1. (10 pts) State Rademacher’s Theorem and the Area and Co-area Formulas.

Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^m \) is Lipschitz and that \( A \) is an \( \mathcal{H}^n \) measurable subset of \( \mathbb{R}^n \).

**Rademacher’s Thm.** Then \( f \) is differentiable \( \mathcal{H}^n \) almost everywhere.

**Area Formula.** If \( n \leq m \), then

\[
\int_A [[Df(x)]] \, d\mathcal{H}^n x = \int_{f(A)} \mathcal{H}^0(A \cap f^{-1}\{y\}) \, d\mathcal{H}^m y.
\]

**Co-area Formula.** If \( n \geq m \), then

\[
\int_A [[Df(x)]] \, d\mathcal{H}^n x = \int_{f(A)} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, d\mathcal{H}^m y.
\]

2. (10 pts) Suppose \( A \) is an \( \mathcal{H}^1 \) measurable subset of \( \mathbb{R}^2 \) with \( \mathcal{H}^1(A) < \infty \). True or False (No proofs necessary.)

(a) \[
\lim_{r \downarrow 0} \frac{\mathcal{H}^1[A \cap B(a,r)]}{2r} = 1
\]

for \( \mathcal{H}^1 \) almost all \( a \in A \). FALSE (See the example from the 1st week.)

(b) \[
\mathcal{H}^1(A) = \inf \{\mathcal{H}^1(U) : U \text{ is an open neighborhood of } A\}.
\]

FALSE Nonempty open sets have infinite \( \mathcal{H}^1 \) measure.

(c) \[
\mathcal{H}^1(A) = \sup \{\mathcal{H}^1(K) : K \text{ is a compact subset of } A\}. TRUE
\]

3. (10 pts) Find the approximate Hausdorff outer measures

\[
\mathcal{H}^1_{\infty}(S^1) \quad \text{and} \quad \mathcal{H}^1_{\sqrt{2}}(S^1)
\]

where \( S^1 \) is the unit circle in the plane. \( \mathcal{H}^1_{\infty}(S^1) = 2 \) and \( \mathcal{H}^1_{\sqrt{2}}(S^1) = 4\sqrt{2} \).
4. (10 pts) Prove or find a counterexample: For any subsets $A_1, A_2, \ldots$ of $R^n$, the Hausdorff dimension
\[
\dim (\bigcup_{i=1}^{\infty} A_i) = \sup_i \dim A_i .
\]

Proof: Let $s = \sup_i \dim A_i$. Then, for $t > s$,
\[
\mathcal{H}^t (\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^t (A_i) = 0
\]
because $t > \dim A_i$ for all $i$. Also, for $r < s$, there is a $j$ with $r < \dim A_j \leq s$ so that
\[
\mathcal{H}^r (\bigcup_{i=1}^{\infty} A_i) \geq \mathcal{H}^r (A_j) = \infty .
\]
Thus, $\dim (\bigcup_{i=1}^{\infty} A_i) = s$.

5. (10 pts) Let $S_0$ be the unit square $[0, 1] \times [0, 1]$ in the plane. Dividing $S_0$ into 9 congruent squares, let $S_1$ be obtained from $S_0$ by removing the middle third square:
\[
S_1 = S_0 \setminus ([1/3, 2/3] \times [1/3, 2/3]).
\]
Similarly, let $S_2$ be obtained from $S_1$ by omitting the middle third squares from each of the 8 remaining squares. Continuing, we let $S_{i+1}$ be obtained from $S_i$ by omitting the middle third squares from the remaining squares of $S_i$. What is the Hausdorff dimension of $S = \bigcap_{i=1}^{\infty} S_i$? Does $S$ have finite Hausdorff measure in this dimension?

This is a self-similar set consisting of 8 pieces each of which can be expanded by a factor of 3 to give a congruent copy of the original set. Its dimension is $t = \frac{\log 8}{\log 3}$.

For each $i$, we may cover $S$ by $8^i$ squares of diameter $\sqrt{3}^{-i}$ so that
\[
\mathcal{H}^t (S) \leq 8^i \alpha_t (\sqrt{3}^{-i})^t = \alpha_t (\sqrt{2})^t (8 \cdot 3^{-t})^i = \alpha_t (\sqrt{2})^t < \infty .
\]
Thus $\dim S \leq t$. The fact that $\mathcal{H}^t (S) > 0$, hence $\dim S \geq t$ follows as in our discussion of the Cantor set.

6. (10 pts) A function $f : R^n \to R$ is approximately continuous at a point $a \in R^n$ if, for every positive $\epsilon$ there is a positive $\delta$ so that
\[
\frac{\lambda \left( \{ x \in B(a, r) : |f(x) - f(a)| > \epsilon \} \right)}{\lambda (B(a, r))} > \epsilon
\]
whenever $0 < r < \delta$.

(a) Prove or find a counterexample: If $a$ is a Lebesgue point of $f$, then $f$ is approximately continuous at $a$.

**Proof**: At a Lebesgue point $a$ of $f$ we may choose, for $\epsilon > 0$, a positive $\delta$ so that, for $0 < r < \delta$,
\[
\left[\lambda(B(a, r))\right]^{-1} \int_{B(a, r)} |f(x) - f(a)| \, dx \leq \epsilon^2,
\]
hence,
\[
\frac{\lambda\{x \in B(a, r) : |f(x) - f(a)| > \epsilon\}}{\lambda(B(a, r))} \geq \epsilon.
\]

(b) Prove or find a counterexample: If $f$ is approximately continuous at $a$, then $a$ is a Lebesgue point of $f$.

For a counterexample one can take
\[
f = \sum_{i=1}^{\infty} \frac{1}{|b_i - a_{i+1}|} \chi_{[a_{i+1}, b_i]}
\]
where positive numbers $a_1 > b_1 > a_2 > b_2 > \ldots$ are chosen inductively so that $\frac{a_i - b_i}{a_i} \to 1$ as $i \to \infty$.

7. (10 pts) Suppose $A$ is a closed subset of $\mathbb{R}^n$ of Lebesgue measure zero, and $f(x) = \text{dist} (x, A) \equiv \inf \{|x - a| : a \in A\}$.

(a) Show that $f$ is differentiable $\lambda$ almost everywhere.

For $x, y \in \mathbb{R}^n$ the triangular inequality implies that
\[
\text{dist} (x, A) \leq |x - y| + \text{dist} (y, A) \quad \text{and} \quad \text{dist} (y, A) \leq |x - y| + \text{dist} (x, A),
\]
so that
\[
|f(x) - f(y)| \leq |x - y|.
\]
Thus $f$ is Lipschitz and so differentiable almost everywhere by Rademacher’s theorem.

(b) Show that for any $g \in L^1(\mathbb{R}^n)$
\[
\int g(x) \, dx = \int_0^{\infty} \int_{f^{-1}(t)} g(y) \, d\mathcal{H}^{n-1}y.
\]
At a point $x$ of differentiability of $f$ we showed that $[[Df(x)]] = 1$, and so the above formula follows from the co-area change of variables formula.
8. (10 pts) Suppose that $K$ is a compact subsets of $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(K) = 0$. Prove that any 2 points in $\mathbb{R}^n \setminus K$ may be connected by a path in $\mathbb{R}^n \setminus K$.

For distinct points $a, b \in \mathbb{R}^n \setminus K$, choose first $0 < \epsilon < \frac{1}{2} |a - b|$ so that the two closed balls $\overline{B}_\epsilon(a)$ and $\overline{B}_\epsilon(b)$ do not intersect $K$. Note that the retraction map

$$f : \mathbb{R}^n \setminus B_\epsilon(a) \to \partial B_\epsilon(a), \quad f(x) = a + \epsilon \frac{x - a}{|x - a|},$$

is Lipschitz with $\text{Lip}(f) \leq \frac{1}{\epsilon}$. Thus

$$\mathcal{H}^{n-1}(f(K)) \leq \frac{1}{\epsilon^k} \mathcal{H}^{n-1}(K) = 0.$$ 

Since also $f(\partial B_\epsilon(b))$ is a nonempty open region in $\partial B_\epsilon(a)$, we may chose a point $\omega \in \partial B_\epsilon(a) \setminus f(\partial B_\epsilon(b))$. Then the half-line $f^{-1}\{\omega\}$ misses $K$ and contains an interval joining the two spheres $\partial B_\epsilon(a)$ and $\partial B_\epsilon(b)$. Joining the endpoints of this interval radially to $a$ and $b$ respectively, gives the desired path in $\mathbb{R}^n \setminus K$ joining $a$ and $b$.

9. (10 points) Suppose $\mu$ and $\nu$ are Borel measures on $\mathbb{R}^n$ with $0 < \mu(\mathbb{R}^n) < \infty$ and $0 < \nu(\mathbb{R}^n) < \infty$. Show that

$$\mu \{ x \in \mathbb{R}^n : \limsup_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} = \infty \} = 0.$$ 

Proof: The above set is the decreasing intersection of the sets

$$E_i = \{ x \in \mathbb{R}^n : \limsup_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} > i \}.$$ 

Thus, if the statement is false, then $\mu(E_i) > \delta$ for some positive number $\delta$ and all $i$ sufficiently large. For each point $x \in E_i$, we can choose a positive $r_x$ so that

$$\frac{\nu(B_{r_x}(x))}{\mu(B_{r_x}(x))} > i.$$ 

Now, for each $i$, we may repeat the proof of the Vitali Covering Theorem to choose from these balls a sequence of balls $B_{r_1}(a_1), B_{r_2}(a_2), \ldots$ covering $E_i$ so that the shrunked balls $B_{r_1/5}(a_1), B_{r_2/5}(a_2), \ldots$ are disjoint. The “essentially largest first” aspect of the proof of Vitali gives us a bound $c_n$ (depending only on $n$) on the number of balls $B_{r_j}(a_j)$ which may intersect $B_{r_k}(a_k)$ with $j \leq k$. Thus, although
the original balls $B_{r_1}(a_1), B_{r_2}(a_2), \ldots$ are not disjoint, any given point lies in at most $c_n$ of them. It follows that

$$c_n \nu(R^n) \geq \sum_{j=1}^{\infty} \nu(B_{r_j}(a_j)) \geq i \sum_{j=1}^{\infty} \mu(B_{r_j}(a_j))$$

$$\geq i \mu(E_i) \geq i \delta \to \infty \text{ as } i \to \infty,$$

contradicting the finiteness of $\nu(R^n)$.

10. (10 pts) P(a) Prove or find a counterexample:

If $\mathcal{I}$ is a (possibly uncountable) family of open rectangles in $R^2$ and $A_I$ is any set with $I \subset A_I \subset \text{Clos} I$ for each $I \in \mathcal{I}$, then $\bigcup_{I \in \mathcal{I}} A_I$ is Lebesgue measurable in $R^2$.

Proof: Replacing $\mathcal{I}$ by $\bigcup_{I \in \mathcal{I}} S_I$ where $S_I = \{\text{open squares } \subset I\}$ and letting

$$A_S = A_I \cap \overline{S} \text{ whenever } I \in \mathcal{I}, S \in S_I,$$

we may assume that $\mathcal{I}$ itself consists of squares which are fine about each point of $\bigcup_{I \in \mathcal{I}} \overline{I}$. Thus by Vitali’s theorem, we may choose a countable disjointed subfamily $\mathcal{C}$ of $\{\overline{I} : I \in \mathcal{I}\}$ which covers almost all of $\bigcup_{I \in \mathcal{I}} \overline{I}$. Since $\bigcup_{I \in \mathcal{I}} A_I$ contains the open set $\bigcup_{I \in \mathcal{I}} \text{Int} I$ and since

$$\lambda(\bigcup_{I \in \mathcal{I}} A_I \setminus \bigcup_{I \in \mathcal{I}} \text{Int} I)$$

$$\leq \lambda(\bigcup_{I \in \mathcal{I}} \overline{I} \setminus \mathcal{C}) + \lambda(\bigcup_{I \in \mathcal{I}} \partial I) = 0 + 0,$$

$\bigcup_{I \in \mathcal{I}} A_I$ is $\lambda$ measurable.

HAPPY SUMMER!