\operatorname{Size}(T) + \epsilon\mathbf{M}(T)$ among rectifiable currents T with $\partial T_\epsilon = \partial T_0$.

Second we observe that the (renormalized) measure

$$\mu_\epsilon \;=\; (\mathcal{H}^m \;\llcorner\; R_{T_\epsilon})(1 \;+\; \epsilon\theta_{T_e})$$

defines a (real) rectifiable varifold, *stationary* [Al] in $\mathbf{R}^{m+1} \setminus \operatorname{spt}\partial T_0$. That is, the first variation of the total mass of the varifold μ_ϵ vanishes for any deformation which fixes $\operatorname{spt}\partial T_0$. Since the total mass of each μ_ϵ is bounded by $\operatorname{Size}(T_0) + \mathbf{M}(T_0)$, we may by [Al], choose a sequence μ_{ϵ_j} weakly convergent to a rectifiable varifold μ. We show that the m density of this varifold is one at $\mathcal{H}^m$ almost every point of $\operatorname{spt}\mu \setminus \operatorname{spt}\partial T_0$. Thus $\mu = \mathcal{H}^m \;\llcorner\; S$ for some rectifiable set S.

Third, by modifying some arguments of Ambrosio, Fusco and Hutchinson [AKH], which proved the minimality of a limit of codimension one minimizing sets, we verify that S is the desired minimizing set.

It is an interesting question to find conditions that will guarantee that the *currents* T_{ϵ_j} converge. While Morgan's example above shows that this will not happen in general, the condition that $\operatorname{spt}\partial T_0$ be a smooth submanifold may be sufficient.

Even for the "soap-film case" of two dimensional size-minimizers in $\mathbf{R}^3$, lack of a priori knowledge of the boundary behavior prevents progress. In the next result, one avoids the boundary behavior problem.

2.4 Theorem. [M], [DH1] *In a compact Riemannian* 3 *manifold, any two dimensional homology class contains a size-minimizing rectifiable current.*

J. Taylor [T] classified the local interior structure of the minimizing set S. Up to a $\mathcal{C}^{1,\alpha}$ diffeomorphism of space, the neighborhood of an interior point is either a plane, three half-planes meeting at equal angles along a line, or 6 planar sectors meeting at equal dihedral angles at a point. This local structure and the compactness of S imply that there exists a global Lipschitz retraction ρ of some open neighborhood of S onto S. Since $\operatorname{spt} T_{\epsilon_j} = \operatorname{spt}\mu_{\epsilon_j}$ converge in Hausdorff metric to S, we find that, for some fixed, sufficiently large j, the retracted current $\rho_\# T_{\epsilon_j}$ is the desired size-minimizing homology representative.

To use this retraction argument for the original Plateau boundary problem, one would need more information about the boundary behavior of minimizing sets or (approximately) size-minimizing currents.

One can also consider a *free boundary* or *obstacle* problem where the minimization occurs among rectifiable currents constrained to have their boundaries lie on a given hypersurface. Here, for two dimensions in three space, the local structure is easy to classify. Again up to a $\mathcal{C}^{1,\alpha}$ diffeomorphism of space, the neighborhood of a free boundary point is either a half-space perpendicular to the given hypersurface or else 3 quadrants at equal angles to each other, meeting the hypersurface orthogonally. Thus one obtains

2.5 Theorem. [DH2] *For any smoothly bounded region* $\Omega \subset \mathbf{R}^3$ *and any* $T_0 \in \mathcal{R}_2$ *with* $\operatorname{spt} T_0 \subset \mathbf{R}^3 \setminus \Omega$ *and* $\operatorname{spt} \partial T_0 \subset \partial\Omega$, *the relative homology class*

$$\{T \in \mathcal{R}_2 \ : \operatorname{spt} T \subset \mathbf{R}^3 \setminus \Omega,\ \operatorname{spt} \partial T_0 \subset \partial\Omega,\ \text{and}$$
$$\operatorname{spt}(T - T_0 - \partial S) \subset \partial\Omega \text{ for some } S \in \mathcal{R}_3\}$$

contains a rectifiable current of least size.

Note that one can imagine Ω as the interior of a thick 3 dimensional wire whose boundary surface is supporting the boundary of a soap film. Unfortunately this last existence theorem provides no obvious uniform mass bounds on the size-minimizing currents for a sequence of such thick wires $\overline{\Omega_i}$ shrinking to a 1 dimensional smooth curve.

There are many other functionals that may also give rise to some minimizing sequences which have unbounded mass and which have no subsequences convergent as currents. Perhaps the simplest are given by "fractional" powers of the density.

2.6 Definition. For any $\alpha \in [0,1]$ and $T \in \mathcal{R}_m$, the α-*mass* of T equals

$$\mathbf{M}_\alpha(T) \ = \ \int_{R_T} [\theta_T(x)]^\alpha \, d\mathcal{H}^m x \ .$$

The case $\alpha = 0$ corresponds to the $\operatorname{Size}(T) = \mathcal{H}^m(R_T)$ and the case $\alpha = 1$ corresponds to the ordinary mass $\mathbf{M}(T)$. The special property of the range $0 \le \alpha \le 1$ is that the functional $\mathbf{M}_\alpha$ is weakly lower semicontinuous on mass bounded sequences. In case $m = 0$ one is only dealing with "collisions of atoms" and the result is elementary, based on the inequality

$$|i + j|^\alpha \ \le \ |i|^\alpha \ + \ |j|^\alpha \text{ for any integers } i \text{ and } j \ ,$$

which is only valid for $\alpha \in [0,1]$. For larger m one may reduce to the case $m = 0$ by slicing which we now review.

§3 SLICING

For any bounded Borel m form Ω on $\mathbf{R}^n$, let $T \llcorner \Omega$ denote the 0 dimensional current given by $(T \llcorner \Omega)(\psi) = T(\psi\Omega)$ for $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$. From [F1], 4.3 we recall that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is Lipschitz and $T \in \mathcal{R}_m$, then, for a.e. $y \in \mathbf{R}^m$ the slice $\langle T, f, y\rangle \in \mathcal{R}_0$ where

$$\langle T, f, y\rangle \ = \ \lim_{r \downarrow 0} T \llcorner f^\# \big(\frac{\chi_{\mathbf{B}_r(y)}}{\omega_m r^m}\Omega_m\big) \text{ with } \Omega_m = dx^1 \wedge \ldots \wedge dx^m \ .$$

In fact, for a.e. $y \in \mathbf{R}^m$, $R_T \cap f^{-1}\{y\}$ is a finite set of points x where the approximate tangent space of R_T exists with $f|R_T$ being approximately differentiable of rank m at x, and the slice is given by the formula

$$\langle T, f, y\rangle \ = \sum_{x \in R_T \cap f^{-1}\{y\}} \sigma(x)\theta_T(x)\delta_x$$

where $\sigma(x) = \operatorname{sgn}\langle \bigwedge_m Df(x)\vec{T}(x), \Omega_m\rangle$. One also has the integral formulas

$$\int_{\mathbf{R}^m} \langle T, f, y\rangle\, dy \ = \ T \llcorner f^{\#}\Omega_m \ ,$$

$$\int_{\mathbf{R}^m} \mathbf{M}_\alpha\langle T, f, y\rangle\, dy \ = \ \mathbf{M}_\alpha[T \llcorner f^{\#}\Omega_m] \ .$$

Recall also the compact space $\mathcal{P}$ of orthogonal projections of $\mathbf{R}^n$ onto $\mathbf{R}^m$, which has an invariant probability measure induced by the transitive action of $\mathbf{O}(n)$. In particular, for each increasing function $\lambda : \{1, \ldots, m\} \to \{1, \ldots, n\}$ we have the coordinate projection

$$p_\lambda : \mathbf{R}^n \to \mathbf{R}^m \ , \quad p_\lambda(x_1, \ldots, x_n) = (x_{\lambda(1)}, \ldots, x_{\lambda(m)}) \ .$$

By writing an m form $\phi \in \mathcal{D}(\mathbf{R}^n)$ in coordinates

$$\phi \ = \ \sum_\lambda \phi_\lambda dx^\lambda \ = \ \sum_\lambda \phi_\lambda p_\lambda^{\#}\Omega_m$$

with each $\phi_\lambda \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, we see how the current T is expressed in terms of its 0 dimensional coordinate slices

$$\begin{aligned} T(\phi) \ &= \ \sum_\lambda T[\phi_\lambda p_\lambda^{\#}\Omega_m] \ = \ \sum_\lambda \big(T \llcorner p_\lambda^{\#}\Omega_m\big)(\phi_\lambda) \\ &= \ \sum_\lambda \int_{\mathbf{R}^m} \langle T, p_\lambda, y\rangle(\phi_\lambda)\, dy \ . \end{aligned}$$

Composing with rotations and integrating over $\mathbf{O}(n)$, we find a formula

$$\begin{aligned} T(\phi) \ &= (\operatorname{vol}\mathbf{O}(n))^{-1} \int_{\mathbf{O}(n)} \sum_\lambda \int_{\mathbf{R}^m} \langle T, p_\lambda \circ g, y\rangle(\phi \cdot (p_\lambda \circ g)^{\#}\Omega_m)\, dy\, dg \\ &= \binom{n}{m}(\operatorname{vol}\mathbf{O}(n))^{-1} \int_{\mathcal{P}} \int_{\mathbf{R}^m} \langle T, p, y\rangle(\phi \cdot p^{\#}\Omega_m)\, dy\, dp \ . \end{aligned}$$

One also has the integral geometric relation

$$\mathbf{M}_\alpha(T) \ = \beta(m, n) \int_{\mathcal{P}} \int_{\mathbf{R}^m} \mathbf{M}_\alpha\langle T, p, y\rangle\, dp\, dy \ .$$

This motivates us to define various classes of *scans* as measurable functions

$$\mathcal{T} \ : \ \mathcal{P} \times \mathbf{R}^m \ \to \ \{0 \text{ dimensional currents}\} \ .$$

In particular, a *rectifiable scan* is a measurable function

$$\mathcal{T} \ : \ \mathcal{P} \times \mathbf{R}^m \ \to \ \mathcal{R}_0$$

corresponding to some rectifiable set $R_{\mathcal{T}}$, some $\mathcal{H}^m$ measurable $\theta_{\mathcal{T}} : R_{\mathcal{T}} \to \{1, 2, \ldots\}$ and some $\mathcal{H}^m$ measurable orientation $\vec{\mathcal{T}}$ of $R_{\mathcal{T}}$ so that, for almost all $(p, y) \in \mathcal{P} \times \mathbf{R}^m$,

$$\mathcal{T}(p, y) \ = \ \sum_{x \in R_{\mathcal{T}} \cap p^{-1}\{y\}} \sigma(x)\theta_{\mathcal{T}}(x)\delta_x$$

where $\sigma(x) = \operatorname{sgn} \langle \bigwedge_m p \vec{\mathcal{T}}(x), \Omega_m \rangle$. One defines

$$\mathbf{M}_\alpha(\mathcal{T}) \ = \ \beta(m, n) \int_{\mathcal{P}} \int_{\mathbf{R}^m} \mathbf{M}_\alpha\big(\mathcal{T}(p, y)\big) \, dp \, dy \ ,$$

and sees that the scan $\mathcal{T}$ corresponds to a rectifiable current if and only if $\mathbf{M}_1(\mathcal{T}) < \infty$. For a rectifiable current $T \in \mathcal{R}_m$, an elementary Fourier transform argument shows that

$$\partial T = 0 \ \text{ if and only if } \ \big\langle T, p, y \big\rangle(1) \ = \ 0 \text{ for almost all } \ (p, y) \in \mathcal{P} \times \mathbf{R}^m \ .$$

Thus, for 2 rectifiable scans $\mathcal{S}, \mathcal{T}$ we say

$$\partial \mathcal{S} = \partial \mathcal{T} \ \text{ if and only if } \ (\mathcal{S} - \mathcal{T})(p, y)(1) \ = \ 0 \text{ for almost all } \ (p, y) \ .$$

3.1 Theorem. [DH1] *Suppose* $0 < \alpha \leq 1$, $T_0 \in \mathcal{R}_m$, *and* $\mathcal{H}^m(\operatorname{spt} \partial T_0) = 0$. *Then there exists a rectifiable scan* $\mathcal{T}$ *with* $\partial \mathcal{T} = \partial T_0$ *and*

$$\mathbf{M}_\alpha(\mathcal{T}) \ = \ \inf\{\mathbf{M}_a(T) \ : \ T \in \mathcal{R}_m, \ \partial T = \partial T_0\} \ .$$

The proof involves getting the convergence as scans of a subsequence of an $\mathbf{M}_\alpha$ minimizing sequence. We must work with a convergence that is weaker than the weak convergence of currents. To see this, consider the following

3.2 Example. Suppose $S_j = j^2 \partial [\![\mathbf{B}_{1/j}]\!]$ in $\mathbf{R}^2$. Thus S_j is an oriented circle of radius $\frac{1}{j}$ with multiplicity j^2. Then, in the weak topology of currents,

$$T_j \ \rightharpoonup \ \partial [\![\delta_{(0,0)} \mathbf{e}_1 \wedge \mathbf{e}_2]\!] \ \neq \ 0 \ ,$$

but, for $0 < \alpha < \frac{1}{2}$,

$$\mathbf{M}_\alpha(T_j) \ = \ j^{2\alpha} \cdot (\frac{2\pi}{j}) \ \to \ 0 \ .$$

An appropriate topology is the α flat distance [Fl2], [W] on $\mathcal{R}_m$

$$\mathcal{F}_\alpha(T_1,T_2) \;=\; \inf\{\mathbf{M}_\alpha(R)+\mathbf{M}_\alpha(S) \;:\; T_1-T_2=R+\partial S,\ R\in\mathcal{R}_m,\ S\in\mathcal{R}_{m+1}\}\ .$$

Recall now from [AK] that a measurable function f from $\mathbf{R}^m$ into a general metric space X has *finite total variation* if $\phi\circ f$ is BV for all Lipschitz $\phi: X\to\mathbf{R}$ and

$$\|Df\|(\mathbf{R}^m) \;\equiv\; \sup\{\int |D(\phi\circ f)| \;:\phi: X\to\mathbf{R},\ \mathrm{Lip}\,\phi\le 1\} \;<\; \infty\ .$$

R. Jerrard [JS] observed that the slice of a normal current was of metric bounded variation with respect to the flat norm. The next result is the analogue for the $\mathcal{F}_\alpha$ distance.

3.3 Theorem. [DH1] *If $T\in\mathcal{R}_m$, $\partial T\in\mathcal{R}_{m-1}$, $\mathbf{M}_\alpha(T)+\mathbf{M}_\alpha(\partial T)<\infty$, and $p\in\mathcal{P}$, then*

$$f:\mathbf{R}^m\to\big(\mathcal{R}_0,\mathcal{F}_\alpha\big),\quad f(y) \;=\; \big\langle T,p,y\big\rangle$$

has finite total variation

$$\|Df\|(\mathbf{R}^m) \;\le\; m\,[\mathbf{M}_\alpha(T)+\mathbf{M}_\alpha(\partial T)\,]$$

Proof : In case $m=1$ the equation, for almost all $s<t$,

$$\big\langle T,p,s\big\rangle \;-\; \big\langle T,p,t\big\rangle \;=(\partial T)\;\llcorner\; p^{-1}[s,t] \;-\; \partial\big(T\;\llcorner\; p^{-1}[s,t]\big)$$

implies that

$$\mathcal{F}_\alpha\big(\big\langle T,p,s\big\rangle,\big\langle T,p,t\big\rangle\big) \;\le\; \mathbf{M}_\alpha\big(T\;\llcorner\; p^{-1}[s,t]\big) \;+\; \mathbf{M}_\alpha\big((\partial T)\;\llcorner\; p^{-1}[s,t]\big)\ .$$

Summing over almost all partitions of $\mathbf{R}$ then gives that the essential variation

$$\mathrm{essvar}\,(f) \;\le\; \mathbf{M}_\alpha(T)+\mathbf{M}_\alpha(\partial T)\ ,$$

which implies the case $m=1$.

For $m=2,3,\ldots$, we use the formula [F1],4.5.9.27

$$\int_{\mathbf{R}^m}|D\psi| \;\le\; \sum_{i=1}^m\int_{\mathbf{R}^{m-1}}\mathrm{essvar}\,\psi(y_1,\ldots,y_{i-1},\cdot,y_{i+1},\ldots,y_m)\,dy_1\ldots dy_{i-1}dy_{i+1}\ldots dy_m$$

with $\psi=\phi\circ f$ to get the desired bound.

This estimate gives the desired variation in y. To get a similar variation bound in (p,y) one may use, for almost all (p,y), the formula

$$\big\langle T,p,y\big\rangle \;=\; \Pi_\#\big\langle T\times[\![\mathcal{P}]\!],P,(y,p)\big\rangle$$

where $\Pi(x,q)=x$ and $P(x,q)=\big(q(x),q\big)$ for $(x,q)\in\mathbf{R}^n\times\mathcal{P}$.

From the usual BV compactness for real-valued functions it is not difficult to derive the following metric-space-valued version:

3.4 Theorem. [DH1] *Suppose N is a Riemannian manifold, Y is a separable metric space, $M : Y \to \mathbf{R}^+$ is lower semicontinuous, and $M^{-1}[0,R]$ is sequentially compact for all $R > 0$. If $f_j : N \to Y$ is measurable with*

$$\Lambda \;=\; \sup_j \|Df_j\|(N) \;+\; \int_N M\big(f_j(x)\big) \;<\; \infty \;,$$

then some subsequence $f_{j'}$ converges pointwise a.e. to a measurable $f : N \to Y$ with

$$\|Df\|(N) \;+\; \int_N M\big(f(x)\big) \;\leq\; \Lambda \;.$$

To prove the existence of a suitable scan $\mathcal{T}$ for Theorem 3.1, we can now apply Theorem 3.3, the remark after, and Theorem 3.4 with

$$N = \mathcal{P} \times \mathbf{R}^m,\; Y = (\mathcal{R}_0, \mathcal{F}_\alpha),\; M = \mathbf{M}_\alpha,\; f_j(p,y) = \langle T_j, p, y\rangle.$$

The lower semicontinuity follows from Fatou's lemma and the lower semicontinuity of $\mathbf{M}_a$ on $(\mathcal{R}_0, \mathcal{F}_\alpha)$.

To verify the rectifiability of the limiting scan, we obtain a rectifiable varifold as in the proof of Theorem 2.3 and then show that our scan is necessarily concentrated on an $\mathcal{H}^m$-finite rectifiable concentration set of this varifold. Almost every projection is transverse a.e. to this rectifiable set and one works with the convergence of the slices in these directions to eventually get the desired consistently defined multiplicity $\theta_{\mathcal{T}}$ and orientation $\vec{\mathcal{T}}$.

In [DH2] we obtain a general compactness theorem in the class of rectifiable scans having rectifiable boundaries and having $\mathbf{M}_\alpha + \mathbf{M}_a\partial$ uniformly bounded. This gives an existence theory for various Plateau problems. We also obtain the optimal interior partial regularity estimate for $\mathbf{M}_\alpha$ minimizers:

3.5 Theorem. [DH2] *For any m dimensional $\mathbf{M}_\alpha$ minimizing rectifiable scan $\mathcal{T}$ in $\mathbf{R}^n$ and $\epsilon > 0$*

$$\mathcal{H}^{m-1+\epsilon}\big(\mathrm{Sing}\,(\mathrm{spt}\,\mathcal{T} \setminus \mathrm{spt}\,\partial\mathcal{T})\big) \;=\; 0 \;.$$

Moreover, $\mathcal{T} \llcorner K$ has finite mass for every compact $K \subset \mathbf{R}^n \setminus \mathrm{spt}\,\partial\mathcal{T}$.

The proof uses the following (roughly stated)

3.6 Lemma. *If $0 < \alpha < 1$, then, near any point a of $\mathrm{spt}\,\mathcal{T} \setminus \mathrm{spt}\,\partial\mathcal{T}$ where $\mathcal{T}$ has a multiplicity ν tangent plane, the scan $\mathcal{T}$ is mostly a multiplicity ν graph of a single function.*

In shrinking cylinders about such a point, the ν excess [F1], 5.3, as well as the ordinary excess, approaches zero. This is in sharp contrast to the case $\alpha = 1$ of mass-minimizing currents. For example, the complex cusp

$$\{(z,w) \in \mathbf{C} \times \mathbf{C}\;: w^2 = z^3\}$$

supports a multiplicity one mass-minimizing rectifiable current with a multiplicity two tangent plane at $(0,0)$. The proof of Lemma 3.6 is based on a multi-valued graphical approximation followed by a squashed comparison current.

§4. ANOTHER FRACTIONAL INTEGRAND

In the works [HR1], [HR2] treating various energy-bounded sequences of Sobolev mappings, one encounters rectifiable currents with bounds on the integral of fractional powers of the (ordinary) mass of slice. In our notations, the analogous situation is to consider for $T \in \mathcal{R}_m$ and $0 < \alpha \leq 1$, the integral

$$\tilde{\mathbf{M}}_\alpha(T) \;=\; \beta(m,n) \int_{\mathcal{P}} \int_{\mathbf{R}^m} \big(\mathbf{M}\langle T, p, y\rangle\big)^\alpha dp\, dy \;.$$

Note the inequalities

$$\tilde{\mathbf{M}}_\alpha(T) \;\leq\; \mathbf{M}_\alpha(T), \quad \mathbf{M}(T) \leq \tilde{\mathbf{M}}_1(T) \leq \mathbf{M}(T) \;.$$

4.1 Example. Consider in $\mathbf{R}^2$ the concentric, multiplicity one circles $T_j = \sum_{i=1}^{j} \partial[\![\mathbf{B}_{1/i}]\!] \in \mathcal{R}_1$. Then, as $j \to \infty$,

$$\mathbf{M}_\alpha(T_j) \;=\; \mathbf{M}(T_j) \to \infty \text{ , but } \sup_j \tilde{\mathbf{M}}_\alpha(T_j) \;<\; \infty \;.$$

Here the limit is a rectifiable scan with $\tilde{\mathbf{M}}_\alpha$ finite, but with a concentration set of infinite Hausdorf measure.

As in [HR1], [HR2], one may again work with $(\mathcal{R}_0, \mathcal{F}_\alpha)$ to obtain the lower semicontinuity of $\tilde{\mathbf{M}}_\alpha$. One can obtain the existence of $\tilde{\mathbf{M}}_\alpha$ minimizing scans by using a weak-type bound in the Lorentz space $L^{\frac{1}{\alpha},\infty}$. See [HR1], §9. However the partial regularity or even rectifiability of $\tilde{\mathbf{M}}_\alpha$ minimizers is unknown. See [HR1], §.8.2.

§5. RELATED PROBLEMS

In the study of $\mathbf{M}_a$ minimizers, one may replace the power function θ^α by a smooth concave unbounded increasing function $H(\theta)$ with $H(0) = 0$ and $H(1) = 1$. Use of scans accommodates as well treatment of the case when $H = H(\theta, x)$ is also allowed to depend smoothly on the space variable x.

By using $H(\theta)$ as an alternate norm on the group of integers, a rectifiable minimizer may also be found in a generalized class of flat chains following the works of Fleming [F2] and White [W]. The close relation between rectifiability and slicing explained in [AK] and [W] was an important motivation for the definition of scans in [HR1] and [DH1].

One dimensional *flat* $\mathbf{M}_\alpha$ minimizers are also applied to describe transport paths in [X1], [X2]. The $\mathbf{M}_\alpha$ functional reflects the efficiency of combining paths in various distribution systems, such as mail delivery, the circulatory system, etc. Here one obtains, for any two probability measures μ_0, μ_1 in $\mathbf{R}^n$, a *flat* 1 chain T which minimizes $\mathbf{M}_\alpha$ subject to the constraint $\partial T = \mu_1 - \mu_0$ as 0 dimensional currents. In [X1], the transport path T has positive *real* density function θ_T with values in $(0,1]$ because $\mathbf{M}(\mu_1) = 1 = \mathbf{M}(\mu_2)$. So one here has the inequality

$$\mathbf{M}_\alpha(T) \geq \mathbf{M}(T) \ ,$$

which is just the opposite of the inequality that we had in our study [DH1], [DH2] of integer-multiplicity rectifiable currents. In contrast to [DH1], [DH2], $\mathbf{M}_\alpha$ minimization for transport paths always gives finite mass *currents*. In [X2], Q.Xia proves the precise local interior regularity of an $\mathbf{M}_\alpha$ minimizing path: that, in $\mathbf{R}^n \setminus (\operatorname{spt}\mu_1 \cup \operatorname{spt}\mu_2)$, T is locally a finite collection of oriented intervals with multiplicities.

Finally we have begun work in [DH3] on carrying over various results of geometric measure theory to rectifiable scans. There we introduce the notion of a rectifiable scan in a metric space X, and give some results generalizing the work of L.Ambrosio and B. Kirchheim [AK] on currents in metric spaces. The idea is that an m dimensional rectifiable scan in X is a measurable function

$$\mathcal{T} \ : \ \operatorname{Lip}(X, \mathbf{R}^m) \times \mathbf{R}^m \ \to \ \mathcal{R}_0(X)$$

which admits a representation in terms of an m dimensional rectifiable [AK] set $R_{\mathcal{T}} \subset X$, and integer density function $\theta_{\mathcal{T}}$, and an orientation $\vec{\mathcal{T}}$ of $R_{\mathcal{T}}$. (The notion of orientation requires some effort to describe.) For a Lipschitz map $g : X \to Y$ of metric spaces and m dimensional rectifiable scan $\mathcal{T}$ on X, the push-forward $g_\# \mathcal{T}$ defined by

$$(g_\# \mathcal{T})(f, y) \ = \ g_\# \big(\mathcal{T}(f \circ g, y)\big)$$

for almost all $(f, y) \in \operatorname{Lip}(X, \mathbf{R}^m) \times \mathbf{R}^m$, is a rectifiable scan on Y. An $m-1$ dimensional scan $\mathcal{S}$ is the *boundary*, $\partial \mathcal{T}$, of $\mathcal{T}$ if, for all points $\alpha \in X$,

$$\mathcal{S}(g, z) \ = \ \lim_{r \downarrow 0} \mathcal{T}\big((g, \operatorname{dist}(\cdot, a)), (r, z)\big)$$

for almost all $(g, z) \in \operatorname{Lip}(X, \mathbf{R}^{m-1}) \times \mathbf{R}^{m-1}$. One may again impose bounds on $\mathbf{M}_\alpha$ and $\mathbf{M}_\alpha \partial$ along with suitable topological bounds on supports to obtain scan compactness theorems.

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