Connecting Topological Hopf Singularities

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§0. Introduction.

There are many interesting questions and works concerning the relation between the topology of Riemannian manifolds M and N and the structure of the various Sobolev spaces $W^{s,p}(M,N)$ of maps between them. For example, the space $W^{1,p}(M,\mathbf{S}^p)$ of finite p energy maps to the p sphere and issues concerning the possible approximability by smooth maps have been well-studied by F. Bethuel [Be1] and others using the notion of topological degree, which is associated with $\pi_p(\mathbf{S}^p)$. For $\dim M < p$, these Sobolev maps are automatically continuous. In the critical dimension $\dim M = p$, one has the phenomenon of *bubbling* whereby a weakly convergent sequence of smooth maps may, in the limit, drop energy and topological degree and produce, in a suitable space, auxiliary objects (bubbles) accounting for topological changes near a finite set of points. In case $\dim M > p$, the limiting map itself may have essential *topologically singularities*, detected by degree, which are topologically connected by a bubbling set of dimension $\dim M - [p]$. These are particularly well-understood for $\dim M = 3, p = 2$ [HL1], [BCL], [Be2], [BBC], [GMS1] where, for example, the bubbling set carries a 1 dimensional finite mass rectifiable current whose boundary is the topological singularities of the limit map.

In general, the homotopy group $\pi_p(N)$ should be used to study the Sobolev spaces $W^{1,p}(M,N)$. In the present paper we work with the Hopf invariant, which is associated with $\pi_3(\mathbf{S}^2)$, to understand spaces $W^{1,3}(M,\mathbf{S}^2)$ where $\dim M = 4$. We discover some new phenomena. Examples in §2.5 show that now the bubbled object can possibly have infinite

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one dimensional mass and that the singularities that appear in weak limits of sequences of smooth maps may possibly not bound *any* finite mass current. We define in §2 a new object, a *scan*, which generalizes a current but still occurs naturally in bubbling while automatically providing the topological connection between the singularities of the limit map. The bubbled scans, which are found in §6 via a new compactness theorem, again enjoy a representation in §7 using a finite measure 1 rectifiable set and an integer density function which is now however only $L^{3/4}$ integrable (rather than L^1 integrable).

Background.

With the target manifold N viewed as isometrically embedded in a Euclidean space $\mathbf{R}^k$, one may define, for positive numbers p and s, with $p \geq 1$, the Sobolev space

$$W^{s,p}(M,N) \;=\; \{u \in W^{s,p}(M,\mathbf{R}^k) \;:\; u(x) \in N \text{ for a.e. } x \in M\}\ ,$$

where the vector space $W^{s,p}(M,\mathbf{R}^k)$ is obtained from $W^{s,p}_{loc}(\mathbf{R}^{\dim M},\mathbf{R}^k)$ using local coordinate charts for M.

In contrast to the vector-space case $N = \mathbf{R}^k$, some Sobolev maps $u \in W^{s,p}(M,N)$ do not admit approximation by a sequence of smooth maps $u_n \in \mathcal{C}^\infty(M,N)$ in the strong or even in the weak $W^{s,p}(M,N)$ topologies. Questions about density of $\mathcal{C}^\infty(M,N)$ in $W^{s,p}(M,N)$ arise naturally for example from the study of variational problems among manifolds such as with harmonic maps, etc. [SU], [W1], [W2], [HL1], [BZ], [Be1], [BCL], [BBC], [GMS2]. Recently the path-connectness of $W^{1,p}(M,N)$ has been studied in [BL] and [HaL].

As a first approach to the notion of topological singularity, with M being the open unit ball $\mathbf{B}^m$ in $\mathbf{R}^m$, we may define the *topological singular set* of a map $u \in W^{s,p}(\mathbf{B}^m,N)$ as the largest open subet of $\mathbf{B}^m$ on which u is $W^{s,p}$ strongly approximable. The obstruction to the strong approximation is characterized by the appearance, locally around the singularities of u, of nonzero elements of $\pi_k(N)$ where $k = [sp]$, the integer part of sp. For example, the fact [SU] that the map $u : \mathbf{B}^3 \to \mathbf{S}^2$, $u(x) = x/|x|$, is not strongly approximable in $W^{1,p}(\mathbf{B}^3,\mathbf{S}^2)$ (for $2 \leq p \leq 3$) by regular maps is due to the realization of a nonzero element of $\pi_2(\mathbf{S}^2)$ on spheres surrounding the singularity 0. More generally, one has, for $s = 1$,

Theorem 0.1. [SU], [BZ], [Be1] *The space $\mathcal{C}^\infty(\mathbf{B}^m,N)$ is strongly dense in $W^{1,p}(\mathbf{B}^m,N)$ if and only if*

$$p \geq m \quad \textit{or} \quad \pi_{[p]}(N) \;=\; 0\ .$$

As recently observed by F. Hang and F.H.Lin [HgL], this sufficient condition for strong density does not extend to an arbitrary domain. The map v from $\mathbf{C}P^3$ to $\mathbf{C}P^2$, defined in homogeneous coordinates by $v\big([z_1,z_2,z_3,z_4]\big) = [z_1,z_2,z_3]$, has a singularity at $a = [0,0,0,1]$ and admits no global strong approximation by smooth maps. While the above theorem gives the existence of local obstructions due only to $\pi_{[p]}(N)$, this counterexample illustrates global obstruction. Here the singularity is delocalized in the sense that one may very well approximate the above v strongly in $W^{1,3}(\mathbf{C}P^3, \mathbf{C}P^2)$ by maps smooth in a fixed neighborhood of a because, one may, with arbitrarily small energy, "order the globally essential singularity to reappear somewhere else." By contrast, the local obstructions are fixed in space: it is impossible to strongly approximate $u(x) = x/|x|$ in $W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$ by a sequence of maps smooth in a fixed neighborhood of the limit singularity 0. This is the phenomenon that we wish to study here, and we will thus restrict especially to the domain $M = \mathbf{B}^m$. We also restrict to the case $s = 1$ although certain results below extend to fractional Sobolev spaces (see [Be3], Ri2]).

Whenever $\pi_{[p]}(N) \neq 0$, $\mathcal{C}^\infty(M, N)$ is too small to "cover by strong density" all of $W^{1,p}(\mathbf{B}^m, N)$, and one uses the following larger space

$$\begin{aligned} R^{\infty,p}(\mathbf{B}^m, N) \;=\; \{u \in \mathcal{C}^\infty(\mathbf{B}^m \setminus A, N) \cap W^{1,p} \;:\; & A \text{ is an } n-[p]-1 \text{ dimensional} \\ & \text{submanifold of } \mathbf{B}^m \text{ and } [u|\mathbf{S}_x K] \neq 0 \text{ in } \pi_{[p]}(N) \text{ for all } x \in A\} \end{aligned}$$

where SA is a normal sphere bundle of A embedded in a small neighborhood of A in $\mathbf{B}^m$ and $u|\mathbf{S}_x A$ is the restriction of u to the ($[p]$ dimensional) sphere over x. One then has the following:

Theorem 0.2. [Be1] *For* $[p] > 1$, $\overline{R^{\infty,p}(\mathbf{B}^m, N)}^{W^{1,p}} = W^{1,p}(\mathbf{B}^m, N)$.

For example, $R^{\infty,2}(\mathbf{B}^3, \mathbf{S}^2)$ consists of maps $u \in W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$ that are smooth away from a finite set A and whose restriction to any small spheres about a point of K has nonzero degree. In particular, $u(x) = x/|x| \in R^{\infty,2}(\mathbf{B}^3, \mathbf{S}^2) \setminus \overline{\mathcal{C}^\infty(\mathbf{B}^3, \mathbf{S}^2)}^{W^{1,p}}$.

Definition. For $u \in R^{\infty,p}(\mathbf{B}^m, N)$ one defines the *topological singularity* of u, $\mathrm{Sing}_{\mathrm{top}}\, u$ as the flat $\pi_{[p]}(N)$ chain obtained from the singular set K by assigning to each point $x \in K$ the multiplicity $[u|\mathbf{S}_x K]$ in $\pi_{[p]}(N)$. Flat G chains are defined in [Fl] (see also [F], [GMS2], [W3]. For example the topological singularity of an element of $R^{\infty,1}(\mathbf{B}^3, \mathbf{R}P^2)$ is a sum of disjoint unoriented curves in $\mathbf{B}^3$ because $\pi_1(\mathbf{R}P^2) = \mathbf{Z}_2$.

The general question motivating the present paper is the following: Being given a sequence u_n in $R^{\infty,p}(\mathbf{B}^m, N)$, converging strongly in $W^{1,p}(\mathbf{B}^m)$ to a limit u, may one experience some convergence of the flat $\pi_{[p]}(N)$ chains $\mathrm{Sing}_{\mathrm{top}}\, u_n$ to a limit "object $\mathrm{Sing}_{\mathrm{top}}\, u$" which will depend only on u and will characterize the approximability of u by smooth maps in $W^{1,p}$ $\big($ in particular, if $\mathrm{Sing}_{\mathrm{top}}\, u = 0$, then $u \in \overline{\mathcal{C}^\infty(\mathbf{B}^m, N)}^{W^{1,p}}$ $\big)$.

As we will see below, the understanding of the behavior of the topological singularities of maps strongly convergent in $W^{1,p}$ is linked to the problem of weak sequential density.

A Well-Understood Case: $\Pi_p(\mathbf{S}^p)$.

One may consider $W^{1,p}(\mathbf{B}^m, \mathbf{S}^p)$ where $m > p$ are positive integers. For simplicity we treat the specific case $p = 2, m = 3$, keeping in mind that the set of results below extends to the general case.

So consider $u_n \in R^{\infty,2}(\mathbf{B}^3, S^2)$ strongly convergent in $W^{1,2}$ to $u \in W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$. Then $\mathrm{Sing_{top}}\, u_n$ is simply a finite sum of integer multiples of point masses $\sum_{a\in A_n} \mathbf{m}_{n,a}[[a]]$. It isn't difficult to see that these distributions are characterized by the formula

$$\sum_{a\in A_n} \mathbf{m}_{n,a}[[a]] \;=\; *d\,u_n^{\#}\left(\frac{\omega_{\mathbf{S}^2}}{2\pi}\right)$$

where $\omega_{\mathbf{S}^2}$ is the volume form of $\mathbf{S}^2$. From the strong $W^{1,2}$ convergence of u_n one deduces without difficulty the convergence

$$\mathrm{Sing_{top}}\, u_n \;=\; *d\,u_n^{\#}\left(\frac{\omega_{\mathbf{S}^2}}{2\pi}\right) \;\to\; *d\,u^{\#}\left(\frac{\omega_{\mathbf{S}^2}}{2\pi}\right) \;\text{ in }\; \mathcal{D}'(\mathbf{B}^3)\ ,$$

independent of u_n, which is the desired topological singularity of u. On has also the

Theorem 0.3. [Be2] $d\,u^{\#}\omega_{\mathbf{S}^2} = 0 \iff u \in \overline{\mathcal{C}^\infty(\mathbf{B}^3, \mathbf{S}^2)}^{W^{1,2}}$.

The relation between the topological singularities and the *weak* convergence of smooth maps is understood by means of the following

Theorem 0.4. [Be2], [BCL], [GMS2] *For $u \in W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, there exists a 1 dimensional rectifiable current such that $\partial I = *d(u^{\#}\omega_{\mathbf{S}^2})$ and*

$$8\pi\mathbf{M}(I) \;\le\; \int_{\mathbf{B}^3} |\nabla u|^2$$

where $\mathbf{M}(I)$ is the mass (or length) of I.

In order to approximate a map u having $d(u^{\#}\omega) \neq 0$ weakly by smooth maps, it suffices to "withdraw" the topological singularities using a finite amount of $W^{1,2}$ energy. This is accomplished (see [Be2]) by inserting some coverings of $\mathbf{S}^2$ along I which, by the above estimate, costs exactly $8\pi\mathbf{M}(I) + \epsilon$ (with ϵ being arbitrarily small). One thus obtains the sequential weak density:

Theorem 0.5. [Be2] *For any u in $W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$ there exists u_n in $\mathcal{C}^\infty(\mathbf{B}^3, \mathbf{S}^2)$ which converge weakly to u in $W^{1,2}$.*

Finally there is another elegant method of characterizing the topological singularity of a map in $W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, constant on $\partial\mathbf{B}^3$, as being the "holes" of its graph:

Theorem 0.6. [GMS1] *For any sequence of maps $u_n \in \mathcal{C}^\infty(\mathbf{B}^3, \mathbf{S}^2)$ that is $W^{1,2}$ weakly convergent to $u \in W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, there exists a subsequence $u_{n'}$ and a one dimensional rectifiable current I, so that one has the weak convergence of the three dimensional rectifiable currents*

$$\mathrm{Graph}\,(u_{n'}) \;\to\; \mathrm{Graph}\,(u) \;+\; I \times [[\mathbf{S}^2]]\ .$$

Moreover,

$$\partial\mathrm{Graph}\,(u) \;=\; \partial I \times [[\mathbf{S}^2]] \;\text{ with }\; \partial I \;=\; *d\,u^{\#}\left(\frac{\omega_{\mathbf{S}^2}}{2\pi}\right)\ .$$

An Example of a More Complex Case: $\pi_3(\mathbf{S}^2)$.

This is the first case of an infinite homotopy group of spheres which is different from $\pi_p(\mathbf{S}^p)$. Thus in the present paper we take $N = \mathbf{S}^2$, $m = 4$ and $p = 3$ and work with the Sobolev space $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$. The space $R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ which now consists of maps in $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ which are smooth outside a finite set of points and realize the nontrivial elements of $\pi_3(\mathbf{S}^3) \simeq \mathbf{Z}$ on sufficiently small spheres centered at these points. Once again the topological singularity is identified with a finite atomic measures having integer multiplicities. Also $R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ is again strongly dense in $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ [Be1]. Criteria for a given map in $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ to be strongly approximable by smooth maps have been obtained by Zhou [Z] and Isobe [I1] who also considered [I2] gap phenomena [HL1] for this space. Being given a sequence u_n of elements of $R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ strongly convergent to a map u in $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ one again poses the question about the limit of the topological singularities $\text{Sing}_{top} u_n$. Recall that the homotopy class in $\pi_3(\mathbf{S}^2)$ of a regular map $\psi : \mathbf{S}^3 \to \mathbf{S}^2$ is given by the Hopf degree of ψ, which is topologically the linking number of the inverse images of two regular values of ψ and is analytically given by the integral

$$\text{Hopf degree}\,(\psi) \;=\; \frac{1}{4\pi^2}\int \eta \wedge \psi^{\#}\omega_{\mathbf{S}^2}$$

where η is any 1-form on $\mathbf{S}^3$ verifying $d\eta = \psi^{\#}\omega_{\mathbf{S}^2}$. A simple integration by parts then shows us that, similar to the case of $R^{\infty,2}(\mathbf{B}^3, \mathbf{S}^2)$, the topological singularity of the map $u_n \in R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ may be written

$$\text{Sing}_{\text{top}}\, u_n \;=\; \sum_{a \in A_n} \mathbf{m}_{n,a}[[a]] \;=\; *d\big[\eta_n \wedge u_n^{\#}\big(\frac{\omega_{\mathbf{S}^2}}{2\pi}\big)\big]$$

where η_n is any 1-form on $\mathbf{B}^4 \setminus A_n$ verifying $d\eta_n = u_n^{\#}\omega_{\mathbf{S}^2}$. Our principal preoccupation is then to study a possible convergence of $*d\big(\eta_n \wedge u_n^{\#}\omega_{\mathbf{S}^2}\big)$ and for example to verify whether, as in the case of $W^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, or not there exists a sequence of 1 dimensional currents I_n having $\partial I_n = \sum_{a \in A_n} \mathbf{m}_{n,a}[[a]]$ and having uniformly bounded masses (i.e. $\|\sum_{a\in A_n} \mathbf{m}_{n,a}[[a]]\|_{W^{-1,1}} \le C$ independent of n). However, such an attempt runs into the basic problem of the actual*failure* of suitable bounds for these convergences.

To see this failure, one starts with $u \in W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ and first proves without difficulty that $du^{\#}\omega_{\mathbf{S}^2} = 0$ and that the 1-form η verifying $d\eta = u^{\#}\omega_{\mathbf{S}^2}$ of "maximal" regularity is a priori the Coulomb guage which is the solution of

$$\begin{aligned} d\eta \;&=\; u^{\#}\omega_{\mathbf{S}^2} \text{ in } \mathcal{D}^2(\mathbf{B}^4) \\ d^*\eta \;&=\; 0 \qquad\quad \text{in } \mathcal{D}^1(\mathbf{B}^4) \\ \iota^{\#}_{\partial\mathbf{B}^4}\eta \;&=\; 0 \end{aligned}$$

where $\iota_{\partial\mathbf{B}^4}$ is the inclusion of $\partial\mathbf{B}^4$ into $\mathbf{R}^4$. Since the form $u^{\#}\omega_{\mathbf{S}^2}$ is only in $L^{3/2}(\mathbf{B}^4)$, the solution η of this problem is in $L^{12/5}(\mathbf{B}^4) \supset\supset L^3(\mathbf{B}^4)$, thus a priori $\eta \wedge u^{\#}$ is not in

$L^1_{loc}(\mathbf{B}^4)$, and it seems difficult to give this a meaning even in $\mathcal{D}'(\mathbf{B}^4)$. In [R1] the second author showed in fact that the above small calculation is optimal in establishing that

$$\log \inf\{\int_{\mathbf{S}^3} |\psi|^3 \, d\mathcal{H}^3 \ : \ \psi : \mathbf{S}^3 \to \mathbf{S}^3, \ \text{Hopf degree}\,(\psi) = d\} \ \approx \ \frac{3}{4}\log d \qquad (0.2)$$

as $d \to \infty$. This $\frac{3}{4}$, which replaces the 1 that occurs in minimizing p-energy among (topological) degree d maps from $\mathbf{S}^p$ to $\mathbf{S}^p$, appears when one expresses the Hopf degree by means of the above Coulomb guage. One shows that it is optimal by using maps whose inverse images are self-linked (see [R1]). This is the source of all the difficulties encountered below in this paper. Using the argument of §2.5, this $\frac{3}{4}$ estimate allows us to construct a sequence $u_n \in R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ such that

$$u_n \ \to \ u \quad \text{strongly in } W^{1,3}$$

but

$$\inf\{\mathbf{M}(I_n) \ : \ \partial I_n = \mathrm{Sing}_{\mathrm{top}}\, u_n\} \ \to \ +\infty \ ,$$

and one does not see a priori how $\mathrm{Sing}_{\mathrm{top}}\, u_n$ may converge in $\mathcal{D}'(\mathbf{B}^4)$. It is necessary to envision some convergences in larger spaces for some objects whose masses may tend to infinity.

Introduction of "Scans".

In face of the impossibility of getting convergence in $\mathcal{D}'(\mathbf{B}^4)$ of our topological singularities of maps $u_n \in R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ strongly convergent to a $u \in W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$, we will adapt the approach that Giaquinta, Modica, and Soucek [GMS1], [GMS2] used for the case $\pi_p(\mathbf{S}^p)$, and we will be interested in a possible convergence of a sequence of *graphs* of smooth maps in $\mathcal{C}^\infty(\mathbf{B}^4, \mathbf{S}^2)$.

Therefore let $u_n \in \mathcal{C}^\infty(\mathbf{B}^4, \mathbf{S}^2)$ converge $W^{1,3}$ weakly to $u \in W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$. One will suppose for simplicity that u_n and u are constant on $\partial\mathbf{B}^4$ (see §2.3). It is not difficult to see that, for all $u \in W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$, the graph of u is a rectifiable current satisfying

$$\partial\, \mathrm{Graph}\,(u) \ = \ 0 \text{ in } \mathbf{B}^4 \ .$$

In fact u may be approximated strongly by a map $v \in R^{\infty,3}(\mathbf{B}^4, \mathbf{S}^2)$ and the 3 dimensional flat current $\partial\, \mathrm{Graph}\,(v)$, being supported in $\mathbf{B}^4$ in the 2 dimensional set $\mathrm{sing}\,(v) \times \mathbf{S}^2$, must vanish [F],4.1.21.

The boundary of the graph thus does *not* characterize, in this case, the failure of the strong approximability by smooth maps. On the other hand, one can prove, from the vanishing of $\pi_1\big(\mathbf{B}^4 \setminus \mathrm{Sing}_{\mathrm{top}}\,(v)\big)$, the existence of a *Hopf lifting* $\tilde{v}$ of v for the Hopf map $\Pi : \mathbf{S}^3 \to \mathbf{S}^2$ (i.e. $\tilde{v} : \mathbf{B}^4 \to \mathbf{S}^3$ and $\Pi \circ \tilde{v} = v$). For such a v one has that

$$\partial\, \mathrm{Graph}\,(\tilde{v}) \ = \ \mathrm{Sing}_{\mathrm{top}}\, v \times [[\mathbf{S}^2]]$$

so that the boundary of the graphs of Hopf lifts do characterize the topological singularities. One is therefore led to take a smooth Hopf lifting $\tilde{u}_n$ of the map u_n and study the possible convergence of $\mathrm{Graph}\,\tilde{u}_n$ to a limit object in the form "$\mathrm{Graph}\,\tilde{u} + I \times [[\mathbf{S}^3]]$" where I will be a "reasonable" object connecting the topological singularities of u. There exist a lifting operation for the Hopf fibration which is associated (§2.1) with the extraction of the Coulomb guage described above. Let $\tilde{u}_n$ denote such a Coulomb lift for which one then has control in $W^{1,5/12}$ but not in $W^{1,3}$ as seen by the example of §2.5. While Example 2.5 shows the possibility that $\mathbf{M}\big(\mathrm{Graph}\,(\tilde{u}_n)\big) \to \infty$, we nevertheless establish in §2.4 an $L^{3/4}$ bound for the mass of hyperplanar slices.

More precisely, for each unit vector $v \in \mathbf{S}^3$ and $t \in \mathbf{R}$, we have the corresponding hyperplane $h(v,t) = \{x \in \mathbf{R}^4 \ : \ x \cdot v = t\}$ oriented by the normal vector v. Intersecting the 4 dimensional current $\mathrm{Graph}\,(\tilde{u}_n)$ by $h(v,t) \times [[\mathbf{S}^3]]$, or equivalently slicing [F],4.3, by the projection $(x,y) \mapsto x \cdot v$, gives the 3 dimensional current $\mathrm{Graph}\,\big(\tilde{u}_n | h(v,t)\big)$ corresponding to restricting $\tilde{u}_n$ to the hyperplane $h(v,t)$. We show

$$\sup_{v \in \mathbf{S}^3} \int_{-1}^{1} \mathbf{M}^{3/4}\big[\mathrm{Graph}\,\big(\tilde{u}_n | h(v,t) \cap \mathbf{B}^4\big)\big]\, dt \ \leq \ C\big(1 + \int_{\mathbf{B}^4} |\nabla u|^3\, dx\big)\ . \tag{0.1}$$

Such control on the integral of the masses of the slices to a power less than 1 suggest characterizing an object as a collection of all (or almost all) its slices. This is the notion of a scan which we will define below.

As motivation, consider the following simplified problem where one is given a sequence of unions of immersed oriented closed curves $\Gamma_n = \cup_k \Gamma_n^k$ in the closed unit ball in $\mathbf{R}^2$ satisfying the bound

$$\sup_{v \in \mathbf{S}^1} \int_{-1}^{1} \mathrm{Card}^{\,\alpha}\big(\Gamma_n \cap L(v,t)\big)\, dt \ \leq \ C \quad \text{independent of } n\ . \tag{0.2}$$

where $L(v,t)$ denotes the line $\{x \in \mathbf{R}^2 \ : \ x \cdot v = t\}$. If $\alpha = 1$, then this bound gives us control on the mass (or total length) of the 1 dimensional current Γ_n, independent of n. Knowing that $\partial \Gamma_n = 0$, one is then in position to apply the Compactness Theorem of Federer-Fleming and deduce that, after passing to a subsequence, the Γ_n converge to a limit rectifiable current Γ.

When $\alpha < 1$, (0.2) does not guarantee control of the total length $\mathbf{M}(\Gamma_n)$, and there is no reasoning that allows us to deduce some convergence of the Γ_n as distributions. One thus introduces a map μ_n from the space of oriented lines $\mathbf{S}^1 \times \mathbf{R}$ to the space $\mathcal{M}$ of atomic measures on $\mathbf{R}^2$ which at almost every (v,t) associates the 0 dimensional intersection current

$$\mu_n(v,t) \ = \ \Gamma_n \cap L(v,t)$$

which is a sum of point masses with integer multiplicities. Being given a reference frame $\{e_1, e_2\}$ of $\mathbf{R}^2$, one equips $\mathcal{M}$ with the following metric

$$d(\mu,\mu') \;=\; \inf\{\mathbf{M}^\alpha(S) \;+\; \sum_{j=1}^{2}\int_{-1}^{1} \mathbf{M}^\alpha\big(\Gamma_n \cap L(e_j,s)\big)\, ds \quad : \; \mu-\mu' = S+\partial T\} \;,$$

and one verifies that the above μ_n is a measurable function from $\mathbf{S}^1\times\mathbf{R}$ to $\mathcal{M}$ equipped with the topology induced from the metric d.

The current equation $\partial\Gamma_n = 0$ translates to a new boundary zero condition for the corresponding scan μ_n (see §1) which is a compatibility condition allowing one to see that μ_n is the scan of an underlying closed object in the plane. Also estimate (0.2) and this boundary zero condition imply the following regularity estimate:

$$d\big(\mu_n(v,t), \mu'_n(v,t')\big) \;\le\; F_n(t)|t-t'|^\alpha$$

for all $v\in\mathbf{S}^1$ and some F_n in $L^{1/\alpha}(\mathbf{R})$weak $\;=\; L^{\frac{1}{\alpha},\infty}$ with

$$\|F_n\|_{L^{\frac{1}{\alpha},\infty}} \;\le\; C\sup_{v\in\mathbf{S}^1}\int_{-1}^{1}\mathbf{M}^\alpha\big(\Gamma_n\cap L(v,s)\big)\, ds \;.$$

Such uniform control of this regularity permits us then to establish, after passing to a subsequence, convergence a.e. of μ_n to a scan limit μ, a limiting object at least which, though a priori strange, is convenient to study in the particular cases we consider. When $\alpha = 1$, we recover the characterization by Ambrosio and Kirscheim of rectifiable objects by means of a weakly BV maps with values in metric spaces. See also White's rectifiability proof [W3].

Returning now to the original problem of the sequence u_n of maps in $\mathcal{C}^\infty(\mathbf{B}^4,\mathbf{S}^2)$ converging weakly in $W^{1,3}$ to u. To each u_n one may associate the scan of its Coulomb lift

$$\mathbf{G}_{\tilde{u}_n} \;:\; \mathbf{S}^2\times\mathbf{R} \;\to\; \mathcal{R}_3(\mathbf{B}^4\times\mathbf{S}^2)\;, \qquad \mathbf{G}_{\tilde{u}_n}(v,t) \;=\; \text{Graph}\,\big(\tilde{u}_n|h(v,t)\big)\;,$$

the space $\mathcal{R}_3(\mathbf{B}^4\times\mathbf{S}^2)$ denoting the 3 dimensional rectifiable currents in $\mathbf{B}^4\times\mathbf{S}^2$. On $\mathcal{R}_3(\mathbf{B}^4\times\mathbf{S}^2)$ one considers the distance

$$d_{\mathbf{e}}(P,Q) = \inf\big\{\mathbf{M}(S) + \sum_{j=1}^{4}\int \mathbf{M}\big(T\cap h(e_j,t)\big)^{3/4} dt \;:\; P-Q = S+\partial T\big\}.$$

where $\mathbf{e} = (e_1,e_2,e_3,e_4)$ is a fixed frame of $\mathbf{R}^4$. This time the control of (0.1) translates to a regularity for the scan $\mathbf{G}_{\tilde{u}_n}$:

$$d\big(\mathbf{G}_{\tilde{u}_n}(v,t), \mathbf{G}_{\tilde{u}_n}(v,t')\big) \;\le\; F_n(t)|t-t'|^{\frac{3}{4}}$$

for all $v \in \mathbf{S}^3$ and some F_n in $L^{\frac{4}{3},\infty}$ with

$$\|F_n\|_{L^{\frac{4}{3},\infty}} \ \le\ C\big(1+\int_{\mathbf{B}^4} |\nabla u_n|^3\, dx\big)\ .$$

Rather than referring to general properties of the space $L^{\frac{4}{3},\infty}$, we prove in §9, for the reader's convenience, the appropriate precise compactness statement needed. With this, we then establish the following result which is the analogue for $W^{1,3}(\mathbf{B}^4,\mathbf{S}^2)$ of Theorem 0.6.

Theorems 6.1, 7.2. *Suppose that $u_n \in \mathcal{C}^\infty(\mathbf{B}^4,\mathbf{S}^2)$ converge weakly in $W^{1,3}$ to u. Then, after passing to a subsequence, one has the convergence almost everywhere of scans of Coulomb lifts*

$$\mathbf{G}_{\tilde u_n}\ \to\ \mathbf{G}_{\tilde u}\ +\ I\times[[\mathbf{S}^3]]$$

where $I\times[[\mathbf{S}^3]]$ is the scan of a rectifiable set $R\times\mathbf{S}^3$ in $\mathbf{B}^4\times\mathbf{S}^3$ equipped with an integer multiplicity θ, measurable on Γ, such that

$$\int_R |\theta|^{\frac{3}{4}}\, d\mathcal{H}^1\ <\ \infty\ .$$

In the sense of scans,

$$\partial\big(\mathbf{G}_{\tilde u}\ +\ I\times[[\mathbf{S}^3]]\big)\ =\ 0\quad \text{in }\ \mathbf{B}^4\ .$$

While it is still unknown whether an arbitrary map $u\in W^{1,3}(\mathbf{B}^4,\mathbf{S}^2)$ is such a weak limit of smooth maps, we can nevertheless still use the scan $\mathbf{G}_{\tilde u}$ of the graph of its Coulomb lift to express the strong approximability criterium

Lemma 2.7.

$$u\in\overline{\mathcal{C}^\infty(\mathbf{B}^4,\mathbf{S}^2)}^{W^{1,3}}\qquad\Longleftrightarrow\qquad \partial\mathbf{G}_{\tilde u}=0\ \text{in }\ \mathbf{B}^4\ .$$

In any case, this scan boundary may again be capped off by a vertical scan:

Theorem 8.1.

$$\partial\big(\mathbf{G}_{\tilde u}\ +\ I\times[[\mathbf{S}^3]]\big)\ =\ 0\ \text{ in }\ \mathbf{B}^4$$

where, for all $v\in\mathbf{S}^3$ and a.e. $t\in\mathbf{R}$,

$$\big(I\times[[\mathbf{S}^3]]\big)\big(h(v,t)\big)\ =\ \sum_{a\in A_{v,t}}\mathbf{m}_{v,t}[[a]]\times[[\mathbf{S}^3]]\ ,$$

for some finite subset $A_{v,t}$ of $h(v,t)$ and non-zero integers $\mathbf{m}_{v,t}$ with

$$\int_{\mathbf{R}}\big(\sum_{a\in A_{v,t}}\mathbf{m}_{v,t}\big)^{3/4}dt\le C\big(1+\int|\nabla u|^3\,dx\big)\ .$$

We can use this estimate to show only that the I of Theorem 8.1 is carried by a set of finite $\mathcal{H}^{4/3}$ measure, and not, as in the case of Theorem 7.2, carried by a 1 rectifiable set. In fact the optimal structure of such an I seems related to the question of the weak sequential density of $\mathcal{C}^\infty(\mathbf{B}^4, \mathbf{S}^2)$ in $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$. For general Sobolev spaces of mappings, strong approximability by smooth maps has been well-studied (see [Be1] and HgL]), but the same problems for the weak topology are still largely open. (see [PR]).

As we have argued, the scans defined and used in this work allow one to study Sobolev mappings via their graphs by exploiting estimates valid on restriction to hyperplanar subspaces. Approximation properties characterized by restricting to lower dimensional subspaces also occurs in the work [M] of Mucci. Our limiting objects however are no longer currents, and the scans we introduce thus strictly extend and generalize the Cartesian currents of [GMS2]. General rectifiable currents (not necessarily related to smooth mappings) have also been understood and well-studied through slicing [AK], [W3]. So general scans (as in the motivating example (0.2)) should provide a useful extension of various classes of currents. The second author and T. DePauw [HD] have studied some compactness, rectifiability, and variational problems for such scans.

Recently we have also realized another argument for $W^{1,3}(\mathbf{B}^4, \mathbf{S}^2)$ for producing the key connecting "bubbled" scan I which avoids the passage to lifted maps into $\mathbf{S}^3$. One such approach, whose description is beyond the scope of this paper, allows us then to envision, in the general case of an arbitrary manifold N, similarly identifying connections of topological singularities issuing from the infinite part of $\pi_p(N)$, $\pi_p(N) \otimes \mathbf{Q}$, given through the Novikov integral expressions [Nov] by means of scans. The scans seem to be necessary to expedite such an approach. One expects, in fact, because of considerations which led to the above $\frac{3}{4}$ and to the exponents of Gromov [Gr], that this power should be replaced by other powers strictly less than 1 (except in the simple case $\pi_p(\mathbf{S}^p)$ described above) and that therefore the masses of these connections I should again be infinite.

§1. Hyperplanes in $\mathbf{R}^4$ and Scans.

We identify $\mathbf{S}^3 \times \mathbf{R}$ with the space H of *oriented hyperplanes in* $\mathbf{R}^4$ by associating with each pair $(v, t) \in \mathbf{S}^3 \times \mathbf{R}$ the hyperplane

$$h(v, t) \;\equiv\; \{x \in \mathbf{R}^4 \;:\; x \cdot v = t\;\}$$

oriented by the normal vector v. Thus H is equipped with the standard metric of $\mathbf{S}^3 \times \mathbf{R}$ and the 4 dimensional Hausdorff measure $(\mathcal{H}^3 | \mathbf{S}^3) \times \mathcal{H}^1$, that is, $dh(v, t) = d\mathcal{H}^3 v\, dt$.

We also occasionally let $h = h(v, t)$ denote the corresponding 3 dimensional *current* (See [F], [S], [GMS2] for notations.) that is the boundary of the standardly-oriented half-space,

$$h \;=\; \partial\, [[\{x \in \mathbf{R}^4 \;:\; x \cdot v < t\;\}]]$$

The orientation is described by either the constant tangent 3 vector $\vec{h}$ or by the dual normal 1 vector $\vec{h}^* = v$.

We will study a smooth map $w \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^3)$ in terms of its restrictions to hyperplanes. In particular, for any $h \in H$, we consider the *oriented graph of w restricted to H* as the 3 dimensional current

$$G_{w\#}h \ = \ \text{where} \ \ G_w(x) \ = \ \big(x, w(x)\big) \ .$$

Thus, $G_{w\#}h \in \mathcal{R}_{3,loc}$ where we here use the abbreviations

$$\mathcal{R}_i \ = \ \mathcal{R}_i(\mathbf{R}^4 \times \mathbf{S}^3) \ , \ \mathcal{R}_{i,loc} \ = \ \mathcal{R}_{i,loc}(\mathbf{R}^4 \times \mathbf{S}^3) \ ,$$

for the groups of i dimensional integer-multiplicity *rectifiable* and *locally rectifiable currents* in $\mathbf{R}^4 \times \mathbf{S}^3$ ([F], [S], [GMS2]).

We also need various projections:

$$\begin{aligned}
&p \ : \ \mathbf{R}^4 \times \mathbf{S}^3 \ \to \ \mathbf{R}^4 \ , \ p(x,y) \ = \ x \ , \\
&q \ : \ \mathbf{R}^4 \times \mathbf{S}^3 \ \to \ \mathbf{S}^3 \ , \ q(x,y) \ = \ y \ , \\
&\pi_v \ : \ \mathbf{R}^4 \ \to \ \mathbf{R} \ , \ \pi_v(x) \ = \ v \cdot x \ , \\
&p_v = \pi_v \circ p \ : \ \mathbf{R}^4 \times \mathbf{S}^3 \ \to \ \mathbf{R} \ , \ p_v(x,y) \ = \ v \cdot x \ ,
\end{aligned}$$

for $x \in \mathbf{R}^4$, $y \in \mathbf{S}^3$, and $v \in \mathbf{S}^3$.

In terms of boundary or slicing (see [F],4.3 or [S],),

$$G_{w\#}h(v,t) \ = \ \partial\left(G_{w\#}[[\pi_v^{-1}(-\infty,t)]]\right) \ = < G_{w\#}[[\mathbf{R}^4]], \, p_v, \, t > \ .$$

Note that

$$\partial G_{w\#}h \ = \ G_{w\#}\partial h \ = \ 0 \ ,$$

$$p_\# G_{w\#}h \ = \ h \ , \quad q_\# G_{w\#}h \ = \ w_\# h \ .$$

Moreover, for any two hyperplanes $h, \, h' \in H$, we have the compatibility property that

$$\big(G_{w\#}h\big) \cap \big(h' \times [[\mathbf{S}^3]]\big) \ = \ \big(G_{w\#}h'\big) \cap \big(h \times [[\mathbf{S}^3]]\big)$$

because

$$\begin{aligned}
\big(G_{w\#}h(v,t)\big) \cap \big(h(v',t') \times [[\mathbf{S}^3]]\big) \ &= << G_{w\#}[[\mathbf{R}^4]], \, p_v, \, t >, \, p_{v'}, \, t' > \\
&= << G_{w\#}[[\mathbf{R}^4]], \, p_{v'}, \, t' >, \, p_v, \, t > \\
&= \ \big(G_{w\#}h(v',t')\big) \cap \big(h(v,t) \times [[\mathbf{S}^3]]\big) \ .
\end{aligned}$$

In general, we define a *scan* to be *any* function

$$S \ : \ H \ \to \ \mathcal{R}_3$$

satisfying

$$S(h) \cap \big(h' \times [[\mathbf{S}^3]]\big) \;=\; S(h') \cap \big(h \times [[\mathbf{S}^3]]\big)$$

for almost every pair h, $h' \in H$. The special scan $S = G_{w\#}$ is called the *scan of the map* $w \in \mathcal{C}^\infty(\mathbf{B}^4, \mathbf{S}^3)$. More generally, for any current $T \in \mathcal{R}_4$ with $p_\# T = [[\mathbf{R}^4]]$, there is an associated scan, $\mathrm{Scan}\, T$, defined by the hyperplanar intersection

$$(\mathrm{Scan}\, T)(h) \;\equiv\; T \cap \big(h \times [[\mathbf{S}^3]]\big) \;\text{ for a.e. }\; h \in H,$$

so that, in terms of slicing, $(\mathrm{Scan}\, T)\big(h(v,t)\big) =< T, p_v, t >$ for a.e. $t \in R$.

Thus a scan may be considered as a generalization of a *Cartesian current* ([GMS2]).

The compatibility condition indicates that scans may be determined by their values on a smaller family of hyperplanes, for example, the coordinate hyperplanes associated with some orthonormal frame. This will be first illustrated in §6 where use of standard coordinate hyperplanes will be sufficient to establish the convergence of a sequence of scans of smooth maps. Actually, our limiting scan will only be determined at *almost every* $h \in H$, but will nevertheless inherit some properties from the scans of rectifiable currents.

In particular, we may define the notion of a *scan cycle*, that is, what it means for a general scan to have *zero boundary.* For the scan of a smooth map $w \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^3)$ and any subset U of $\mathbf{R}^4$ of locally finite perimeter

$$\partial G_{w\#}\partial[[U]] = G_{w\#}\partial\partial[[U]] = 0 \;\text{ and }\; G_{w\#}\partial[[U]]\big(q^\# \omega_{\mathbf{S}^3}\big) = G_{w\#}[[U]]\big(q^\# d\omega_{\mathbf{S}^3}\big) = 0\ .$$

A definition suitable for general scans may be made by using *polyhedral domains.*

A *polyhedral frontier* is a current $\partial[[U]]$ where U is an open polyhedral domain in $\mathbf{R}^4$. For a polyhedral domain U with k distinct 3-dimensional faces, we may represent

$$\partial[[U]] \;=\; \sum_{i=1}^k h_i \llcorner \partial U$$

where each h_i is supported by the hyperplane containing some 3-face of ∂U and $\vec{h_i}^{\,*}$ is the outward unit normal of this face. We now say that

$$\partial\, S \;=\; 0$$

(or that S is a *span cycle*) if, for *almost all* polyhedral frontiers $\partial[[U]] = \sum_{i=1}^k h_i \llcorner \partial U$ as above, the rectifiable current

$$S_{\partial U} \;\equiv\; \sum_{i=1}^k S(h_i) \llcorner p^{-1}(\partial U)$$

satisfies the two conditions

$$\partial\big(S_{\partial U}\big) \;=\; 0 \quad\text{and}\quad S_{\partial U}\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; 0\,.$$

Here, "almost all" means that an exceptional set Z of polyhedral frontiers has *measure zero* in the sense that

$$\{(h_1, h_2, \ldots, h_n) \in H^n \;:\; \partial[[U]] \in Z \text{ for some component } U \text{ of } \mathbf{R}^4 \setminus \cup_{i=1}^n h_i\}$$

has measure zero in H^n for all n. The necessity of the second condition in the definition is shown by the oriented graph of $\frac{x}{|x|}$, which is a current in $\mathcal{R}_4$ with nonzero boundary $[[0]] \times [[\mathbf{S}^3]]$ whose corresponding scan satisfies the first, but not the second condition. In fact, as we will see later, the graph of any map in $W^{1,3}(\mathbf{R}^4, \mathbf{S}^3)$ satisfies the first condition. For the scan of a rectifiable current, we have the following:

Lemma 1.1. *If $T \in \mathcal{R}_{4,loc}$ and $\mathbf{M}(\partial T) < \infty$, then $\partial T = 0$ if and only if $\partial(\operatorname{Scan} T) = 0$.*

Proof. For almost all polyhedral domains U as above, [F],4.3 gives the formula

$$(\operatorname{Scan} T)_{\partial U} \;=\; \partial\big(T \,\llcorner\, p^{-1}(U)\big) \;-\; \big(\partial T\big) \,\llcorner\, p^{-1}(U))\,.$$

Thus if $\partial T = 0$, then $\partial\big((\operatorname{Scan} T)_{\partial U}\big) = \partial\partial\big(T \,\llcorner\, p^{-1}(U)\big) = 0$ and

$$(\operatorname{Scan} T)_{\partial U}\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; \partial\big(T \,\llcorner\, p^{-1}(U)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; \big(T \,\llcorner\, p^{-1}(U)\big)\big(q^{\#}d\omega_{\mathbf{S}^3}\big) \;=\; 0\,.$$

Conversely, suppose $\partial(\operatorname{Scan} T) = 0$. Then, for almost all $h(v,t) \in H$,

$$< \partial T, p_v, t >= 0$$

because, we may, for any form $\phi \in \mathcal{D}^2(\mathbf{R}^4 \times \mathbf{S}^3)$, choose a large polyhedral domain U with $\partial U \cap \operatorname{spt}\phi = h(v,t) \cap \operatorname{spt}\phi$, hence, $< \partial T, p_v, t > (\phi) = \partial\big((\operatorname{Scan} T)_{\partial U}\big)(\phi) = 0$.

It follows that, for $\|\partial T\|$ almost all points z, the approximate tangent 3 plane L_z associated with $\overrightarrow{\partial T}(z)$ has $p(L_z) = 0$. In fact, if $p(L_z)$ contained a line, then $p_v|L_z$ would have rank one for a.e. $v \in \mathbf{S}^3$. By [F],4.3, this would give

$$z \;\in\; \operatorname{spt} \,< \partial T, p_v, p_v(z) >$$

for $\|\partial T\|$ almost all such z, contradicting the vanishing of $< \partial T, p_v, t >$ for a.e. t.

Thus $\overrightarrow{\partial T}(z) \;=\; \pm\big(0\,, \overrightarrow{[[\mathbf{S}^3]]}\big(q(z)\big)\big)$ for $\|\partial T\|$ almost all z, and, since $\partial\partial T \;=\; 0$, an elementary argument [H],Th.1, shows that

$$\partial T \;=\; \sum_{a \in A} m_a[[a]] \times [[\mathbf{S}^3]]$$

for some finite subset A of $\mathbf{R}^4$ and some integers m_a. Using now the second condition of $\partial\big(\operatorname{Scan} T\big) = 0$ with almost any polyhedral domain U with $U \cap A = \{a\} = \overline{U} \cap A$, we deduce that

$$m_a \;=\; \big((\partial T) \,\llcorner\, p^{-1}(U)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; \big((\operatorname{Scan} T)_{\partial U} \,+\, \partial(T \,\llcorner\, p^{-1}(U)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; 0 + 0\,.$$

Thus $\partial T = 0$. ∎

§2. The Hopf Map and Coulomb Lifting.

Recall that the *Hopf map*

$$\Pi \;:\; \mathbf{S}^3 \;\to\; \mathbf{S}^2 \;.$$

may be described explicitly by the formula $\Pi(z,w) = z/\,\overline{w}$ where we identify the domain $\mathbf{S}^3$ with

$$\{(x_1 + \mathbf{i}x_2, x_3 + \mathbf{i}x_4) \in \mathbf{C}^2 \;:\; |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = 1\}$$

and the range $\mathbf{S}^2$ with the extended complex plane $\hat{\mathbf{C}}$ via the usual stereographic projection. One readily checks that

$$\Pi(z,w) = \Pi(z',w') \;\text{ if and only if }\; (z,w) = e^{\mathbf{i}\theta}(z',w') \text{ for some } \theta \in \mathbf{R} \;.$$

Also, pulling back the volume form $\omega_{\mathbf{S}^2}$ via Π gives

$$\Pi^{\#}\omega_{\mathbf{S}^2} \;=\; 4(dx_1\,dx_2 \;+\; dx_3\,dx_4) \;=\; 2d\alpha$$

where $\alpha \;=\; x_1\,dx_2 \;-\; x_2\,dx_1 \;+\; x_3\,dx_4 \;-\; x_4\,dx_3$.

Let $M = \mathbf{S}^3$, $\mathbf{R}^3$, $\mathbf{R}^4$ (or any oriented simply 2-connected Riemannian manifold). For any smooth map $u : M \to \mathbf{S}^2$, a smooth map $\hat{u} : M \to \mathbf{S}^3$ satisfying

$$\Pi \circ \hat{u} \;=\; u$$

is called a *Hopf lift of* u and a smooth 1 form η on M satisfying

$$d\eta \;=\; u^{\#}\omega_{\mathbf{S}^2}$$

is called a *guage for* u. For a Hopf lift $\hat{u}$ of u, the formula

$$\eta \;=\; 2\hat{u}^{\#}\alpha$$

clearly defines a guage for u. Conversely,

Lemma 2.1. *Any guage η for $u \in \mathcal{C}^\infty(M,\mathbf{S}^2)$ equals $2\hat{u}^{\#}\alpha$ for some Hopf lift $\hat{u}$ of u. The lift $\hat{u}$ for η is unique up to multiplication by $e^{\mathbf{i}\theta}$ for some constant $\theta \in \mathbf{R}$, and*

$$|\nabla\hat{u}|^2 \;=\; \frac{1}{4}|\eta|^2 \;+\; |\nabla u|^2 \;.$$

Proof. First note that the u pull-back of the bundle $\Pi : \mathbf{S}^3 \to \mathbf{S}^2$ is a trivial $\mathbf{S}^1$ bundle over M. Using a trivializing map, we readily find some smooth Hopf lift $\check{u}$ of. For a Hopf lift $\hat{u}$ of u, the formula

$$\eta \;=\; 2\hat{u}^{\#}\alpha$$

clearly defines a guage for u. Conversely,

Lemma 2.1. *Any guage η for $u \in \mathcal{C}^\infty(M, \mathbf{S}^2)$ equals $2\hat{u}^\# \alpha$ for some Hopf lift $\hat{u}$ of u. The lift $\hat{u}$ for η is unique up to multiplication by $e^{\mathbf{i}\theta}$ for some constant $\theta \in \mathbf{R}$, and*

$$|\nabla \hat{u}|^2 \;=\; \frac{1}{4}|\eta|^2 \;+\; |\nabla u|^2 \;.$$

Proof. First note that the u pull-back of the bundle $\Pi : \mathbf{S}^3 \to \mathbf{S}^2$ is a trivial $\mathbf{S}^1$ bundle over M. Using a trivializing map, we readily find some smooth Hopf lift $\breve{u}$ of u. Since

$$d\big(2\breve{u}^\# \alpha \;-\; \eta\big) \;=\; 0 \;,$$

$\breve{u}^\# \alpha \;-\; \frac{1}{2}\eta \;=\; d\phi$ for some smooth $\phi : \mathbf{R}^4 \to \mathbf{R}$. Then we readily verify that

$$\hat{u} \;\equiv\; e^{-\mathbf{i}\phi}\breve{u}$$

is a Hopf lift of u satisfying

$$2\hat{u}^\# \alpha \;=\; \big(e^{-\mathbf{i}\phi}\breve{u}\big)^\# \alpha \;=\; 2\big(\breve{u}^\# \alpha \;-\; d\phi\big) \;=\; \eta \;.$$

Also if $\hat{\hat{u}}$ is another Hopf lift of u, then

$$\hat{\hat{u}} \;=\; e^{-\mathbf{i}\theta}\hat{u}$$

for some smooth $\theta : M \to \mathbf{R}$ because M is simply connected. Assuming in addition that

$$2\,\hat{\hat{u}}^{\,\#} \alpha \;=\; \eta \;,$$

we compute, as above, that

$$d\theta \;=\; \hat{u}^\# \alpha \;-\; e^{-\mathbf{i}\theta}\hat{u}^\# \alpha \;=\; \frac{1}{2}\eta \;-\; \hat{\hat{u}}^{\,\#}\alpha \;= 0 \;,$$

so that θ is a constant.

For the Hopf map $\Pi : \mathbf{S}^3 \to \mathbf{S}^2$, observe that the restriction of $D\Pi$ to the orthogonal complement of the tangent space of any fiber is an isometry. Since α orients every fiber, we find that, for each $a \in M$ and unit tangent vector v at a,

$$\begin{aligned}
|D\tilde{u}_a[v]|^2 \;&=\; |\alpha_{\tilde{u}(a)}\big(D\tilde{u}_a[v]\big)|^2 \;+\; |D\Pi_{\tilde{u}(a)}\big(D\tilde{u}_a[v]\big)|^2 \\
&=\; |(\tilde{u}^\# \alpha)_a[v]\big)|^2 \;+\; |Du_a[v]|^2 \\
&=\; |\,\frac{1}{2}\eta_a[v]\,|^2 \;+\; |Du_a[v]\big)|^2 \;,
\end{aligned}$$

and the Lemma follows. ∎

For a smooth $u : \mathbf{S}^3 \to \mathbf{S}^2$, the *Hopf degree* of u is the integer

$$\frac{1}{2\pi^2}\int_{\mathbf{S}^3}\eta\wedge d\eta \;=\; \text{degree}\,(\tilde u) \;=\; \frac{1}{2\pi^2}\int_{\mathbf{S}^3}\tilde u^\#\omega_{\mathbf{S}^3} \;=\; \frac{1}{2\pi^2}G_{\tilde u\#}[[\mathbf{S}^3]]\big(q^\#\omega_{\mathbf{S}^3}\big)$$

for any any guage η of u or Hopf lift $\tilde u$ of u. Incidentally, the normalizing constant here,

$$\mathcal{H}^3(\mathbf{S}^3) \;=\; 2\pi^2\ ,$$

will be frequently used.

For a smooth u from $\mathbf{R}^3$ (respectively, $\mathbf{R}^4$) to $\mathbf{S}^2$ which is constant near infinity, we can use the special *Coulomb guage* [R1]

$$\tilde\eta \;\equiv\; d^*\big[-\frac{1}{8\pi|x|}*u^\#\omega_{\mathbf{S}^2}\big]\ \big(\text{respectively},\ d^*\big[-\frac{1}{4\pi^2|x|^2}*u^\#\omega_{\mathbf{S}^2}\big]\big) \tag{2.1}$$

which, besides being a guage for u, has the additional properties that

$$d^*\tilde\eta \;=\; 0 \ \text{ and } \ |\nabla\tilde\eta| \;=\; |d\tilde\eta| \;=\; |u^*\omega_{\mathbf{S}^2}|\ . \tag{2.2}$$

A Hopf lift $\tilde u$ corresponding to the Coulomb guage $\tilde\eta$ will be called a *Coulomb lift of* u. Also for a smooth map $u : \mathbf{S}^3 \to \mathbf{S}^2$, using stereographic projection from $\mathbf{S}^3\setminus\{(0,0,0,1)\}$ to $\mathbf{R}^3$ readily gives a corresponding Coulomb guage and Coulomb lift.

For a map $u : \mathbf{S}^3 \to \mathbf{S}^2$ that is only $W^{1,3}$, the Hopf degree is still well-defined as the degree of a $W^{1,3}$ lifting [R1] or via the associated guage. It can also be given by approximation since smooth maps are strongly dense in $W^{1,3}(\mathbf{S}^3,\mathbf{S}^2)$ [Be1]. Under this strong approximation, the corresponding Coulomb guages converge strongly in $W^{1,1}$ to a Coulomb guage of u (as defined by 2.1) and the Coulomb lifts converge strongly in $W^{1,3}$ to a lift $\tilde u$ of u that satisfies (2.2) weakly and hence pointwise a.e. in $\mathbf{R}^3$.

We need the following important lower bound [R1]:

Lemma 2.2.

$$\delta_0 \;\equiv\; \inf\{\int_{\mathbf{S}^3}|\nabla u|^3\,d\mathcal{H}^3 \ :\ u\in W^{1,3}(\mathbf{S}^3,\mathbf{S}^2),\ \text{Hopf deg}\,(u)\neq 0\} \;>\; 0\ .$$

Proof. We assume $u \in \mathcal{C}^\infty(\mathbf{S}^3,\mathbf{S}^2)$ with Coulomb lift $\hat u : \mathbf{S}^3 \to \mathbf{S}^3$ and with 3 energy $\int_{\mathbf{S}^3}|\nabla u|^3\,d\mathcal{H}^3 \le 1$. By Lemma 2.1, Sobolev embedding, and Hölder's inequality,

$$\begin{aligned}
\big(2\pi^2\big)^{1/3} \;&\le\; \Big|\int_{\mathbf{S}^3}\tilde u^\#\omega_{\mathbf{S}^3}\Big|^{1/3} \;\le\; \|\nabla\tilde u\|_{L^3}\\
&\le\; \|\frac{1}{2}\tilde\eta\|_{L^3} \;+\|\nabla u\|_{L^3}\\
&\le\; c\|\nabla\tilde\eta\|_{L^{3/2}(h)} \;+\; \|\nabla u\|_{L^3}\\
&=\; c\|u^\#\omega_{\mathbf{S}^2}\|_{L^{3/2}} \;+\; \|\nabla u\|_{L^3} \;\le\; (c+1)\|\nabla u\|_{L^3}\ ,
\end{aligned}$$

for some absolute constant c. ■

Now we turn to maps from a 4 dimensional domain to $\mathbf{S}^2$. It will be notationally simpler to work with mappings defined on all of $\mathbf{R}^4$, that are constant near infinity. The next extension lemma indicates how we may reduce to this situation.

Suppose Ω_0 is a bounded $\mathcal{C}^1$ domain in $\mathbf{R}^4$. We may construct $\mathcal{C}^1$ domains Ω_t so that $\overline{\Omega_s} \subset \Omega_t$ for $-1 \le s < t \le 1$ along with a retraction map $\rho : \overline{\Omega_1} \setminus \Omega_{-1} \to \partial\Omega_0$ so that the induced map sending $x \in \partial\Omega_t$ to $\big(\rho(x), t\big)$ gives a $\mathcal{C}^1$ correspondence between the tubular neighborhood $\overline{\Omega_1} \setminus \Omega_{-1}$ and $\partial\Omega_0 \times [-1, 1]$.

Lemma 2.3.(compare [He],p.146) *Any $u \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbf{S}^2)$ admits an extension $\mathcal{U}_u \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$ such that $\mathcal{U}_u \equiv (0,0,1)$ on $\mathbf{R}^4 \setminus \Omega_1$ and*

$$\int_{\mathbf{R}^4} |\nabla \mathcal{U}_u|^3 \, dx \;\le\; C_{\Omega_0}\big(1 \;+\; \int_{\Omega_0} |\nabla u|^3 \, dx\big)$$

where C_{Ω_0} depends only on Ω_0 .

Proof. Letting $c_1, c_2, \ldots$ denote constants depending only on Ω_0, we first choose, by Fubini's Theorem, a number $r \in (0, 1]$ so that

$$\|\nabla u\|_{L^3(\partial\Omega_{-r})} \;\le\; c_1 \|\nabla u\|_{L^3(\Omega_0)} \ ,$$

and, for $x \in \Omega_t$ with $t \in [0, r]$ define $v(x) = u(\bar{x})$ where $\bar{x} \in \partial\Omega_{-t}$ and $\rho(\bar{x}) = \rho(x)$. Then

$$\|\nabla v\|_{L^3(\Omega_r \setminus \Omega_0)} \;\le\; c_2 \|\nabla u\|_{L^3(\Omega_0)} \ ,$$

and

$$\|\nabla v\|_{L^3(\partial\Omega_r)} \;\le\; c_3 \|\nabla u\|_{L^3(\partial\Omega_{-r})} \ .$$

By Sobolev embedding and the bound $|v| \le 1$, we have

$$\|v\|_{W^{\frac{3}{4},4}(\partial\Omega_r)} \;\le\; c_4 \|v\|_{W^{1,3}(\partial\Omega_r)} \;\le\; c_4\big(1 \;+\; \|\nabla v\|_{L^3(\partial\Omega_r)}\big)$$

Since $u\big|\,\overline{\Omega_{-r}}$ is continuous, $\deg\,(v|\partial\Omega_r) = \deg\,(u|\partial\Omega_{-r}) = 0$, and the class

$$\mathcal{W} \;=\; \big\{w \in W^{1,4}\big(\Omega_1 \setminus \overline{\Omega_r}\ \big) \;:\; w = v \text{ on } \partial\Omega_r,\ w \equiv (0,0,1) \textit{ on } \partial\Omega_1\big\}$$

is nonempty. Thus we may find $w \in \mathcal{W}$ of minimum *4-energy,*

$$\int_{\Omega_1 \setminus \overline{\Omega_r}} |\nabla w|^4 \, dx \ ,$$

which is, by [HL2], bounded by

$$c_4 \|v\|^4_{W^{3/4,4}(\partial\Omega_r)} \ .$$

Hölder's inequality then gives

$$\|\nabla w\|_{L^3(\Omega_1\setminus\overline{\Omega_r})} \ \le \ c_6\|\nabla w\|_{L^4(\Omega_1\setminus\overline{\Omega_r})} \ .$$

Letting

$$\begin{cases} \zeta \ = \ u & \text{on } \Omega_0 \\ \zeta \ = \ v & \text{on } \Omega_r\setminus\Omega_0 \\ \zeta \ = \ w & \text{on } \Omega_1\setminus\Omega_r \\ \zeta \ = \ (0,0,1) & \text{on } \mathbf{R}^4\setminus\Omega_1 \end{cases}$$

we deduce from the above inequalities that

$$\|\zeta\|_{L^3(\mathbf{R}^4)} \ \le \ \big(1 \ + \ c_2 \ + c_6c_5^{1/4}c_4c_3c_1\big)\|\nabla u\|_{L^3(\Omega_0)} \ + \ c_6c_5^{1/4}c_3 \ .$$

Qualitatively, we deduce from the interior and boundary regularity of 4-energy minimizers [HL2] that ζ is continuous on $\mathbf{R}^4$. Letting ζ_ϵ denote a standard $\mathbf{R}^4$-valued smoothing of ζ on the closed region $\overline{\Omega_1}\setminus\Omega_0$, preserving boundary data, so that, in particular,

$$\|\nabla\zeta_\epsilon\|_{L^3} \ \to \ \|\nabla\zeta\|_{L^3} \ \text{ and } \ |\zeta_\epsilon|\to 1 \text{ uniformly as } \epsilon\to 0 \ ,$$

we may complete the proof by taking $\mathcal{U}_u = \zeta_\epsilon/|\zeta_e|$ for ϵ sufficiently small. ■

For the remainder of the paper, we will, for simplicity, restrict to mappings that are constant outside of a fixed compact set.

For map $u \in W^{1,3}(\mathbf{R}^4,\mathbf{S}^2)$, constant near infinity, one again has the notion of a Coulomb guage $\tilde{h}\in W^{1,1}$ defined by (2.1) which may alternately be obtained using the strong $W^{1,3}$ approximation [Be1] of u by maps smooth away from a finite singuar set and homogeneous near each singularity. The complement of each finite set being simply-connected, we readily obtain corresponding Coulomb lifts that converge strongly in $W^{1,12/5}$ to a Coulomb lift $\tilde{u}$ satisfying (2.2) a.e. Our main estimate for Coulomb lifts on $\mathbf{R}^4$ is the following:

Lemma 2.4. *Suppose $u\in W^{1,3}(\mathbf{R}^4,\mathbf{S}^2)$, and u is constant outside of a compact set $K\subset\mathbf{R}^4$. Then there is a constant c_K depending only on K so that any Coulomb lift $\tilde{u}$ of u satisfies*

$$\begin{aligned}
\|\nabla\tilde{u}\|_{L^{\frac{12}{5}}(\mathbf{R}^4)} \ &\le \ c_K\big(1 \ + \ \int|\nabla u|^3\,dx\big)^{\frac{2}{3}} \ , \\
\|\nabla(\tilde{u}|h)\|_{L^3(h)} \ &\le \ c_K\big(1 \ + \ \int_h|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{2}{3}} \text{ for each } h\in H \ , \\
\mathbf{M}\big(G_{\tilde{u}\#}[h \llcorner K]\big) \ &\le \ c_K\big(1 \ + \ \int_h|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{4}{3}} \text{ for each } h\in H \ , \\
\int_{-\infty}^{\infty}\big(\int_{h(v,t)}|\nabla\big(\tilde{u}|h(v,t)\big)|^3\,d\mathcal{H}^3\big)^{\frac{1}{2}}\,dt \ &\le \ c_K\big(1 \ + \ \int|\nabla u|^3\,dx\big) \text{ for each } v\in\mathbf{S}^3 \ , \\
\int_{-\infty}^{\infty}\mathbf{M}\big(G_{\tilde{u}\#}[h(v,t) \llcorner K]\big)^{\frac{3}{4}}\,dt \ &\le \ c_K\big(1 \ + \ \int|\nabla u|^3\,dx\big) \text{ for each } v\in\mathbf{S}^3 \ .
\end{aligned}$$

Proof. By Lemma 2.1, Sobolev embedding in $\mathbf{R}^4$, and Hölder's inequality,

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^{12/5}(\mathbf{R}^4)} &\le \|\frac{1}{2}\tilde{\eta}\|_{L^{12/5}} + \|\nabla u\|_{L^{12/5}} \le c_1\|\nabla\tilde{\eta}\|_{L^{3/2}} + c_2\|\nabla u\|_{L^3}\\ &= c_1\|u^{\#}\omega_{\mathbf{S}^2}\|_{L^{3/2}} + c_2\|\nabla u\|_{L^3} \le c_3\big(1 + \int|\nabla u|^3\,dx\big)^{\frac{2}{3}} ,\end{aligned}$$

for some constants c_1, c_2, c_3, depending only on K.

Similarly Lemma 2.1, Sobolev embedding in the 3 dimensional h, and Hölder's inequality imply

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^3(h)} &\le \|\frac{1}{2}\tilde{\eta}\|_{L^3(h)} + \|\nabla u\|_{L^3(h)} \le c_4\|\nabla\tilde{\eta}\|_{L^{3/2}(h)} + \|\nabla u\|_{L^3(h)}\\ &= c_4\|u^{\#}\omega_{\mathbf{S}^2}\|_{L^{3/2}(h)} + \|\nabla u\|_{L^3(h)} \le c_5\big(1 + \int_h|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{2}{3}} ,\end{aligned}$$

for some constants c_4, c_5, depending only on K.

To prove the third conclusion, observe first that

$$\mathbf{M}\big(G_{\tilde{u}\#}[h \llcorner K]\big) = \mathcal{H}^3\big(G_{\tilde{u}}(h\cap K)\big) = \int_{h\cap K} JG_{\tilde{u}|h}\,d\mathcal{H}^3 ,$$

where $JG_{\tilde{u}|h}$ is the 3 dimensional Jacobian $\|\Lambda_3 DG_{\tilde{u}|h}\|$. Also, in the expansion of $|JG_{\tilde{u}|h}|^2$ on h, the only term involving the square of the product of 3 derivatives of $\tilde{u}$ is the square of

$$\begin{aligned}|J(\tilde{u}|h)| &\le |(J\tilde{u})| = |\tilde{u}^{\#}\omega_{\mathbf{S}^3}| = |\tilde{u}^{\#}\big(\alpha\wedge\Pi^{\#}\omega_{\mathbf{S}^2}\big)|\\ &= |\tilde{u}^{\#}\alpha\wedge\tilde{u}^{\#}\Pi^{\#}\omega_{\mathbf{S}^2}| = \frac{1}{2}|\tilde{\eta}\wedge u^{\#}\omega_{\mathbf{S}^2}| .\end{aligned}$$

Thus we may use Sobolev embedding in h and Hölder's inequality several times to estimate

$$\begin{aligned}\mathbf{M}\big(G_{\tilde{u}\#}[h \llcorner K]\big) &\le c_6\int_{h\cap K}\big(|\tilde{\eta}\wedge u^{\#}\omega_{\mathbf{S}^2}| + |\nabla\tilde{u}|^2 + |\nabla\tilde{u}| + 1\big)\,d\mathcal{H}^3\\
&\le c_6\big(\int_{h\cap K}|\tilde{\eta}|^3\,d\mathcal{H}^3\big)^{\frac{1}{3}}\big(\int_{h\cap K}|u^{\#}\omega_{\mathbf{S}^2}|^{\frac{3}{2}}\,d\mathcal{H}^3\big)^{\frac{2}{3}} + c_7 + c_8\int_{h\cap K}|\nabla\tilde{u}|^2\,d\mathcal{H}^3\\
&\le c_6\big(\int_{h\cap K}|\nabla\tilde{\eta}|^{\frac{3}{2}}\,d\mathcal{H}^3\big)^{\frac{2}{3}}\big(\int_{h\cap K}|u^{\#}\omega_{\mathbf{S}^2}|^{\frac{3}{2}}\,d\mathcal{H}^3\big)^{\frac{2}{3}} + c_7 + c_8\int_{h\cap K}\big(|\tilde{\eta}|^2 + |\nabla u|^2\big)\,d\mathcal{H}^3\\
&\le c_6\big(\int_{h\cap K}|u^{\#}\omega_{\mathbf{S}^2}|^{\frac{3}{2}}\,d\mathcal{H}^3\big)^{\frac{4}{3}} + c_9 + c_{10}\big(\int_{h\cap K}|\tilde{\eta}|^3\,d\mathcal{H}^3\big)^{\frac{2}{3}} + c_{11}\big(\int_{h\cap K}|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{1}{3}}\\
&\le c_6\big(\int_{h\cap K}|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{4}{3}} + c_{12} + c_{13}\big(\int_{h\cap K}|\nabla\tilde{\eta}|^{\frac{3}{2}}\,d\mathcal{H}^3\big)^{\frac{4}{3}} + c_{11}\big(\int_{h\cap K}|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{4}{3}}\\
&\le c_{14}\big(\int_{h\cap K}|\nabla u|^3\,d\mathcal{H}^3\big)^{\frac{4}{3}} + c_{12} ,\end{aligned}$$

where $c_6, \ldots, c_{14}$ depend only on K.

Finally, taking $h = h(v,t)$, raising the second and third inequalities to the $\frac{3}{2}$ and $\frac{4}{3}$ powers, integrating with respect to t, and noting that

$$\int_{-\infty}^{\infty}\int_{h(v,t)\cap K}|\nabla u|^3\,d\mathcal{H}^3 dt = \int_K|\nabla u|^3 dx$$

gives the fourth and fifth conclusions. ∎

Example 2.5. One cannot improve $\frac{12}{5}$ to 3 in the above estimate for $\tilde{u}$. In fact, we here describe a sequence of smooth maps $u_n \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$, constant outside of $\mathbf{B}_1$, so that

$$\sup_n \int |\nabla u_n|^3 \, dx \; < \; \infty \, ,$$

but

$$\lim_{n\to\infty} \inf\{\int |\nabla \tilde{u}_n|^3 \, dx \; : \; \tilde{u}_n \text{ is a Hopf lift of } u_n\} \; = \; \infty \, .$$

We obtain the u_n by a suitable dipole construction. First consider in $\mathbf{R}^4$ the oriented intervals

$$I_j = [(0,0,0,\frac{1}{2j}),(0,0,\frac{1}{2j^2},\frac{1}{2j})]$$

for $j = 1, 2, \ldots$ There is a map $u_n \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$ which is a constant $(0,0,1)$ outside of small disjoint closed tubular neighborhoods of I_1, I_2, ..., I_n and which, has on each 3 dimensional perpendicular slice D_j^t of the neighborhood of I_j, *Hopf degree* j (as a map from $(D_j^t, \partial D_j^t)$ to $(\mathbf{S}^2, \{(0,0,1)\})$), for $j = 1, 2, \ldots, n$. Using the crucial observation of [R1] that

$$\inf\{\int_{\mathbf{S}^3} |\nabla g|^2 \, d\mathcal{H}^3 \; : \; g \in \mathcal{C}^\infty(\mathbf{S}^3, \mathbf{S}^2), \text{ Hopf deg}\, g = j\} \; \leq \; c j^{\frac{3}{4}} \, ,$$

we see that we may obtain such a u_n with 3 energies uniformly bounded

$$\int_{\mathbf{R}^4} |\nabla u_n|^3 \, dx \; \leq \; c \sum_{j=1}^{n} j^{\frac{3}{4}} \text{length}\,(I_j) \; \leq \; c \sum_{j=1}^{\infty} j^{\frac{3}{4}} (\frac{1}{2j^2}) \; < \; \infty.$$

On the other hand, for any smooth Hopf lift $\tilde{u}_n$ of u_n the restriction of $\tilde{u}_n$ to each 3 dimensional slice D_j^t has (as in §2) *topological degree* j, when viewed as a map from $(D_j^t, \partial D_j^t)$ to $(\mathbf{S}^2, \{(0,0,1)\})$. Thus we have the lower energy bound [R1] on each slice

$$\int_{D_j^t} |\nabla(\tilde{u}_n | D_j^t)|^3 \, d\mathcal{H}^3 \; \geq \; 3^{3/2} 4\pi^2 j \, ,$$

Integrating over all such slices we find that

$$\int_{\mathbf{R}^4} |\nabla \tilde{u}_n|^3 \, dx \; \geq \; 3^{3/2} 4\pi^2 \sum_{j=1}^{n} j \cdot \text{length}\,(I_j) \; \geq \; 3^{3/2} 4\pi^2 \sum_{j=1}^{n} j (\frac{1}{2j^2}) \; \to \; \infty \text{ as } n \to \infty \, .$$

Similarly, concerning the graphs $G_{\tilde{u}_n}(\mathbf{R}^4) \subset \mathbf{R}^4 \times \mathbf{S}^3$, we find that, for each slice,

$$\mathcal{H}^3\big(G_{\tilde{u}_n}(D_j^t)\big) \; \geq \; \int_{\mathbf{S}^3} \mathcal{H}^0\big(D_j^t \cap \tilde{u}_n^{-1}\{y\}\big) \, d\mathcal{H}^3 y \; \geq \; \frac{4}{3}\pi j \, ,$$

and integrating over all the slices gives

$$\mathcal{H}^4\big(G_{\tilde{u}_n}(\mathbf{B}_4)\big) \; \geq \; \frac{4}{3}\pi \sum_{j=1}^{n} j (\frac{1}{2j^2}) \; \to \; \infty \text{ as } n \to \infty \, ,$$

so that these graphs do not converge as Cartesian currents [GMS2].

Remark 2.6. For any fixed 3 dimensional oriented hyperplane $h \subset \mathbf{R}^4$ and (not necessarily smooth) map ψ in $W^{1,3}(h, \mathbf{S}^3)$, the graph of ψ, $G_\psi(h)$, is still a countably 3 rectifiable subset of $h \times \mathbf{S}^3$ whose approximate tangent planes project onto h and are thus oriented by the orientation $\vec{h}$ of h. We will use the notation $\mathbf{G}_\psi$ for the resulting locally rectifiable current defined by

$$\mathbf{G}_\psi(\phi) \ = \ \int_{G_\psi(h)} < \Lambda_3 DG_\psi(x)\vec{h}, \phi(x,y) > d\mathcal{H}^3(x,y)$$

for $\phi \in \mathcal{D}^3(\mathbf{R}^4 \times \mathbf{S}^3)$. Note that $\mathbf{G}_\psi = G_{\psi\#}h$ in case ψ is smooth. Inasmuch as smooth maps are strongly dense in $W^{1,3}(h, \mathbf{S}^3)$ [Be1], say $\psi = \lim_{\epsilon \to 0} \psi_\epsilon$ with $\psi_\epsilon \in W^{1,3}(h, \mathbf{S}^3) \cap \mathcal{C}^\infty$, we readily verify that

$$\partial \mathbf{G}_\psi \ = \ \lim_{\epsilon \to 0} \partial \mathbf{G}_{\psi_\epsilon} \ = \ \lim_{\epsilon \to 0} \partial G_{\psi_\epsilon \#} h \ = \lim_{\epsilon \to 0} G_{\psi_\epsilon \#} \partial h \ = 0 \ .$$

Similarly, for any Lipschitz domain $\Omega \subset \mathbf{R}^4$ and $\psi \in W^{1,3}(\partial\Omega, \mathbf{S}^3)$ one may define $\mathbf{G}_\psi$ and verify that $\partial \mathbf{G}_\psi = 0$.

For $u \in W^{1,3}(\mathbf{R}^4, \mathbf{S}^2)$, constant near infinity, we infer from Lemma 2.4 that the function sending

$$h \in H \ \to \mathbf{G}_{\tilde{u}|h} \in \mathcal{R}_{3,loc}$$

is a scan which we denote $\mathbf{G}_{\tilde{u}}$.

This scan may be used to give another criteria [Z], [I1] for the strong $W^{1,3}$ approximability by smooth maps.

Lemma 2.7. *A map $u \in W^{1,3}(\mathbf{R}^4, \mathbf{S}^2)$, constant near ∞, is in the strong $W^{1,3}$ sequential closure of $\mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$ if and only if $\partial \mathbf{G}_{\tilde{u}} = 0$ for some lift $\tilde{u}$ of u.*

Proof. Suppose that $u_n \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$ converges strongly in $W^{1,3}$ to u with $u_n \equiv (0,0,1)$ near infinity. Consider corresponding Coulomb guages $\tilde{\eta}_n$ and Coulomb lifts $\tilde{u}_n$ with $\tilde{u}_n \equiv (0,0,0,1)$ near infinity. Passing to a subsequence, we obtain from Lemma 2.4 a pointwise a.e. limit

$$\tilde{u} \ = \ \lim_{n \to \infty} \tilde{u}_n \in W^{1,\frac{5}{12}}(\mathbf{R}^4, \mathbf{S}^3) \ .$$

For all $v \in \mathbf{S}^3$, Fatou's Lemma and Fubini's Theorem,

$$\int_{-\infty}^{\infty} \liminf_{n \to \infty} \int_{h(v,t)} |\nabla(u_{n'} - u)|^3 \, d\mathcal{H}^3 \, dt \ \le \ \liminf_{n \to \infty} \int_{\mathbf{R}^4} |\nabla(u_{n'} - u)|^3 \, dx \ = \ 0 \ ,$$

then give, for a.e. $t \in \mathbf{R}$, strong convergence in $W^{1,3}\big(h(v,t)\big)$ of a subsequence (depending on t) $u_{n'}|h(v,t)$ to $u|h(v,t)$. Then the Coulomb guages $\tilde{\eta}_{n'}$ converge strongly in $W^{1,1}\big(h(v,t)\big)$ and the Coulomb lifts $\tilde{u}_{n'}$ converge strongly in $W^{1,3}\big(h(v,t)\big)$. The graphs then converge weakly as currents

$$\lim_{n \to \infty} G_{\tilde{u}_{n'}\#} h(v,t) \ = \ \lim_{n \to \infty} \mathbf{G}_{\tilde{u}_{n'}|h(v,t)} \ = \ \mathbf{G}_{\tilde{u}|h(v,t)} \ .$$

Slicing theory [F],4.3, then gives, for almost all polyhedral domains $U \subset \mathbf{R}^4$, the current convergence of $G_{\tilde{u}_{n'}\#}\partial[[U]]$ to the rectifiable current $(\mathbf{G}_{\tilde{u}})_{\partial U}$. We thus deduce the vanishing of $\partial\big[(\mathbf{G}_{\tilde{u}})_{\partial U}\big]$, $(\mathbf{G}_{\tilde{u}})_{\partial U}(q^{\#}\omega_{\mathbf{S}^3})$, and hence $\partial\mathbf{G}_{\tilde{u}}$.

The converse follows essentially from arguments of [Be1]. In the proof of [Be1], Theorem 2 (with n=4, $p = 3$), the singularities of the approximating map only arise in making a homogeneous extension on some 4 dimensional cubes C where the 3 energy on ∂C is controlled. But now second condition of the of $\partial\mathbf{G}_{\tilde{u}} = 0$ implies that the Hopf invariant on ∂C is zero so that one may, with arbitrarily small extra energy, remove the singularity by modifying this homogeneous extension in a very small ball about the center. ■

To effectively use Lemma 2.4 to get information about the span $\mathbf{G}_{\tilde{u}}$, we require a new topology on $\mathcal{R}_3$, which we now describe.

§3. The $d_{\mathbf{e}}$ metric on $\mathcal{R}_3$.

For any orthonormal frame $\mathbf{e} = (e_1, e_2, e_3, e_4)$ of $\mathbf{R}^4$, and $P, Q \in \mathcal{R}_{3,loc}$ with spt $(P-Q)$ compact, we now define

$$d_{\mathbf{e}}(P,Q) = \inf\big\{\mathbf{M}(S) + \sum_{j=1}^{4}\int \mathbf{M} < T, p_{e_j}, t >^{3/4} dt \ : \ P - Q = S + \partial T,\ S \in \mathcal{R}_3,\ T \in \mathcal{R}_4\big\}.$$

This should be compared to the *flat* distance where one uses instead the quantity $\mathbf{M}(S) + \mathbf{M}(T)$ while noting that

$$\mathbf{M}(T) \ \leq \ \sum_{j=1}^{4}\int \mathbf{M} < T, p_{e_j}, t > dt \leq \ 4\mathbf{M}(T) \ .$$

Lemma 3.1. *$d_{\mathbf{e}}$ is a metric on $\mathcal{R}_3$.*

Proof. The function $d_{\mathbf{e}}$ clearly satisfies the conditions $d_{\mathbf{e}}(P,Q) = d_{\mathbf{e}}(Q,P)$, $d_{\mathbf{e}}(P,P) = 0$, and the triangular inequality because $(A+B)^{3/4} \leq A^{3/4} + B^{3/4}$.

It remains to prove that the assumption

$$d_{\mathbf{e}}(P,Q) = 0 \ \text{ implies } \ P \ = \ Q \ .$$

For this we first find $S_i \in \mathcal{R}_3$ and $T_i \in \mathcal{R}_4$, so that

$$P - Q \ = \ S_i + \partial T_I$$

and

$$\mathbf{M}(S_i) \ + \ \sum_{j=1}^{4}\int \mathbf{M} < T_i, p_{e_j}, t >^{3/4} dt \ \to 0 \ \text{ as } \ i \to \infty \ .$$

For each $j \in \{1,2,3,4\}$, Fatou's Lemma implies that

$$\int \liminf_{i\to\infty} \mathbf{M} < T_i, p_{e_j}, t >^{3/4}\, dt \quad = \ 0\ .$$

Thus, for almost every $t \in \mathbf{R}$, there is a subsequence i' (depending on j, t) so that

$$\mathbf{M} < T_{i'}, p_{e_j}, t > \to\ 0 \text{ as } i' \to \infty\ .$$

Since $\mathbf{M}(P - Q - \partial T_{i'}) \to 0$, we find from [F],4.3 that, for such j, t, and any 2 form $\phi \in \mathcal{D}^2(\mathbf{R}^4 \times \mathbf{S}^3)$,

$$< P - Q, p_{e_j}, t > (\phi)\ =\ \lim_{i'\to\infty} < \partial T_{i'}, p_{e_j}, t > (\phi)\ =\ -\lim_{i'\to\infty} \partial < T_{i'}, p_{e_j}, t > (\phi)\ =\ 0\ .$$

It follows that, for $\|P - Q\|$ almost all z, the approximate tangent 3 plane L_z associated with $\overrightarrow{P - Q}(z)$ has $p(L_z) = 0$. In fact, otherwise $p_{e_j}|L_z$ would have rank one for some $j \in \{1,2,3,4\}$ which would, as in the proof of Corollary 1.1, contradict the vanishing of $< P - Q, p_{e_j}, t >$ for a.e. t.

We deduce again, as in the proof of Corollary 1.1, that

$$P - Q\ =\ \sum_{a\in A} m_a[[a]] \times [[\mathbf{S}^3]]$$

for some finite subset A of $\mathbf{B}_R$ and some nonzero integers m_a.

If $A \neq \emptyset$, we may fix one point $b \in A$, a positive $\delta \leq 1$ so that

$$\mathbf{B}_\delta(b) \cap (A \cup \partial\mathbf{B}_R)\ =\ \{b\}\ ,$$

and i sufficiently large so that

$$\mathbf{M}(S_i)\ +\ \sum_{j=1}^{4} \int \mathbf{M} < T_i, p_{e_j}, t >^{3/4}\, dt\ <\ \left(\frac{\pi^2}{8}\right)^{3/4} \delta \cos\left(\frac{3\pi}{8}\right)\ . \qquad (*)$$

Since $\int_{\mathbf{S}^3} \mathbf{M} < S_i, q, v > d\mathcal{H}^3 v \leq \mathbf{M}(S_i)$ by [F],4.3 and $< S_i, q, v >$ is an integer-multiplicity 0 chain for $\mathcal{H}^3$ almost all $v \in \mathbf{S}^3$, we first note that

$$V\ =\ \{v \in \mathbf{S}^3\ :< S_i, q, v >= 0\}\ =\ \{v \in \mathbf{S}^3\ :\ \mathbf{M} < S_i, q, v > < 1\}$$

has $\mathcal{H}^3(V) \geq \pi^2 = \frac{1}{2}\mathcal{H}^3(\mathbf{S}^3)$.

For $\mathcal{H}^3$ almost all $v \in V$,

$$-\partial < T_i, q, v > = < P - Q, q, v > = \sum_{a\in A} m_a[[(a, v)]]\ ,$$

and $< T_i, q, v >$ contains an oriented integer-multiplicity curve Γ_v in $\mathbf{R}^4 \times \{v\}$ joining (b, v) to (a_v, v) for some point $a_v \in (A \setminus \{b\}) \cup \partial \mathbf{B}_R$. Then the corresponding direction

$$\frac{b - a_v}{|b - a_v|} \cdot \iota_v e_{j_v} \ \leq \ \cos(\frac{3\pi}{8})$$

for some integers $\iota_v \in \{-1, 1\}$, $j_v \in \{1, 2, 3, 4\}$. The current $p_{e_{j_v}\#}\Gamma_v$ is a nonzero integer multiple of the projected interval $I_v = p_{e_{j_v}}\big([(b, v), (a_v, v)]\big)$ which has length at least δ. So, using Fubini's tTheorem and [F],4.3, we see that, for $\mathcal{H}^3$ almost all $v \in V$ and $\mathcal{H}^1$ almost all $t \in I_v$,

$$\mathbf{M} << T_i, p_{e_{j_v}}, t >, q, v > = \ \mathbf{M} << T_i, q, v > p_{e_{j_v}}, t > \ \geq \ 1 \ .$$

Now we simply chose $\iota \in \{-1, 1\}$ and $j \in \{1, 2, 3, 4\}$ so that

$$W \ = \ \{v \in V \ : \ \iota_v = \iota \text{ and } j_v = j\}$$

has $\mathcal{H}^3(W) \geq \frac{1}{8}\mathcal{H}^3(V) \geq \frac{\pi^2}{8}$. Then the interval I joining $e_j \cdot b$ and $e_j \cdot b + \iota_v \delta$ is contained in each I_v for all $v \in W$, and we may use [F],4.3 to deduce that

$$\begin{aligned}
\sum_{j=1}^{4} \int \mathbf{M} < T_i, p_{e_j}, t >^{3/4} dt \ &\geq \ \sum_{j=1}^{4} \int \big(\int_W \mathbf{M} << T_i, p_{e_j}, t >, q, v > dv\big)^{3/4} dt \\
&= \ \int_I \big(\int_W \mathbf{M} << T_i, q, v > p_{e_v}, t > dv\big)^{3/4} dt \\
&\geq \ \big(\mathcal{H}^3(W)\big)^{3/4} \delta \ \geq \ \big(\frac{\pi^2}{8}\big)^{3/4} \delta \ ,
\end{aligned}$$

which contradicts (*). Thus, $A = \emptyset$, and $(P - Q) \llcorner p^{-1}(\mathbf{B}_R) = 0$. ∎

As in [F], let $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$ for a current T.

Lemma 3.2. *For each $R > 0$, $\mathbf{N}$ is, with respect to the $d_{\mathbf{e}}$ metric, lower semi-continuous on*

$$\mathcal{R}_R \ = \ \{P \in \mathcal{R}_3 \ : \ \operatorname{spt} P \subset \overline{\mathbf{B}}_R(0) \times \mathbf{S}^3\} \ .$$

For each $\Lambda > 0$, $\{P \in \mathcal{R}_R \ : \mathbf{N}(P) \leq \Lambda\}$ is $d_{\mathbf{e}}$ sequentially compact.

Proof. By the Federer-Fleming compactness theorem [F],4.2.16, any $\mathbf{N}$ bounded sequence in $\mathcal{R}_R$ has a subsequence P_i that is convergent to some $P \in \mathcal{R}_R$ in the $\mathbf{F}_{\overline{\mathbf{B}}_R(0) \times \mathbf{S}^3}$ norm, that is, $P_i - P = S_i + \partial T_i$ of some rectifiable currents S_i, T_i with supports in $\overline{\mathbf{B}}_R(0) \times \mathbf{S}^3$ so that

$$\mathbf{M}(S_i) \ + \ \mathbf{M}(T_i) \ \to \ 0 \text{ as } i \to \infty \ .$$

This flat convergence implies the weak (current) convergence of P_i to P and ∂P_i to ∂P so that

$$\begin{aligned} \mathbf{N}(P) \;=\; \mathbf{M}(P) + \mathbf{M}(\partial P) \;&\le\; \liminf_{i\to\infty} \mathbf{M}(P_i) \;+\; \liminf_{i\to\infty} \mathbf{M}(\partial P_i) \\ &\le\; \liminf_{i\to\infty} \big[\mathbf{M}(P_i) + \mathbf{M}(\partial P_i)\big] \;=\; \liminf_{i\to\infty} \mathbf{N}(P_i)\ . \end{aligned}$$

But the flat convergence also implies the $d_{\mathbf{e}}$ convergence of P_i to P because Hölder's inequality and [F],4.3 show that, for each $j \in \{1,2,3,4\}$,

$$\begin{aligned} \int \mathbf{M} < T_i, p_{e_j}, t >^{3/4} dt \;&\le\; \big(\int \mathbf{M} < T_i, p_{e_j}, t > dt\big)^{3/4} (2R)^{1/4} \\ &\le\; \mathbf{M}(T_i)^{3/4}(2R)^{1/4} \;\to\; 0 \text{ as } i \to \infty\ . \end{aligned}$$

This establishes the desired compactness.

Moreover, to show the lower-semicontinuity that $\mathbf{N}(P) \le \liminf_{i\to\infty} \mathbf{N}(P_i)$ for any $d_{\mathbf{e}}$ convergent sequence $P_i \to P$ in $\mathcal{R}_R$, we may assume first that the righthand side is finite and second, by passing to a subsequence that $\lim_{i\to\infty} \mathbf{N}(P_i) < \infty$. Then as above we find the flat (and hence weak current) convergence of a subsequence $P_{i'} \to Q$. But then this implies $d_{\mathbf{e}}$ convergence so that $Q = P$ and

$$\mathbf{N}(P) \;= \mathbf{N}(Q) \;\le\; \liminf_{i\to\infty} \mathbf{N}(P_{i'}) \;\le\; \liminf_{i\to\infty} \mathbf{N}(P_i)\ .$$

■

4. Energy Concentration Associated with Bubbles in 3 Dimensions.

For a general $W^{1,3}$ weakly convergent sequences of maps in $\mathcal{C}^\infty(\mathbf{R}^3, \mathbf{S}^3)$, the graphs may subconverge to a current that includes not only the graph of the limit but additional terms, called "bubbles". These are studied in the work [GMS2] of Giaquinta, Modica, and Soucek on Cartesian Currents, which shows the following:

Lemma 4.1. (3d Bubbling) *For any sequence of maps $\psi_n \in \mathcal{C}^\infty(\mathbf{R}^3, \mathbf{S}^3)$ which are constant $(0,0,0,1)$ outside of a fixed bounded subset of $\mathbf{R}^3$ and have a uniform 3-energy bound*

$$\sup_n \int_{\mathbf{R}^3} |\nabla \psi_n|^3 \, d\mathcal{H}^3 \;<\; \infty\ ,$$

the oriented graphs $\mathbf{G}_{\psi_n}$ locally have uniformly bounded masses, and a subsequence of them converges weakly to a locally rectifiable cycle. Assuming that the mappings $\psi_n \rightharpoonup \psi$ weakly in $W^{1,3}$, any such limiting current has the form

$$\mathbf{G}_\psi \;+\; \sum_{a\in A} \mathbf{m}(a)\big([[a]] \times [[\mathbf{S}^3]]\big)$$

for some finite subset A of $\mathbf{R}^3$ and nonzero integers $\mathbf{m}(a)$ for $a \in A$. Moreover,

$$\lim_{r\to 0} \liminf_{n\to\infty} \int_{\mathbf{B}_r(a)} |\nabla \psi_n|^3 \, dx \;\ge\; 2 \cdot 3^{3/2} \pi^2 |\mathbf{m}(a)|$$

for all $a \in A$.

Thus the finite supporting set A of the "bubbles" is contained in the 3-*energy concentration set*,

$$\{a \in \mathbf{R}^3 \;:\; \lim_{r\to 0}\liminf_{n\to\infty} \int_{\mathbf{B}_r(a)} |\nabla \psi_n|^3\, dx \;>\; 0\}\ ,$$

of the sequence ψ_n.

We are here interested in these phenomena for *Hopf lifts* of given maps from $\mathbf{R}^3$ or from $\mathbf{R}^4$ to $\mathbf{S}^2$. Of course, each map $\psi_n : h \to \mathbf{S}^3$ above can be viewed as a Hopf lift of the map

$$\Pi \circ \psi_n : h \to \mathbf{S}^2\ .$$

It is important that the supporting set A of bubbles for the sequence ψ_n is actually contained in the 3-energy concentration set of this "downstairs" sequence $\Pi \circ \psi_n$. Specifically, under the hypotheses of Lemma 4.1,

$$\lim_{r\to 0}\liminf_{n\to\infty} \int_{h\cap\mathbf{B}_r(a)} |\nabla(\Pi\circ\psi_n)|^3\, d\mathcal{H}^3 \;\geq\; \varepsilon_0$$

for all $a \in A$ and some absolute positive constant ε_0. This will follow from Lemma 4.2 below, which is actually a stronger result because it has no hypotheses concerning uniform energy bounds.

The reason we will need this stronger version is that, in our application, we start with a 3-energy bounded sequence of maps from $\mathbf{R}^4$ to $\mathbf{S}^2$. This sequence may unfortunately not have a single subsequence with Hopf lifts of bounded 3 energy on almost all hyperplanes. In fact, any single subsequence may itself fail to have bounded 3 energy on almost all hyperplanes, despite Fatou's Lemma which only guarantees bounded energy subsequences *depending on the hyperplane*. Nevertheless, we will find, in §6, a single subsequence of graphs of Coulomb lifts that is $d_{\mathbf{e}}$ convergent on almost all hyperplanes, thus giving the desired limiting scan. The next lemma, which is crucial for proving the *rectifiability* of this limit in §7, shows that the appearence of a bubble under this weak $d_{\mathbf{e}}$ convergence (even in the absence of energy or mass bounds) is enough to guarantee energy concentration.

Lemma 4.2. *Suppose h is a fixed hyperplane in $\mathbf{R}^4$ and $u_n = \Pi\circ\tilde{u}_n$ where $\tilde{u}_n \in \mathcal{C}^\infty(h, \mathbf{S}^3)$ and the $\tilde{u}_n$ are constant $(0,0,0,1)$ outside of some fixed bounded subset of h. If*

$$(d_{\mathbf{e}})\lim_{n\to\infty} \mathbf{G}_{\tilde{u}_n} \;=\; \mathbf{G}_{\tilde{u}} \;+\; \sum_{a\in A} \mathbf{m}(a)\big([[a]]\times[[\mathbf{S}^3]]\big)$$

for some $\tilde{u} \in W^{1,3}(h, \mathbf{S}^3)$, some finite subset A of h, and some nonzero integers $\mathbf{m}(a)$, then

$$\lim_{r\to 0}\liminf_{n\to\infty} \int_{h\cap\mathbf{B}_r(a)} |\nabla u_n|^3\, d\mathcal{H}^3 \;\geq\; \varepsilon_0$$

for all $a \in A$ for some absolute positive constant ε_0.

Proof. We begin by deriving the desired constant ε_0.

First, on the unit 3 dimensional ball $\mathbf{B} = h \cap \mathbf{B}_1(0)$, any smooth map $v : \mathbf{B} \to \mathbf{S}^2$ has a *local Coulomb lift* $\hat{v} : \mathbf{B} \to \mathbf{S}^3$ obtained from an associated guage (as in §2) $\hat{\eta}$ which satisfies

$$\begin{cases} d\hat{\eta} \;=\; \hat{v}^{\#}\omega_{\mathbf{S}^2} & \text{on } \mathbf{B} \\ d^*\hat{\eta} \;=\; 0 & \text{on } \mathbf{B} \\ \iota^{\#}_{\partial\mathbf{B}}\hat{\eta} \;=\; 0 & \text{where } \iota_{\partial\mathbf{B}} \text{ is the inclusion map of } \partial\mathbf{B}. \end{cases}$$

Arguing as in §2.2, using a Poincaré instead of Sobolev inequality, we find constants c_1, c_2 so that

$$\begin{aligned} \|\nabla\hat{v}\|_{L^3(\mathbf{B})} &\le \|\frac{1}{2}\hat{\eta}\|_{L^3(\mathbf{B})} + \|\nabla v\|_{L^3(\mathbf{B})} \\ &\le c_1\|\nabla\hat{\eta}\|_{L^{3/2}(\mathbf{B})} + \|\nabla v\|_{L^3(\mathbf{B})} \\ &= c_1\|v^{\#}\omega_{\mathbf{S}^2}\|_{L^{3/2}(\mathbf{B})} + \|\nabla v\|_{L^3(\mathbf{B})} \le c_2\|\nabla v\|_{L^3(\mathbf{B})} \end{aligned}$$

assuming $\int_{\mathbf{B}} |\nabla v|^3\, d\mathcal{H}^3 \le 1$.

Second, since $\Pi^{-1}\{y\}$ is a great circle in $\mathbf{S}^3$ for every $y \in \mathbf{S}^2$, we readily find a positive constant ρ_0 so that

$$\mathcal{H}^3\big(\Pi^{-1}\big[\mathbf{B}_{\rho_0}(y) \cap \mathbf{S}^2\big]\big) \;<\; \pi^2 \;=\; \frac{1}{2}\mathcal{H}^3(\mathbf{S}^3)\ . \tag{4.1}$$

Third, on any convex 2-*dimensional* region $\Omega \subset \mathbf{R}^2$ with $\mathbf{B}^2_{\frac{1}{8}} \subset \Omega \subset \mathbf{B}^2_4$, a function $f \in W^{1,3}(\Omega, \mathbf{R}^4)$ is, by Sobolev embedding, Hölder continuous, and there is a positive constant ε_1 small enough so that

$$\int_\Omega |\nabla f|^3\, d\mathcal{H}^2 \;\le\; \epsilon_1 \quad \text{implies} \quad f\big(\Omega\big) \subset \mathbf{B}_{\rho_0/3}(y) \tag{4.2}$$

for some $y \in \mathbf{R}^4$. Now we choose

$$\varepsilon_0 \;=\; \min\{1,\ \frac{1}{6}\varepsilon_1,\ \frac{2\pi^2}{4(c_2^3+6)}\}\ . \tag{4.3}$$

Assuming for contradiction that the lemma is false with this ε_0 , we find (by passing to a subsequence without changing notation) $\tilde{u}_n$, $\tilde{u}$, A, $\mathbf{m}(a)$ satisfying the hypothesis, a point $a \in A$, and a positive r so that

$$\int_{h\cap\mathbf{B}_r(a)} |\nabla u_n|^3\, d\mathcal{H}^3 \;<\; \varepsilon_0$$

for all n. Since A is discrete and $\tilde{u} \in W^{1,3}$, we may also assume that r is small enough to insure that $A \cap \mathbf{B}_r(a) = \{a\}$ and

$$\int_{h\cap\mathbf{B}_r(a)} |\nabla\tilde{u}|^3\, d\mathcal{H}^3 \;<\; \varepsilon_0\ .$$

For convenience, we now rescale from $\mathbf{B}_r(a)$ to $\mathbf{B} \equiv h \cap \mathbf{B}_1(0)$ by defining

$$v_n(x) = u_n(a+rx)\ ,\ \tilde{v}_n(x) = \tilde{u}_n(a+rx)\ ,\ \tilde{v}(x) = \tilde{u}(a+rx)\ ,$$

and noting that

$$\int_{\mathbf{B}} |\nabla v_n|^3 \, d\mathcal{H}^3 \ = \ \int_{h\cap \mathbf{B}_r(a)} |\nabla u_n|^3 \, d\mathcal{H}^3 \ \le \ \varepsilon_0\ , \tag{4.4}$$

$$\int_{\mathbf{B}} |\nabla \tilde{v}|^3 \, d\mathcal{H}^3 \ = \ \int_{h\cap \mathbf{B}_r(a)} |\nabla \tilde{u}|^3 \, d\mathcal{H}^3 \ \le \ \varepsilon_0\ , \tag{4.5}$$

While the given sequence of lifts $\tilde{v}_n$ of v_n may have unbounded 3-energy on the ball $\mathbf{B}$, the local Coulomb lifts $\hat{v}_n$ described above satisfy

$$\int_{\mathbf{B}} |\nabla \hat{v}_n|^3 \, d\mathcal{H}^3 \ \le \ c_2^3 \int_{\mathbf{B}} |\nabla v_n|^3 \, d\mathcal{H}^3 \ \le \ c_2^3 \varepsilon_0\ . \tag{4.6}$$

We may homologically connect the graphs of the two lifts $\hat{v}_n$ and $\tilde{v}_n|\mathbf{B}$ because they are homotopic through lifts. Specifically, each circle fiber $\Pi^{-1}\{y\}$ of the Hopf map is oriented (by the 1 form α). Let $\Gamma(\cdot,0) : [0,1] \to h \times \mathbf{S}^3$ be the unique shortest constant-speed, positively-oriented curve in the circle $\{0\}\times\Pi^{-1}\{v_n(0)\}$ from $\{(0,\hat{v}_n(0))\}$ to $\{(0,\tilde{v}_n(0))\}$. By the simple- connectedness of $\mathbf{B}$, there is a unique smooth extension $\Gamma : [0,1]\times\mathbf{B} \to h\times\mathbf{S}^3$ so that, for all $x \in \mathbf{B}$, $\Gamma(\cdot,x)$ is a constant-speed, positively-oriented curve in $\{x\}\times\Pi^{-1}\{v_n(x)\}$ from $\{(x,\hat{v}_n(x))\}$ to $\{(x,\tilde{v}_n(x))\}$. The current $\hat{T} \equiv \Gamma_{\#}\big([[0,1]]\times[[\mathbf{B}]]\big)$ then gives the homology

$$\partial \hat{T}_n \ = \ \mathbf{G}_{\tilde{v}_n|\mathbf{B}} \ - \ \mathbf{G}_{\hat{v}_n} \ - \Gamma_{\#}\big([[0,1]]\times\partial[[\mathbf{B}]]\big)\ , \tag{4.7}$$

with the last term having support in $p^{-1}(\partial\mathbf{B})$. This current may have high multiplicity at various points, and we have no control on the $\mathbf{M}(\hat{T}^n)$ as $n\to\infty$.

Next, using the $d_{\mathbf{e}}$ convergence, we may write

$$\mathbf{G}_{\tilde{v}} \ + \ \sum_{b\in A} \mathbf{m}(b)\big(\ [[\ \frac{b-a}{r}\ ]]\times[[\mathbf{S}^3]]\ \big) \ - \ \mathbf{G}_{\tilde{v}_n} \ = \ \tilde{S}_n \ + \ \partial\tilde{T}_n\ , \tag{4.8}$$

for some $\tilde{S}_n \in \mathcal{R}_3$ and $\tilde{T}_n \in \mathcal{R}_4$ such that

$$\mathbf{M}(\tilde{S}_n) \ + \ \sum_{j=1}^{4} \int \mathbf{M} < \tilde{T}_n, p_{e_j}, t >^{3/4} dt \ \to \ 0 \text{ as } n\to\infty\ .$$

We now fix an integer n sufficiently large to guarantee that

$$\mathbf{M}(\tilde{S}_n) \ + \ \sum_{j=1}^{4} \int \mathbf{M} < \tilde{T}_n, p_{e_j}, t >^{3/4} dt \ \le \ \epsilon_0\ . \tag{4.9}$$

To use this estimate we need to choose polyhedral regions whose boundaries lie in such hyperplanes. For our fixed frame $\mathbf{e} = (e_1, e_2, e_3, e_4)$ of $\mathbf{R}^4$, we note that the angle between the hyperplane h and three of the e_i, say e_1, e_2, e_3, is at most $\frac{\pi}{4}$. So using the norm

$$\mu(x) = \max\{e_1 \cdot x, e_2 \cdot x, e_3 \cdot x\}$$

on h, we find that, for $\frac{1}{2} \leq t \leq 1$, the open parallellopiped $\Omega_t \equiv \{x \in h : \mu(x) < t\}$ has

$$h \cap \mathbf{B}_{\frac{1}{8}}(0) \ \subset \ \Omega_t \ \subset \ h \cap \mathbf{B}_4(0) \ .$$

Each of the six 2 dimensional faces of $\partial\Omega_t$,

$$\{x \in \partial\Omega_t \ : \ x \cdot e_j = \pm t\}$$

similarly satisfy this interior-exterior ball property.

Since $p^{-1}\big(\partial\Omega_t\big) \subset \cup_{j=1}^{3} p_{e_j}^{-1}\{-t, t\}$ and

$$\int_0^1 \Big[\int_{\partial\Omega_t} |\nabla v_n|^3 \, d\mathcal{H}^2 \ + \ \sum_{j=1}^{4} \big(\mathbf{M} < \tilde{T}_n, p_{e_j}, -t >^{\frac{3}{4}} + \mathbf{M} < \tilde{T}_n, p_{e_j}, t >^{\frac{3}{4}} \big)\Big] \, dt \ \leq \ 3\varepsilon_0$$

by (4.4) and (4.9), we may now fix a number $t \in [\frac{1}{2}, 1]$ so that

$$\int_{\partial\Omega_t} |\nabla v_n|^3 \, d\mathcal{H}^2 \ \leq \ 6\varepsilon_0 \ , \tag{4.10}$$

and the slice $\tilde{R}_n \equiv < \tilde{T}_n, \mu \circ p, t >$ has

$$\mathbf{M}(\tilde{R}_n) \leq \ 6\varepsilon_0 \ . \tag{4.11}$$

Also since $\hat{T}_n$ is constructed by a homotopy through liftings, one deduces from (4.2), (4.3), and (4.10) that the other slice $\hat{R}_n \equiv < \hat{T}_n, \mu \circ p, t >$ has

$$\operatorname{spt} \hat{R}_n \ \subset \ \partial\Omega_t \times \Pi^{-1} v_n(\partial\Omega_t) \ \subset \ \partial\Omega_t \times \Pi^{-1}\big(\mathbf{S}^2 \cap \mathbf{B}_{\rho_0}(y)\big) \tag{4.12}$$

for some $y \in \mathbf{S}^2$.

Combining (4.7) and (4.8) with usual slicing formulas, we now have the equation of rectifiable currents

$$\begin{aligned} &\partial\big(\tilde{T}_n \llcorner p^{-1}\Omega_t \ + \ \hat{T}_n \llcorner p^{-1}\Omega_t\big) \\ &\quad = \ \mathbf{G}_{\tilde{v}|\Omega_t} \ + \ \mathbf{m}(a)\big([[0]] \times [[\mathbf{S}^3]]\big) \ - \mathbf{G}_{\hat{v}_n|\Omega_t} - \ \tilde{S}_n \llcorner p^{-1}\Omega_t \ + \ \tilde{R}_n + \ \hat{R}_n \ . \end{aligned} \tag{4.13}$$

We may apply $q_\#$ to project onto $\mathbf{S}^3$ and then restrict to the region,

$$U \ \equiv \ \mathbf{S}^3 \setminus \Pi^{-1}\big(\mathbf{S}^2 \cap \mathbf{B}_{\rho_0}(y)\big) \ . \tag{4.14}$$

and compute the resulting masses. Checking each term using (4.1), (4.3), (4.5), (4.4), (4.6), (4.9), (4.11), (4.12), and (4.14),

$$
\begin{aligned}
\mathbf{M} q_\# \partial\big(\tilde{T}_n \llcorner p^{-1}\Omega_t \ + \ \hat{T}_n \llcorner p^{-1}\Omega_t\big) \llcorner U \ &\le \ \mathbf{M}\big(\partial q_\#(4\,\text{dimensional current}\,)\big) \ = \ 0 \ , \\
\mathbf{M}\big(q_\# \mathbf{m}(a)\big([[0]] \times [[\mathbf{S}^3]]\big) \llcorner U\big) \ &= \ |\mathbf{m}(a)| \mathcal{H}^3(U) \ \ge \ \mathcal{H}^3(U) \ > \ \pi^2 \ , \\
\mathbf{M}(q_\# \mathbf{G}_{\tilde{v}|\Omega_t} \llcorner U) \ &\le \ \int_{\mathbf{B}} |\nabla \tilde{v}|^3 \, d\mathcal{H}^3 \ \le \ \varepsilon_0 \ \le \ \frac{\pi^2}{24} \ , \\
\mathbf{M}(q_\# \mathbf{G}_{\hat{v}_n|\Omega_t} \llcorner U) \ &\le \ \int_{\mathbf{B}} |\nabla \hat{v}_n|^3 \, d\mathcal{H}^3 \ \le \ c_2^3 \varepsilon_0 \ \le \ \frac{\pi^2}{4} \ , \\
\mathbf{M}\big(q_\#(\tilde{S}_n \llcorner p^{-1}\Omega_t) \llcorner U\big) \ &\le \ \mathbf{M}(\tilde{S}_n) \ \le \ \varepsilon_0 \ \le \ \frac{\pi^2}{24} \ , \\
\mathbf{M}\big(q_\#(\tilde{R}_n) \llcorner U\big) \ &\le \ \mathbf{M}(\tilde{R}_n) \ \le \ 6\varepsilon_0 \ \le \ \frac{\pi^2}{4} \ , \\
\mathbf{M}\big(q_\# \hat{R}_n \llcorner U\big) \ &= \ 0
\end{aligned}
$$

which gives the desired contradiction and completes the proof. ■

§5 A Fractional Maximal Function Estimate for the Scan of a Smooth Map.

In this section we assume w is a smooth map from $\mathbf{R}^4$ to $\mathbf{S}^3$ which is constant outside of a fixed ball $\mathbf{B}_R = \mathbf{B}_R(0)$. We first note that the corresponding scan

$$
G_{w\#} : \ H \to \mathcal{R}_3 \ \ \textit{is continuous, and hence, measurable} \ ,
$$

with respect to the $d_{\mathbf{e}}$ metric on $\mathcal{R}_3$. In fact, for oriented hyperplanes $h, k \in H$, we may use a geodesic path from h to k in $H = \mathbf{S}^3 \times \mathbf{R}$ to define a 4 dimensional locally rectifiable current $T \in \mathcal{R}_4$ with $\partial T = h - k$ and $\mathbf{M}(T \llcorner \mathbf{B}_R) \le CR^3|h-k|$, hence,

$$
\begin{aligned}
G_{w\#}h \ - \ G_{w\#}k \ &= \ \partial G_{w\#}T \\
\mathbf{M}(G_{w\#}(T \llcorner \mathbf{B}_R)) \ &\le C(w) R^3 |h-k| \ .
\end{aligned}
$$

Thus $G_{w\#}$ is a continuous map from H into $\mathcal{R}_3$, with the locally flat, and hence (as in §3), $d_{\mathbf{e}}$ topology. ■

Theorem 5.1. *Suppose that $w \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^3)$, and w is constant outside of $\mathbf{B}_R$. If $\mathbf{e} = (e_1, e_2, e_3, e_4)$ is an orthonormal frame for $\mathbf{R}^4$, $0 < \alpha < 1$, $v \in \mathbf{S}^3$, and I is a subinterval of $[-R, R]$, then the fractional maximal function*

$$
\mathcal{M}_I(t) \ \equiv \ \operatorname{esssup}_{t \ne s \in I} \frac{d_{\mathbf{e}}\big(G_{w\#}h(v,s)\,, G_{w\#}h(v,t)\big)}{|s-t|^\alpha} \ \ \text{for a.e. } t \in I \ ,
$$

satisfies the weak-type measure estimate

$$
\big|\{t \in I \, : \, \mathcal{M}_I^{1/\alpha}(t) > \lambda\}\big| \ \le \ \lambda^{-1} \mu_w(I)
$$

where

$$\mu_w(I) \ = \ \alpha^{-1} 5^{\frac{1}{\alpha}} \sum_{j=1}^{4} \Big(\int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R \cap \pi_v^{-1}(I)]\big)^{\alpha} \, d\tau\Big)^{\frac{1}{\alpha}} \ .$$

Remark 5.2. Note that the quantity μ_w satisfies the super-additivity relation

$$\mu_w(I) + \mu_w(J) \ \leq \ \mu_w(K)$$

whenever $I, \ J$ are nonoverlapping subintervals of an interval K in $[-R, R]$.

Proof of Theorem 5.1. For numbers $s, t \in [-R, R]$, let $\overline{st}$ denote the closed interval joining s and t and observe that the 4 dimensional locally rectifiable current

$$T_{s,t} \ = \ G_{w\#}[[\pi_v^{-1}(\,\overline{st}\,)]]$$

has

$$G_{w\#}h(v,s) \ - \ G_{w\#}h(v,t) \ = \ \pm \partial T_{s,t}$$

so that we may estimate

$$\begin{aligned} d_{\mathbf{e}}\big(G_{w\#}h(v,s)\,, G_{w\#}h(v,t)\big) \ &\leq \ \sum_{j=1}^{4} \int \mathbf{M}\big(< T_{s,t}, p_{e_j}, \tau > \ \llcorner \ p^{-1}(\mathbf{B}_R)\big)^{\alpha} \, d\tau \\ &= \ \sum_{j=1}^{4} \int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R \cap \pi_v^{-1}(\,\overline{st}\,)]\big)^{\alpha} d\tau \ . \end{aligned}$$

We can now use a covering argument to estimate the measure of

$$E_\lambda \ \equiv \ \{t \in I \, : \, \mathcal{M}_I^{1/\alpha}(t) > \lambda\} \ .$$

For each $t \in E_\lambda$, we may choose a number $s_t \neq t$ in I so that

$$\begin{aligned} \lambda^\alpha \, |t - s_t|^\alpha \ &< \ d_{\mathbf{e}}\big(G_{w\#}h(s_t,v)\,, G_{w\#}h(v,t)\big) \\ &\leq \ \sum_{j=1}^{4} \int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R \cap \pi_v^{-1}(\,\overline{s_t t}\,)]\big)^{\alpha} d\tau \ . \end{aligned}$$

The (1 dimensional) Besicovitch covering Lemma allows us to find, for $i = 1, \ldots, 5$ and $k = 1, 2, \ldots$, points $t_{i,k} \in E_\lambda$ with corresponding closed intervals

$$I_{i,k} \ = \ \overline{s_{t_{i,k}} t_{i,k}}$$

that altogether cover E_λ,

$$E_\lambda \ \subset \ \cup_{i=1}^{5} \cup_{k=1}^{\infty} I_{i,k} \ ,$$

while each of the five separate families

$$\{I_{i,1},\, I_{i,2},\, \ldots\}\ ,$$

corresponding to $i \in \{1, \ldots, 5\}$, consists of disjoint intervals. In particular,

$$|E_\lambda| \ \le\ \sum_{i=1}^{5}\sum_{k=1}^{\infty} r_{i,k}$$

where $r_{i,k} = |t_{i,k} - s_{t_{i,k}}|$.

For nonnegative $A_{i,k}$ we have the elementary estimate

$$\sum_{i,k} r_{i,k}^{1-\alpha} A_{i,k} \ \le\ \big(\sum_{i,k} r_{i,k}\big)^{1-\alpha}\big(\sum_{i,k} A_{i,k}\big)\ .$$

One may check this by reducing to the case $\sum_{i,k} A_{i,k} = 1$ and noting that, on the simplex

$$\{(x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}, x_{2,1}, x_{2,2}, \ldots) \in \mathbf{R}^\infty \ :\ x_{i,k} \ge 0,\ \sum_{i=1}^{5}\sum_{k=1}^{\infty} x_{i,k} = 1\}\ ,$$

the linear function $\sum_{i,k} r_{i,k}^{1-\alpha} x_{i,k}$ is bounded by the number $\big(\sum_{i,k} r_{i,k}\big)^{1-\alpha}$ because it is trivially bounded by this number at all the vertices $(1, 0, \ldots)$, $(0, 1, 0, \ldots)$, $\ldots$.

Now taking

$$A_{i,k} \ =\ \sum_{j=1}^{4}\int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R \cap \pi_v^{-1}(I_{i,k})]\,\big)^\alpha d\tau\ ,$$

we see that $A_{i,k} \ \ge\ \lambda^\alpha r_{i,k}^\alpha$ by the choice of $s_{t_{i,k}}$, and, using the disjointness of each $\{I_{i,1},\, I_{i,2},\, \ldots\}$, we conclude that

$$\begin{aligned}
\lambda^\alpha |E_\lambda|^\alpha \ &\le\ \lambda^\alpha\big(\sum_{i,k} r_{i,k}\big)^\alpha \ =\ \big(\sum_{i,k} r_{i,k}\big)^{\alpha-1}\big(\sum_{i,k} r_{i,k}\lambda^\alpha\big)\\
&\le\ \big(\sum_{i,k} r_{i,k}\big)^{\alpha-1}\sum_{i,k} r_{i,k}^{1-\alpha} A_{i,k} \ \le\ \sum_{i,k} A_{i,k}\\
&=\ \sum_{i=1}^{5}\sum_{k=1}^{\infty}\sum_{j=1}^{4}\int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R] \llcorner p_v^{-1}(I_{i,k})\big)^\alpha d\tau\\
&\le\ 5\sum_{j=1}^{4}\int \mathbf{M}\big(G_{w\#}[h(e_j,\tau) \llcorner \mathbf{B}_R] \llcorner p_v^{-1}(I)\,\big)^\alpha d\tau\ ,
\end{aligned}$$

and then raise to the $\frac{1}{\alpha}$ power to complete the proof. ■

§6. Limits of Scans of Coulomb Lifts of Weakly Convergent Smooth Maps.

Here we will prove our main existence theorem by using the estimates of §2-5 in the general compactness lemma of the appendix §9.

Theorem 6.1. *Suppose $u_n \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$, $u_n \equiv (0,0,1)$ on $\mathbf{R}^4 \setminus \mathbf{B}_2^4$, and*

$$\sup \int |\nabla u_n|^3 \, dx \ < \ \infty \ .$$

If $\tilde{u}_n : \mathbf{R}^4 \to \mathbf{S}^3$ is a Coulomb lifting of u_n as in §2, then there is a subsequence n' of n, a pointwise a.e. limit

$$\tilde{u} \ = \ \lim_{n\to\infty} \tilde{u}_{n'} \in W^{1,\frac{5}{12}}(\mathbf{R}^4, \mathbf{S}^3) \ ,$$

and a scan cycle S so that, for all $v \in \mathbf{S}^3$,

$$S\big(h(v,t)\big) \ = \ (d_\epsilon) \lim_{n\to\infty} G_{\tilde{u}_{n'}} \# h(v,t)$$

for almost all $t \in \mathbf{R}$.

Proof. By Lemma 2.4, we may pass to a subsequence, without changing notations, to get weak convergence in $W^{1,\frac{12}{5}}(\mathbf{R}^4, \mathbf{S}^3)$ of the Coulomb lifts

$$\tilde{u}_n \rightharpoonup \tilde{u} \in W^{1,\frac{12}{5}}(\mathbf{R}^4, \mathbf{S}^3) \ .$$

This sequence converges pointwise a.e. on $\mathbf{R}^4$, and the limit $\tilde{u}$ is a Hopf lift of

$$u \ \equiv \ \Pi \circ \tilde{u} \ \in \ W^{1,3}(\mathbf{R}^4, \mathbf{S}^2) \ .$$

Turning now to the corresponding scans, we proceed in two steps.

STEP I. *Convergence a.e. on one family of parallel hyperplanes.*

Here we will show that

for each fixed direction $v \in \mathbf{S}^3$, there is a a subsequence $n_v(i)$ (depending on v) so that, for almost all $t \in \mathbf{R}$, each sequence

$$G_{\tilde{u}_{n_v(i)}} \# h(v,t)$$

is $d_{\mathbf{e}}$ convergent as $i \to \infty$.

For this, we apply the Compactness Theorem 9.1 of the Appendix, with

$$\begin{aligned}
\alpha &= \frac{3}{4}\,, \\
X &= [-2,2]\,, \\
Y &= \{P \in \mathcal{R}_3 \;:\; \partial P = 0,\ \operatorname{spt} P \subset \overline{\mathbf{B}_2} \times \mathbf{S}^3\}\,, \\
\operatorname{dist}_Y &= d_{\mathbf{e}}\,, \\
\mathcal{N}(P) &= \mathbf{N}(P)^{\frac{3}{4}} \;=\; \mathbf{M}(P)^{\frac{3}{4}}\,, \\
f_n(t) &= G_{\tilde{u}_n\#}\big[h(v,t) \mathbin{\llcorner} \mathbf{B}_2\big]\,. \\
\mu_n(I) &= \frac{4}{3}\cdot 5^{\frac{4}{3}} \sum_{j=1}^{4} \Big(\int \mathbf{M}\big(G_{\tilde{u}_n\#}[h(e_j,\tau) \mathbin{\llcorner} \mathbf{B}_2 \cap \pi_v^{-1}(I)]\big)^{\frac{3}{4}}\, d\tau\Big)^{\frac{4}{3}}\,.
\end{aligned}$$

We have in hand all the necessary hypotheses. Note that the $d_{\mathbf{e}}$ lower semi-continuity of $\mathcal{N}$ and $d_{\mathbf{e}}$ sequential compactness of $\mathcal{N}$ bounded sets is provided by Lemma 3.2 while the uniform integral bound of $\mathbf{M}\big(G_{\tilde{u}_n\#}\big[h(v,t) \mathbin{\llcorner} \mathbf{B}_2\big]\big)^{3/4}$ is given by Lemma 2.4. We thus conclude the $d_{\mathbf{e}}$ convergence of a subsequence (depending on v) of $G_{\tilde{u}_n\#}h(v,t)$ for almost every $t \in \mathbf{R}$, and Step I is complete.

STEP II. *Convergence at a.e. coordinate hyperplane is sufficient.*

We first apply Step I four times with v equaling

$$(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)$$

to find a subsequence n' so that, for almost all $s \in \mathbf{R}$, one has the $d_{\mathbf{e}}$ convergences

$$\lim_{n\to\infty} G_{\tilde{u}_{n'}\#}h_s^1 \;=\; P_s^1,\quad \ldots\,,\quad \lim_{n\to\infty} G_{\tilde{u}_{n'}\#}h_s^4 \;=\; P_s^4,$$

at the *coordinate hyperplanes*

$$h_s^1 \equiv h\big((1,0,0,0),s\big),\ \ldots,\ h_s^4 \equiv h\big((0,0,0,1),s\big)\,.$$

It only remains to show that for *any* other direction

$$v \in \mathbf{S}^3 \setminus \{(1,0,0,0),\,(0,1,0,0),\,(0,0,1,0),\,(0,0,0,1)\}$$

this convergence automatically implies $d_{\mathbf{e}}$ convergence of $G_{\tilde{u}_{n'}\#}h(v,t)$ for almost every $t \in \mathbf{R}$ (with the same subsequence n').

Thanks to Step I, *any subsequence* of $G_{\tilde{u}_{n'}\#}h(\cdot,v)$ contains a subsequence, that is $d_{\mathbf{e}}$ convergent a.e. on $\mathbf{R}$. It thus suffices to show that any subsequence of $G_{\tilde{u}_{n'}\#}h(\cdot,v)$ contains a subsequence, having pointwise a.e. $d_{\mathbf{e}}$ limit $P(\cdot)$ that is *uniquely determined* by the currents P_s^i already obtained from the coordinate hyperplanes.

By Fubini's Theorem, the exceptional set

$$Z \ \equiv \ \big\{t \in \mathbf{R} \ : \ \mathcal{H}^3\{x \in h(v,t) \ : \ \lim_{n\to\infty} \tilde{u}_{n'}(x) \neq \tilde{u}(x)\} \ > \ 0\big\}$$

has measure zero. Moreover, Lemma 2.4 and Fatou's Lemma imply that

$$\int_{-\infty}^{\infty} \liminf_{n\to\infty} \big(\int_{h(v,t)} |\nabla\big(\tilde{u}_{n'}|h(v,t)\big)|^3 \, d\mathcal{H}^3\big)^{\frac{1}{2}} \, dt \ \leq \ \sup_n c\big(1 \ + \ \int |\nabla u_{n'}|^3 \, dx\big)$$

so that the set

$$\tilde{Z} \equiv \ \{t \in \mathbf{R} \ : \ \liminf_{n\to\infty} \int_{h(v,t)} |\nabla\big(\tilde{u}_{n'}|h(v,t)\big)|^3 \, d\mathcal{H}^3 v \ = \ \infty\}$$

also has measure zero.

Let $P(\cdot)$ be any pointwise a.e. $d_{\mathbf{e}}$ limit of a subsequence of $G_{\tilde{u}_{n'}\#} h(\cdot, v)$. For $t \in \mathbf{R} \setminus (Z \cup \tilde{Z})$, we see that the limiting map $\tilde{u}\,|\,h(v,t)$ is a weak $W^{1,3}$ limit of some subsequence of maps with bounded 3-energy while the limiting current $P(t)$ is, by Lemma 2.4, the weak limit of some subsequence of locally $\mathbf{M}$ bounded smooth graphs. Thus, the limiting map has finite 3-energy,

$$\int_{h(v,t)} |\nabla\big(\tilde{u}\,|\,h(v,t)\big)|^3 \, d\mathcal{H}^3 \ < \ \infty \ ,$$

and, by Lemma 4.1, the limiting current has the form

$$P(t) \ = \ \mathbf{G}_{\tilde{u}|h(v,t)} \ + \ \sum_{a\in h(v,t)} \mathbf{m}_{v,t}(a)\big([[a]] \times [[\mathbf{S}^3]]\big)$$

for an integer-valued function $\mathbf{m}_{v,t}$ supported in some finite subset of $h(v,t)$. We now only need to show how, *for almost all t, all the integer multiplicities $\mathbf{m}_{v,t}(a)$, for $a \in h(v,t)$, are uniquely determined by our coordinate hyperplane currents P^i_s.*

This will be accomplished by using, for each $a \in h(v,t)$ and almost all $r > 0$, the open coordinate cube

$$\mathbf{Q}_r(a) \ \equiv \ \prod_{i=1}^{4} (a_i - r, a_i + r) \ ,$$

and looking at the limit of the graph of the restriction of $\tilde{u}_{n'}$ to the boundary of the *half-cube*

$$\mathbf{Q}^v_r(a) \ \equiv \ \{x \in \mathbf{Q}_r(a) \ : \ x \cdot v > t\} \ .$$

Since $a \cdot v = t$, one face of $\partial \mathbf{Q}^v_r(a)$ lies in the hyperplane $h(v,t)$ under present consideration while the others lie in the 8 coordinate hyperplanes $h^i_{a_1 - r}, h^i_{a_i + r}, \ldots, h^i_{a_4 - r}, h^i_{a_4 + r}$. The corresponding equation of currents is

$$\partial[[\mathbf{Q}^v_r(a)]] = \sigma h(v,t) \llcorner \mathbf{Q}_r(a) + \sum_{i=1}^{4} \big(h^i_{a_i + r} - h^i_{a_i - r}\big) \llcorner \{x \in \partial \mathbf{Q}^v_r(a) \ : \ x \cdot v > t\}$$

where $\sigma = \operatorname{sign}(t - v \cdot a)$.

As above, we have, for a.e. $r > 0$, pointwise a.e. convergence of $\tilde{u}_{n'}$ to $\tilde{u}$ on the hyperplanes $h^i_{a_i \pm r}$ as well as

$$\liminf_{n\to\infty} \int_{h^i_{a_i \pm r}} |\nabla\big(\tilde{u}_{n'}|h^i_{a_i\pm r}\big)|^3 \, d\mathcal{H}^3 v \ < \infty$$

for $i = 1,2,3,4$. As before, we deduce that, for a.e. $r > 0$, each limiting map $\tilde{u}\,|h^i_{a_i\pm r}$ is a weak $W^{1,3}$ limit of some subsequence of maps with bounded 3-energy. Also by Lemma 2.4, for a.e. $r > 0$, any $d_{\mathbf{e}}$ limit of a subsequence of $G_{\tilde{u}_{n'}\#} h^i_{a_i \pm r}$ the weak current limit of some locally $\mathbf{M}$ bounded subsequence.

Next we recall one relevant consequence of slicing [F],4.3.1. If h is a hyperplane and n^* is a subsequence of n such that the currents $G_{\tilde{u}_{n^*}\#} h$ are locally mass bounded and converge weakly to a current T, then, for any direction $v^* \in \mathbf{S}^3$, not perpendicular to h, one has, for a.e. $s > 0$, the current convergence of the restrictions

$$G_{\tilde{u}_{n\#}\#}\big(h \llcorner \{x : x\cdot v^* > s\}\big) \ \text{ to } \ T \llcorner \{(x,y) : x\cdot v^* > s\} \ .$$

Consider now any two subsequences n'' and n''' of n'. Passing to subsequences of these, we may assume, by Lemma 4.1, that

$$\begin{aligned}
\lim_{n\to\infty} G_{\tilde{u}_{n''}\#} h(v,t) &= \mathbf{G}_{\tilde{u}|h(v,t)} + \sum_{a\in h(v,t)} \mathbf{m}''(a)[[a]] \times [[\mathbf{S}^3]] \\
\lim_{n\to\infty} G_{\tilde{u}_{n'''}\#} h(v,t) &= \mathbf{G}_{\tilde{u}|h(v,t)} + \sum_{a\in h(v,t)} \mathbf{m}'''(a)[[a]] \times [[\mathbf{S}^3]]
\end{aligned}$$

for some integer-valued functions $\mathbf{m}''$ and $\mathbf{m}'''$ supported in finite subsets of $h(v,t)$. We now want to show that $\mathbf{m}''(a) = \mathbf{m}'''(a)$ for each point $a \in h(v,t)$. For this, first choose r_a small enough so that

$$\overline{\mathbf{Q}_{r_a}(a)} \ \cap \ (\operatorname{spt}\mathbf{m}'' \cup \operatorname{spt}\mathbf{m}''') \ \subset \ \{a\} \ .$$

The slicing remark implies that, for almost every $t \in \mathbf{R}$, we may, for almost every $0 < r < r_a$, pass to subsequences, without changing notations, to insure that

$$\begin{aligned}
\lim_{n\to\infty} G_{\tilde{u}_{n''}\#}\big(h(v,t) \llcorner \mathbf{Q}_r(a)\big) &= \mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\mathbf{Q}_r(a) + \mathbf{m}''(a)[[a]] \times [[\mathbf{S}^3]] \\
\lim_{n\to\infty} G_{\tilde{u}_{n'''}\#}\big(h(v,t) \llcorner \mathbf{Q}_r(a)\big) &= \mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\mathbf{Q}_r(a) + \mathbf{m}'''(a)[[a]] \times [[\mathbf{S}^3]] \ ,
\end{aligned}$$

Similarly, cutting by $\{x : x\cdot v > t\}$, we see that the the two current limits

$$\lim_{n\to\infty} G_{\tilde{u}_{n''}\#}\big(\partial[[\mathbf{Q}^v_r(a)]] \llcorner \{x : x\cdot v > t\}\big) \ ,$$

$$\lim_{n\to\infty} G_{\tilde{u}_{n'''}\#}\big(\partial[[\mathbf{Q}_r^v(a)]] \llcorner \{x : x\cdot v > t\}\big)$$

exist and equal the *same* current

$$P_r(a) \equiv \sum_{i=1}^{4} \big(P^i_{a_i+r} - P^i_{a_i-r}\big) \llcorner \{(x,y) : x \in \partial\mathbf{Q}_r^v(a),\, x\cdot v > t\}$$

because of the uniqueness of the 8 coordinate hyperplane limits $P^i_{a_i\pm r}$.

Finally concerning the total boundary, one has, by the smoothness of $\tilde{u}_n$ on $\overline{\mathbf{Q}_r^v(a)}$, that

$$\begin{aligned} G_{\tilde{u}_n\#}\partial[[\mathbf{Q}_r^v(a)]]\big(q^{\#}\omega_{\mathbf{S}^3}\big) &= q_{\#}\partial G_{\tilde{u}_n\#}[[\mathbf{Q}_r^v(a)]]\big(\omega_{\mathbf{S}^3}\big) \\ &= \partial q_{\#}G_{\tilde{u}_n\#}[[\mathbf{Q}_r^v(a)]]\big(\omega_{\mathbf{S}^3}\big) = 0 \end{aligned}$$

because $q_{\#}G_{\tilde{u}_n\#}[[\mathbf{Q}_r^v(a)]]$ is a 4 dimensional current in $\mathbf{S}^3$. Thus,

$$\begin{aligned} &2\pi^2\mathbf{m}''(a) + \sigma\big(\mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\partial\mathbf{Q}_r^v(a)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) + P_r(a)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \lim_{n\to\infty} G_{\tilde{u}_{n''}\#}\big(\partial[[\mathbf{Q}_r^v(a)]] \llcorner h(v,t) + \partial[[\mathbf{Q}_r^v(a)]] \llcorner \{x : x\cdot v > t\}\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \lim_{n\to\infty} G_{\tilde{u}_{n''}\#}\partial[[\mathbf{Q}_r^v(a)]]\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \lim_{n\to\infty} 0 \\ &= \lim_{n\to\infty} G_{\tilde{u}_{n'''}\#}\partial[[\mathbf{Q}_r^v(a)]]\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \lim_{n\to\infty} G_{\tilde{u}_{n'''}\#}\big(\partial[[\mathbf{Q}_r^v(a)]] \llcorner h(v,t) + \partial[[\mathbf{Q}_r^v(a)]] \llcorner \{x : x\cdot v > t\}\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= 2\pi^2\mathbf{m}'''(a) + \sigma\big(\mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\partial\mathbf{Q}_r^v(a)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) + P_r(a)\big(q^{\#}\omega_{\mathbf{S}^3}\big), \end{aligned}$$

hence,

$$\mathbf{m}''(a) = \mathbf{m}'''(a)\ .$$

Arbitrary subsequences of $G_{\tilde{u}_{n'}\#}h(\cdot,v)$ thus have subsequences $d_{\mathbf{e}}$ convergent to a unique limit, determined by the coordinate hyperplane currents P^i_s, and for the *original subsequence* n' (which was chosen independent of v) the currents $G_{\tilde{u}_{n'}\#}h(v,t)$ $d_{\mathbf{e}}$ converge for a.e. $t \in \mathbf{R}$. ¿From Fubini's Theorem, we finally conclude that, for a.e. hyperplane $h \in H$, the $d_{\mathbf{e}}$ limit

$$S(h) = \lim_{n\to\infty} G_{\tilde{u}_{n'}\#}h$$

exists.

To check that S is a scan cycle, we note that, for a.e. $h \in H$, Lemma 2.4 and Fatou's Lemma imply that

$$\liminf_{n'\to\infty} \mathbf{M}\big(G_{\tilde{u}_{n'}\#}(h \llcorner \mathbf{B}^4_2)\big) < \infty\ ,$$

so that $S(h)$ is the limit of a locally $\mathbf{M}$ bounded, weakly convergent, subsequence $G_{\tilde{u}_{n''}\#}h$. In particular, using [F],§4.3.2,

$$S(h) \cap \big(h' \times [[\mathbf{S}^3]]\big) = \big(\lim_{n\to\infty} G_{\tilde{u}_{n''}\#}h\big) \cap \big(h' \times [[\mathbf{S}^3]]\big) = \lim_{n\to\infty} \big((G_{\tilde{u}_{n''}\#}h) \cap (h' \times [[\mathbf{S}^3]])\big)$$

for almost every $h' \in H$. Moreover, §1 and the same argument show that, for almost every $h' \in H$,

$$\lim_{n\to\infty} \big((G_{\tilde{u}_{n''}\#}h) \cap (h' \times [[\mathbf{S}^3]])\big) = \lim_{n\to\infty} \big((G_{\tilde{u}_{n''}\#}h') \cap (h \times [[\mathbf{S}^3]])\big) = S(h') \cap \big(h \times [[\mathbf{S}^3]]\big)$$

for a.e. $h \in H$. Also, for almost every polyhedral frontier $\partial[[U]]$ as in §1, one has, by the previous slicing remark, that $\partial\big(S_{\partial U}\big) = \lim_{n\to\infty} \partial G_{\tilde{u}_{n''}\#}\partial[[U]] = 0$, and

$$\big(S_{\partial U}\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;=\; \lim_{n\to\infty} G_{\tilde{u}_{n''}\#}\partial[[U]]\big(q^{\#}\omega_{\mathbf{S}^3}\big) = \lim_{n\to\infty} G_{\tilde{u}_{n''}\#}[[U]]\big(dq^{\#}\omega_{\mathbf{S}^3}\big) \;=\; 0\ ,$$

so that S is a scan cycle. ■

§7. Structure and Rectifiability of the Limiting Scan.

Here we will show that the limiting scan of Theorem 6.1 is carried by the graph of the limiting map and a set $R_S \times \mathbf{S}^3$ for some 1 rectifiable subset R_S of an energy concentration set of finite measure. The measure estimate is provided by the elementary:

Lemma 7.1. *For $\epsilon > 0$ and any sequence $u_n \in W^{1,3}(\mathbf{R}^4, \mathbf{S}^2)$ with $L \equiv \sup_n \int |\nabla u_n|^3\,dx$ being finite, the ϵ energy concentration set*

$$E_\epsilon \;=\; \{x \in \mathbf{R}^4 \;:\; \lim_{r\to 0}\liminf_{n\to\infty} \frac{1}{r}\int_{\mathbf{B}_r(x)} |\nabla u_n|^3\,dy \geq \epsilon\}\ ,$$

has $\mathcal{H}^1(E_\epsilon) \leq 24\epsilon^{-1}L$.

Proof. Let K be a compact subset of E_ϵ. For each $\delta > 0$ and point $x \in K$, we may choose a positive $r_x < \frac{1}{2}\delta$ so that

$$\liminf_{n\to\infty} \frac{1}{r_x}\int_{\mathbf{B}_{r_x}(x)} |\nabla u_n|^3\,dy \;>\; \frac{1}{2}\epsilon\ .$$

By compactness and the Vitali covering theorem, we may choose a finite subset A of K so that the corresponding balls $\{\mathbf{B}_{r_a}(a) \;:\; a \in A\}$ are disjoint while their triple enlargements $\{\mathbf{B}_{3r_a}(a) \;:\; a \in A\}$ cover K. We may now choose a single integer n sufficiently large to guarantee that

$$\frac{1}{r_a}\int_{\mathbf{B}_{r_a}(a)} |\nabla u_n|^3\,dy \geq \frac{1}{2}\epsilon$$

for all $a \in A$. Thus,

$$\mathcal{H}^1_\delta(K) \;\leq\; \sum_{a\in A} 2(6r_a) \;\leq\; 24\epsilon^{-1}\sum_{a\in A}\int_{\mathbf{B}_{r_a}(a)} |\nabla u_n|^3\,dy \;\leq\; 24\epsilon^{-1}L\ .$$

Letting $\delta \downarrow 0$ and taking the supremum over such K completes the proof. ■

As motivation for our rectifiability Theorem 7.2, recall that a 1 dimensional, finite mass, integer-multiplicity *rectifiable current* T in $\mathbf{R}^4$ is given by 3 things:

a 1 rectifiable set R_T of finite measure,
an $\mathcal{H}^1$ measurable $\vec{T} : R_T \to \mathbf{S}^3$ orienting a.e. the approximate tangent of R_T, and
an $\mathcal{H}^1$ integrable multiplicity function $\mathbf{m}_T : R_T \to \mathbf{Z}^+$

so that

$$T(\phi) \;=\; \int_{R_T} < \vec{T}(x), \phi(x) > \mathbf{m}_T(x)\, d\mathcal{H}^1 x \quad \text{for } \phi \in \mathcal{D}(\mathbf{R}^4)\ .$$

Each of these three may be determined, $\mathcal{H}^1$ a.e., by the 0 dimensional slices $T \cap h$ for a.e. $h \in H$. More, in fact, is true.

For any countably 1 rectifiable set $R \supset R_T$ and $v \in \mathbf{S}^3$, let

$$\Sigma_{R,v} \;\equiv\; \{x \in R \,:\, h(v, x\cdot v) \text{ is not transverse to the approximate tangent line of } R \text{ at } x\}$$

and observe that

$$V_R \;\equiv\; \{v \in \mathbf{S}^3 \;:\; \mathcal{H}^1(\Sigma_{R,v}) \;>\; 0\}$$

is at most countable because $\mathcal{H}^1(\Sigma_{R,v} \cap \Sigma_{R,v'}) > 0$ if and only if $v = \pm v'$.

Now fixing a direction $v \in \mathbf{S}^3 \setminus V_R$, one has, at a.e. point $x \in R_T$, that

$$\mathbf{m}_T(x) \;=\; \mathbf{m}_{T \cap h(v, x\cdot v)}(x) \;>0$$

and that the choice of orientation $\vec{T}(x)$ is determined by the slice condition

$$\operatorname{sgn}\big[\vec{T}(x)\cdot v\big] \;=\; \operatorname{sgn}\big[\big(T \cap h(v, x\cdot v)\big)\big(\chi_{\mathbf{B}_\rho(x)}\big)\big]$$

for $\rho > 0$ small. Also, up to an $\mathcal{H}^1$ null set, the carrying set for T,

$$R_T \;=\; \cup_{t \in \mathbf{R}}\{x \;:\; \mathbf{m}_{T \cap h(v,t)}(x) \neq 0\}\ .$$

Thus, for such a generic $v \in \mathbf{S}^3 \setminus V_R$, the rectifiable current T is completely determined just by its slices by almost all of the parallel hyperplanes $\{h(v,t)\ : t \in \mathbf{R}\}$.

The bubble part of our limiting scan has a similar representation by a rectifiable set, orienting vectorfield, and multiplicity function, except the multiplicity function is only $L^{3/4}$ integrable:

Theorem 7.2. *There is a positive constant ε_1 so that if $u_{n'}, u, \tilde{u}_{n'}, \tilde{u}$, and $S = \lim_{n\to\infty} G_{\tilde{u}_{n'}\#}$ are as in Theorem 6.1, then, there exist an $\mathcal{H}^1$ measurable 1 rectifiable subset R_S of the ε_1 energy concentration set E_{ε_1} for u_n, an $\mathcal{H}^1$ measurable $\vec{S} : R_S \to \mathbf{S}^3$*

orienting a.e. the approximate tangent line of R_S, and a nonzero integer multiplicity function $\mathbf{m}_S$ with $\int_{R_S} \mathbf{m}_S^{3/4} d\mathcal{H}^1 < \infty$ such that for almost every hyperplane $h \in H$,

$$S(h) \ = \mathbf{G}_{\tilde{u}|h} \ + \sum_{a \in R_S \cap h} \operatorname{sgn}\big(\vec{S}(a) \cdot \vec{h}^*\big) \mathbf{m}_S(a)[[a]] \times [[\mathbf{S}^3]] \ .$$

Proof. First to choose ε_1, note that there is a uniform bilipschitz equivalence between each half-cube $\mathbf{Q}_r^v(a)$ (from the proof of Theorem 5.3) and the ball $\mathbf{B}_r$. Thus by Lemma 2.2 there is a positive λ_0 so that

$$\inf\{\int_{\partial \mathbf{Q}_r^v(a)} |\nabla \psi|^3 \, d\mathcal{H}^3 \ : \ \psi \in W^{1,3}(\partial \mathbf{Q}_r^v(a), \mathbf{S}^2), \ \text{Hopf deg}\,(\psi) \neq 0\} \ > \ \lambda_0 \delta_0 \ .$$

Let

$$\varepsilon_1 \ = \ \frac{1}{6} \min\{\varepsilon_0, \ \lambda_0 \delta_0\} \ .$$

Next recall [G], Th.2.2, that, for the $W^{1,3}(\mathbf{R}^4)$ function $\tilde{u}$, the set of energy density points

$$X \ \equiv \ \{x \in \mathbf{R}^4 \ : \ \limsup_{r \to 0} \frac{1}{r} \int_{\mathbf{B}_r(x)} |\nabla \tilde{u}|^3 \, dy \ > \ 0\} \ .$$

has $\mathcal{H}^1(X) = 0$.

By Lemma 4.1 and Theorem 6.1, we know that for any fixed direction $v \in \mathbf{S}^3$, we have, for t off some measure zero subset Z_v of $\mathbf{R}$, that $\tilde{u}|h(v,t) \in W^{1,3}$ and that the sequence $G_{u_{n'}\#}h(v,t)$ is $d_{\mathbf{e}}$ convergent to

$$S\big(h(v,t)\big) \ = \mathbf{G}_{\tilde{u}|h(v,t)} \ + \sum_{a \in A(v,t)} \mathbf{m}_{v,t}(a)[[a]] \times [[\mathbf{S}^3]]$$

for some finite subset $A(v,t)$ of $h(v,t)$ and non-zero integers $\mathbf{m}_{v,t}$. We will next verify that, for all such $t \in \mathbf{R} \setminus Z_v$,

$$A(v,t) \setminus X \ \subset \ E_{\varepsilon_1} \ .$$

Assuming, for contradiction that $a \in A(v,t) \setminus (X \cup E_{\varepsilon_1})$, we choose a positive ρ sufficiently small so that $A(v,t) \cap \mathbf{B}_\rho(a) = \{a\}$ and

$$\int_{h(v,t) \cap \mathbf{B}_\rho(a)} |\nabla \tilde{u}|^3 \, d\mathcal{H}^3 \ + \ \rho^{-1} \int_{\mathbf{B}_\rho(a)} |\nabla \tilde{u}|^3 \, dy \ + \ \liminf_{n \to \infty} \rho^{-1} \int_{\mathbf{B}_\rho(a)} |\nabla u_n|^3 \, dy \ < \ 3\varepsilon_1 \ .$$

By Fatou's Lemma and Fubini's Theorem,

$$\begin{aligned} &\int_0^{\rho/2} \liminf_{n \to \infty} \int_{\partial \mathbf{Q}_r(a)} \big(|\nabla \tilde{u}|^3 + |\nabla u_n|^3\big) d\mathcal{H}^3 dr \\ &\leq \ \liminf_{n \to \infty} \int_{\mathbf{Q}_{\rho/2}(a)} \big(|\nabla \tilde{u}|^3 + |\nabla u_n|^3\big) \, dy \ < \ 3\varepsilon_1 \rho \end{aligned}$$

where, as before, $\mathbf{Q}_r(a) = \prod_{i=1}^4 (a_i - r, a_i + r) \subset \mathbf{B}_{2r}(a)$. Recalling now the proof of STEP II of Theorem 5.3, we see that we may pass to a subsequence and find $r \in [0, \rho/2]$ so that, for all n,

$$\int_{\partial \mathbf{Q}_r(a)} \big(|\nabla \tilde{u}|^3 + |\nabla u_n|^3\big) d\mathcal{H}^3 \ < \ 6\varepsilon_1 \ ,$$

and so that, for each of the 8 hyperplanes $h^i_{a\pm r}$ determined by the faces of $\mathbf{Q}_r(a)$, $\tilde{u}|h^i_{a\pm r} \in W^{1,3}$ and each sequence $G_{u_{n'}\#} h^i_{a\pm r}$ is locally $\mathbf{M}$ bounded and convergent as in Lemma 4.1. As in the proof of Theorem 5.3, r can also be chosen so that these convergences restrict to each face of the half-cube $\mathbf{Q}^v_r(a)$. However, now, by Lemma 4.2 and the last small energy estimate, *no bubbles* occur on any of the faces in the coordinate hyperplanes $h^i_{a\pm r}$, while, on the remaining face in $h(v,t)$, exactly one bubble occurs at a because $A(v,t) \cap \partial \mathbf{Q}^v_r(a) = \{a\}$. Also the small energy estimate

$$\int_{\partial \mathbf{Q}^v_r(a)} |\nabla u|^3 \, d\mathcal{H}^3 \ < \ 6\varepsilon_1$$

implies

$$\big(\mathbf{G}_{\tilde{u}|\partial \mathbf{Q}^v_r(a)}\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \ = \ 4\pi^2 \text{Hopf deg}\,\big(u|\partial \mathbf{Q}^v_r(a)\big) \ = \ 0 \ .$$

Summing over the faces in $\partial \mathbf{Q}^v_r(a)$ now gives the desired contradiction

$$\begin{aligned} 0 \ &= \ \lim_{n\to\infty} \partial\, 0 \ = \ \lim_{n\to\infty} \partial q_{\#} G_{\tilde{u}_{n'}\#}[[\mathbf{Q}^v_r(a)]]\big(\omega_{\mathbf{S}^3}\big) \\ &= \ \lim_{n\to\infty} G_{\tilde{u}_{n'}\#} \partial [[\mathbf{Q}^v_r(a)]]\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \ \pm 2\pi^2 \mathbf{m}_{v,t}(a) \ + \ \big(\mathbf{G}_{\tilde{u}|\partial \mathbf{Q}^v_r(a)}\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \\ &= \ \pm 2\pi^2 \mathbf{m}_{v,t}(a) \ + \ 0 \ , \end{aligned}$$

and establishes the inclusion $A(v,t) \setminus X \subset E_{\varepsilon_1}$.

By Lemma 7.1 and the Besicovitch Structure Theorem [F],3.3.13, the energy concentration set E_{ε_1} contains an $\mathcal{H}^1$ measurable 1 rectifiable set R so that the "unrectifiable visibility" directions

$$Y \ \equiv \ \{v \in \mathbf{S}^3 \ : \ \mathcal{H}^1\big(\pi_v(E_{\varepsilon_1} \setminus R)\big) > 0\}$$

have $\mathcal{H}^3$ measure 0. Also, as before, the "non-generically transverse" directions

$$V_R \ = \ \big\{v \in \mathbf{S}^3 \ : \ \mathcal{H}^1(\Sigma_{R,v}) \ > \ 0\big\}$$

are at most countable.

Ignoring these exceptional directions, we now *fix* one direction $v \in \mathbf{S}^3 \setminus (Y \cup V_R)$ and let

$$R_S \ \equiv \ \bigcup \big\{ \, A(v,t) \ : \ t \in \mathbf{R} \setminus \big[Z_v \cup \pi_v\big(X \cup (E_{\varepsilon_1} \setminus R) \cup \Sigma_{R,v}\big)\big] \, \big\} \ .$$

and, for $a \in R_S$, let

$$\mathbf{m}_S(a) \equiv |\mathbf{m}_{v,v\cdot a}(a)| \ ,$$

and $\vec{S}(a)$ be the unit vector orienting the approximate tangent line of R at a with $\mathrm{sgn}\,\big(\vec{S}(a)\cdot v\big) = \mathrm{sgn}\ \mathbf{m}_{v,v\cdot a}(a)$. These definitions automatically give the desired formula for $S(h)$ in case $h = h(v,t)$ for almost all $t \in \mathbf{R}$.

We will now show that the formula continues to hold for hyperplanes in almost all other directions; i.e. that these definitions are, up to an $\mathcal{H}^1$ null set, independent of our choice of $v \in \mathbf{S}^3 \setminus (Y \cup V_R)$. For this, we now fix a second direction $v' \in \mathbf{S}^3 \setminus (Y \cup V_R)$. Since $R_S \subset R$ and $\mathcal{H}^1(\Sigma_{R,v'}) = 0$, the area formula applied to $\pi_{v'}|R$ shows that

$$R_S \cap \pi_{v'}^{-1}\big[\, Z_{v'} \cup \pi_{v'}\big(X \cup (E_{\varepsilon_1} \setminus R) \cup \Sigma_{R,v'}\big)\big]$$

has $\mathcal{H}^1$ measure zero. It now suffices for us to verify, at each of the remaining points $a \in R_S \setminus \pi_{v'}^{-1}\big[\, Z_{v'} \cup\ p_{v'}\big(X \cup (E_{\varepsilon_1} \setminus R) \cup \Sigma_{R,v'}\big)\big]$, that $a \in A(v', v'\cdot a)$ and that the integer multiplicity $\mathbf{m}_{v',v'\cdot a}(a)$ is correct, that is,

$$|\mathbf{m}_{v',v'\cdot a}(a)| \ = \ |\mathbf{m}_S(a)| \ \text{ and } \ \mathrm{sgn}\ \mathbf{m}_{v',v'\cdot a}(a) = \ \mathrm{sgn}\,\big(\vec{S}(a)\cdot v'\big)\ .$$

We will argue as before by considering the behavior on the boundary of small half-cubes

$$\mathbf{Q}_r^v(a) \ \equiv \ \{x \in \mathbf{Q}_r(a)\ :\ (x-a)\cdot v > 0\}\ , \quad \mathbf{Q}_r^{v'}(a) \ \equiv \ \{x \in \mathbf{Q}_r(a)\ :\ (x-a)\cdot v' > 0\}\ ,$$

To use this same notation, we assume (as can be achieved by rotating coordinates) that none of the four coordinate directions $(1,0,0,0), ..., (0,0,0,1)$ are in the exceptional directions $Y \cup V_R$. We also suppose first, for convenience that

$$\mathrm{sgn}\,\big(\vec{S}(a)\cdot v\big)\ > 0\ \ ,\ \ \mathrm{sgn}\,\big(\vec{S}(a)\cdot v'\big)\ > 0\ .$$

Then the (affine) approximate tangent line $\{a + t\vec{S}(a)\ :\ t \in \mathbf{R}\}$ of R intersects the closed conical region

$$\begin{aligned} C \ &\equiv \ \{x\ :\big(x-a)\cdot v \le 0,\ (x-a)\cdot v' \ge 0\} \ \cup\ \{x\ :\ (x-a)\cdot v \ge 0,\ (x-a)\cdot v' \le 0\} \\ &= \ \bigcup_{r>0}\big[\,\overline{\mathbf{Q}_r^v(a)} \setminus \mathbf{Q}_r^{v'}(a)\big] \cup \big[\,\overline{\mathbf{Q}_r^{v'}(a)} \setminus \mathbf{Q}_r^{v}(a)\big] \end{aligned}$$

only at a and

$$\limsup_{r\to 0} r^{-1}\mathcal{H}^1\big(R \cap C \cap \mathbf{B}_r(a)\big) \ = \ 0\ ,$$

so that, by Fubini's Theorem,

$$J \ \equiv \ \{r > 0\ :\ R \cap C \cap \partial\mathbf{Q}_r(a) \ \neq \ \emptyset\},$$

has density 0 at 0. Also, since both restrictions $\tilde{u}|h(v, v\cdot a)$ and $\tilde{u}|h(v', v'\cdot a)$ are $W^{1,3}$ and $a \notin X$, we may argue as before to choose a positive $r \notin J$ so that

$$\mathbf{B}_r(a) \cap A(v, v\cdot a) \ = \ \{a\}\ , \quad \mathbf{B}_r(a) \cap A(v', v'\cdot a) \ \subset \ \{a\}\ ,$$

$$\int_{h(v,v\cdot a)\cap \mathbf{B}_r(a)} |\nabla \tilde{u}|^3 \, d\mathcal{H}^3 \ + \ \int_{h(v',v'\cdot a)\cap \mathbf{B}_r(a)} |\nabla \tilde{u}|^3 \, d\mathcal{H}^3 \ + \ \int_{\partial \mathbf{Q}_r(a)} |\nabla \tilde{u}|^3 \, d\mathcal{H}^3 \ < \varepsilon_1\ ,$$

$$a_1 \pm r \notin Z_{(1,0,0,0)} \cup \pi_{(1,0,0,0)}\big(X \cup (E_{\varepsilon_1} \setminus R) \cup \Sigma_{R,(1,0,0,0)}\big)\ ,\ \dots\ ,$$
$$a_4 \pm r \notin Z_{(0,0,0,1)} \cup \pi_{(0,0,0,1)}\big(X \cup (E_{\varepsilon_1} \setminus R) \cup \Sigma_{R,(0,0,0,1)}\big)\ .$$

and so that, for a subsequence, there the weak convergence of currents

$$\lim_{n\to\infty} G_{\tilde{u}_{n'}\#}\partial[[\mathbf{Q}_r^v(a)]] \ = \ \mathbf{G}_{\tilde{u}|\partial\mathbf{Q}_r^v(a)} \ + \ \sum_{b\in B} m(b)[[b]] \times [[\mathbf{S}^3]] \tag{7.1}$$

$$\lim_{n\to\infty} G_{\tilde{u}_{n'}\#}\partial[[\mathbf{Q}_r^{v'}(a)]] \ = \ \mathbf{G}_{\tilde{u}|\partial\mathbf{Q}_r^{v'}(a)} \ + \ \sum_{b\in B'} m'(b)[[b]] \times [[\mathbf{S}^3]] \tag{7.2}$$

for some finite subsets B of $\partial\mathbf{Q}_r^v(a)$, B' of $\partial\mathbf{Q}_r^{v'}(a)$, and integer multiplicities m, m'. Then

$$B \cap h(v, v\cdot a) \ = \ \{a\}\ , \ m(a) \ = \ \mathbf{m}_{v,v\cdot a}(a)\ ,$$
$$B' \cap h(v', v'\cdot a) \ \subset \ \{a\}\ , \ m'(a) \ = \ \mathbf{m}_{v',v'\cdot a}(a) \text{ in case } \ a \in A(v', v'\cdot a)\ .$$

For the remaining faces, we infer from Lemma 4.2 and our choice of r that

$$\begin{aligned}(B \cup B') \setminus \{a\} \ &\subset \big(R \cap \partial\mathbf{Q}_r^v(a) \setminus h(v, v\cdot a)\big) \cup \big(R \cap \partial\mathbf{Q}_r^{v'}(a) \setminus h(v', v'\cdot a)\big) \\ &\subset \ \big(\partial\mathbf{Q}_r^v(a) \setminus h(v, v\cdot a)\big) \cup \big(\partial\mathbf{Q}_r^{v'}(a) \setminus h(v', v'\cdot a)\big) \ \setminus \ C \\ &= \ \partial\big(\mathbf{Q}_r^v(a) \cap \mathbf{Q}_r^{v'}(a)\big) \setminus \big(h(v, v\cdot a) \cup h(v', v'\cdot a)\big).\end{aligned}$$

So we may restrict our mappings to the latter set and pass to the limit to conclude that

$$B \setminus \{a\} \ = \ B' \setminus \{a\} \text{ and } m(b) \ = \ m'(b) \text{ for } b \in B \setminus \{a\}\ .$$

As before, the smoothness of the $u_{n'}$ gives

$$G_{\tilde{u}_{n'}\#}\partial[[\mathbf{Q}_r^v(a)]]\big(q^\#\omega_{\mathbf{S}^3}\big) \ = \ \partial G_{\tilde{u}_{n'}\#}[[\mathbf{Q}_r^v(a)]]\big(q^\#\omega_{\mathbf{S}^3}\big) \ = \ 0\ ,$$
$$G_{\tilde{u}_{n'}\#}\partial[[\mathbf{Q}_r^{v'}(a)]]\big(q^\#\omega_{\mathbf{S}^3}\big) \ = \ \partial G_{\tilde{u}_{n'}\#}[[\mathbf{Q}_r^{v'}(a)]]\big(q^\#\omega_{\mathbf{S}^3}\big) \ = \ 0\ ,$$

while the small energy condition

$$\int_{\partial\mathbf{Q}_r^v(a)} |\nabla u|^3 \, d\mathcal{H}^3 \ + \ \int_{\partial\mathbf{Q}_r^{v'}(a)} |\nabla u|^3 \, d\mathcal{H}^3 \ < \ 2\varepsilon_1$$

implies

$$\big(\mathbf{G}_{\tilde{u}|\partial\mathbf{Q}_r^v(a)}\big)\big(q^\#\omega_{\mathbf{S}^3}\big) \ = 0 \ = \ \big(\mathbf{G}_{\tilde{u}|\partial\mathbf{Q}_r^{v'}(a)}\big)\big(q^\#\omega_{\mathbf{S}^3}\big)\ .$$

Plugging the form $q^{\#}\omega_{\mathbf{S}^3}$ into (7.1) and (7.2) now gives

$$\begin{aligned} 0 &= 0 + 2\pi^2 m(a) + \sum_{b\in B\setminus\{a\}} 2\pi^2 m(b) \\ &= 0 + 2\pi^2 m(a) + \sum_{b\in B'\setminus\{a\}} 2\pi^2 m'(b) \\ &= 0 + \sum_{b\in B'} 2\pi^2 m'(b) \ , \end{aligned}$$

and we conclude that $\mathbf{B}_r(a)\cap A(v',v'\cdot a)=\{a\}$ and $\mathbf{m}_{v',v'\cdot a}=m'(a)\ =m(a)$, hence,

$$\begin{aligned} |\mathbf{m}_{v',v'\cdot a}| &= |m(a)| = |\mathbf{m}_S(a)| \ , \\ \operatorname{sgn}\, \mathbf{m}_{v',v'\cdot a}(a) &= \operatorname{sgn}\, m(a) = \operatorname{sgn}\, \mathbf{m}_{v,v\cdot a}(a) = \operatorname{sgn}\big(\vec{S}(a)\cdot v\big) = \operatorname{sgn}\big(\vec{S}(a)\cdot v'\big) \ . \end{aligned}$$

The remaining three cases where

$$\big(\operatorname{sgn}(\vec{S}(a)\cdot v)\ ,\ \operatorname{sgn}(\vec{S}(a)\cdot v')\big) = (-1,+1),\ (+1,-1),\ (-1,-1)\ ,$$

may be treated similarly, and we now have the desired representation formula for $S\big(h(v',t)\big)$ for a.e. $t\in\mathbf{R}$.

The set R_S, being a subset of the 1 rectifiable set R is itself 1 rectifiable. To see that it is also $\mathcal{H}^1$ measurable, it suffices to show that

$$\lim_{r\to 0} r^{-1}\mathcal{H}^1\big(\mathbf{B}_r(a)\cap R_S\big) = 1 \ \text{ for a.e. } a\in R_S \ ,$$

$$\lim_{r\to 0} r^{-1}\mathcal{H}^1\big(\mathbf{B}_r(a)\setminus R_S\big) = 0 \ \text{ for a.e. } a\in R\setminus R_S \ .$$

Both of these may be verified by arguing as before taking $v\in\mathbf{S}^3\setminus(Y\cup V_R)$ and forming the limits of a subsequence of graphs restricted to a half-cube boundary $\partial\mathbf{Q}_r^v(a)$ for a generic point $a\in R\setminus X$ and generic small positive r. Here R is $\mathcal{H}^1$ almost contained in a countable union of $\mathcal{C}^1$ arcs Γ_i, and one may insist that a be a point of density 1 for precisely one Γ_i and be of density 0 for all the others. So as before, r is chosen so that $\partial\mathbf{Q}_r^v(a)$ hits Γ_i transversally at a and at preciesly one other point $a_r\in\partial\mathbf{Q}_r^v(a)$. The limiting current equation guarantees that this point $a_r\in R_S$ if and only if $a\in R_S$ and in which case that $\mathbf{m}_S(a_r)=\mathbf{m}_S(a)$ and that $\vec{S}(a_r)$ and $\vec{S}(a)$ give the same orientation to Γ_i. The set of suitable positive r for both v and $-v$ has density 1 at 0. For the point a, we conclude not only the above density statements, but also the approximate continuity of both $\mathbf{m}_S$ and $\vec{S}$. Thus we verify the $\mathcal{H}^1$ measurabliity of the set R_S as well as the $\mathcal{H}^1 \llcorner R_S$ measurabliity of the functions $\mathbf{m}_S$ and $\vec{S}$.

To establish the multiplicity estimate, we will prove, for a.e. $h\in H$, the bound

$$\sum_{a\in R_S\cap h} \mathbf{m}_S^{3/4}(a) \le C\liminf_{n\to\infty}\int_h |\nabla u_{n'}|^3\, d\mathcal{H}^3 \ . \tag{7.3}$$

This will do it because we may then choose a frame $\{v_1, v_2, v_3, v_4\} \subset \mathbf{S}^3 \setminus (Y \cup V_R)$ and apply the coarea formula to each $p_{v_i}|R_S$ as well as Fatou's Lemma and Fubini's Theorem to deduce that

$$\int_{R_S} \mathbf{m}_S^{3/4} d\mathcal{H}^1 \;\leq\; \sum_{i=1}^{4} \int_{R_S} \mathbf{m}_S^{3/4} |v_i \cdot \vec{S}|\, d\mathcal{H}^1 \;=\; \sum_{i=1}^{4} \int_{-\infty}^{\infty} \sum_{a \in R_S \cap h(v_i,t)} \mathbf{m}_S^{3/4}(a)\, dt$$
$$\leq\; C \sum_{i=1}^{4} \int_{-\infty}^{\infty} \liminf_{n \to \infty} \int_{h(v_i,t)} |\nabla u_{n'}|^3\, d\mathcal{H}^1\, dt \;=\; 4C \liminf_{n \to \infty} \int_{\mathbf{R}^4} |\nabla u_{n'}|^3\, dx \;<\; \infty\ .$$

To verify (7.3) (by contradiction) we may pass to subsequences depending on h, without changing notations. In particular, we may by Fatou's Lemma, assume that $\sup_{n'} \int_h |\nabla u_{n'}|^3\, d\mathcal{H}^3 < \infty$ and that the graphs $G_{\tilde{u}_{n'}\#}$ converge to $S(h)$, as in Lemma 4.1, weakly as currents. For each point $a \in R_S \cap h$,

$$\lim_{r \downarrow 0} \int_{\mathbf{B}_r(a) \cap h} |\nabla u|^3\, d\mathcal{H}^3 \;=\; 0\ ,$$

and we have, for a.e. $r > 0$, strong $W^{1,3}$ convergence of a subsequence on the 2 dimensional sphere $h \cap \partial \mathbf{B}_r(a)$. So there is, for any $\epsilon > 0$, a small $r > 0$ so that $\mathbf{B}_{2r}(a) \cap R_S \cap h = \{a\}$ and so that (for a subsequence) $\sup_{n'} \int_{h \cap \partial \mathbf{B}_r(a)} |\nabla u_{n'}|^3\, d\mathcal{H}^2 < \epsilon$. Then we may obtain extensions $\psi_{n'} : h \to \mathbf{S}^2$ of $u_{n'}|\big(h \cap \mathbf{B}_r(a)\big)$ so that $\psi_{n'}$ is a constant $y_{n'}$ on $h \setminus \mathbf{B}_{2r}(a)$ and $\sup_{n'} \int_{h \cap \left(\mathbf{B}_{2r}(a) \setminus \mathbf{B}_r(a)\right)} |\nabla \psi_{n'}|^3\, d\mathcal{H}^3 < c\epsilon$. It follows that the induced relative map

$$\psi_{n'} : \big(h \cap \overline{\mathbf{B}_{2r}(a)}, h \cap \partial \mathbf{B}_{2r}(a)\big) \;\to\; \big(\mathbf{S}^2, \{y_{n'}\}\big)$$

has a well-defined Hopf degree, which must, for n' sufficiently large, be the multiplicity $\mathbf{m}_S(a)$, by the convergence of $G_{\tilde{u}_{n'}\#}$ and the formula for $S(h)$. From the conformal invariance of the 3 energy in 3 dimensions and the lower bound of [R1], we now conclude that

$$\int_{h \cap \mathbf{B}_r(a)} |\nabla u_{n'}|^3\, dx \;+\; c\epsilon \geq \int_{h \cap \mathbf{B}_{2r}(a)} |\nabla \psi_{n'}|^3\, dx \;\geq\; C^{-1} \mathbf{m}_S(a)^{3/4}\ .$$

Summing over $a \in R_S \cap h$ and letting $n' \to \infty$ and $\epsilon \to 0$ now gives (7.3) and completes the proof. ∎

Remark 7.3. The vanishing of the second term in 7.2 (i.e. no "bubbling") does *not* guarantee that the convergence is strong in $W^{1,3}$. In fact, it is easy to make a construction as in §2.4 of a smooth map $u_n : \mathbf{R}^4 \to \mathbf{S}^2$ which is constant $(0,0,0,1)$ outside the $\frac{1}{4n}$ tubular neighborhoods of the two parallel intervals $[(0,0,0,0),(1,0,0,0)]$, $[(0,\frac{1}{n},0,0),(1,\frac{1}{n},0,0)]$ and which has Hopf degree $+1$ and -1 on the slices of these 2 tubular neighborhoods by the hyperplane $x_2 = t$ for $\frac{1}{n} \leq t \leq 1 - \frac{1}{n}$. One can insist that

$$\big|2\,\delta_0 \;-\; \int_{\mathbf{R}^4} |\nabla u_n|^3\, dx\,\big| \;\leq\; \frac{1}{n}\ .$$

The weak $W^{1,3}$ limit is a constant map, but the energy drop is not detected by the limiting scan, which is the constant $[[\mathbf{R}^4]] \times [[(0,0,0,1)]]$ by cancellation. Note that here one may have the energy concentration on an interval, with the convergence of the positive measures,

$$\lim_{n\to\infty} |\nabla u_n|^3 \, dx \;\to\; 2\cdot 2\pi^2 \mathcal{H}^1 \llcorner [(0,0,0,0),(1,0,0,0)] \,,$$

even though the "topological" part of this concentration vanishes.

§8. Connecting the Singularities of a General Finite Energy Map.

It is still unknown whether an arbitrary map $u \in W^{1,3}(\mathbf{R}^4, \mathbf{S}^2)$ is a *weak* $W^{1,3}$ sequential limit of smooth maps in $\mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^2)$ although some other cases of such weak density have been established [Be1], [PR], [HgL]. In this section, we verify that we can still (as in the conclusion of Theorems 6.1) cap off the scan boundary of the graph of a Coulomb lift of u by addition of an oriented vertical scan.

As motivation for our proof, consider a corresponding construction [GMS2],2.5, for the simpler case of a map $u \in W^{1,2}(\mathbf{R}^3, \mathbf{S}^2)$. Here one can first approximate u strongly in $W^{1,2}$ by maps $u_n \in \mathcal{C}^\infty(\mathbf{R}^3 \setminus A_n) \cap W^{1,2}$ with A_n a finite set. Then as currrents

$$\partial \mathbf{G}_{u_n} \;=\; \partial\big(I_n \times [[\mathbf{S}^2]]\big)$$

where I_n is a "minimal connection", a finite sum of oriented intervals whose boundary gives the points of A_n with multiplicities determined by the local degrees of u_n. Then local mass bounds for $\mathbf{G}_{u_n}$ and $I_n \times [[\mathbf{S}^2]]$ along with the Federer-Fleming Compactness Theorem provides subconvergence of the augmented graphs $\mathbf{G}_{u_n} - I_n \times [[\mathbf{S}^2]]$ to an integer-multiplicity rectifiable current in the form $\mathbf{G}_u - I \times [[\mathbf{S}^2]]$, in particular, $\partial\big(\mathbf{G}_u - I \times [[\mathbf{S}^2]]\big) = 0$.

In our situation, we need to work with the scan cycles of augmented graphs of Coulomb lifts, replacing mass bounds by $L^{\frac{3}{4}}$ slice mass bounds and the Federer-Fleming Compactness Theorem by a version of §6 upgraded to handle such augmented graphs.

Theorem 8.1. *For any map $u \in W^{1,3}(\mathbf{R}^4, \mathbf{S}^2)$ with u constant outside of $\mathbf{B}_2$ and Coulomb lift $\tilde{u} : \mathbf{R}^4 \to \mathbf{S}^3$,*

$$\partial\big(\mathbf{G}_{\tilde{u}} \;-\; T\big) \;=\; 0$$

for some scan T such that, for all $v \in \mathbf{S}^3$ and a.e. $t \in \mathbf{R}$,

$$T\big(h(v,t)\big)\big) \;=\; \sum_{a \in A_{v,t}} \mathbf{m}_{v,t} [[a]] \times [[\mathbf{S}^3]] \;,$$

for some finite subset $A_{v,t}$ of $h(v,t)$ and non-zero integers $\mathbf{m}_{v,t}$ with

$$\int_{\mathbf{R}} \big(\sum_{a \in A_{v,t}} \mathbf{m}_{v,t}\big)^{3/4} dt \le C\big(1 + \int |\nabla u|^3 \, dx\big) \,.$$

Proof. We first recall from [Be1] that there is, for each positive integer n, a finite subset A_n of $\mathbf{B}_1$ and a map $u_n \in \mathcal{C}^\infty(\mathbf{R}^4 \setminus A_n) \cap W^{1,3}$ with $u_n \equiv u$ outside of $\mathbf{B}_2$ and

$$\lim_{n\to\infty} \int |\nabla u_n - \nabla u|^3 \, dx \ = \ 0 \ .$$

In particular, by passing to a subsequence,

$$L \equiv \sup_n \int |\nabla u_n|^3 \, dx \ \leq \ 2 \int |\nabla u|^3 \, dx \ < \ \infty \ .$$

We may also insist that near each point $a \in A_n$, u_n is homogeneous with Hopf degree on small spheres about a being ± 1. Consider now the Coulomb guage $\tilde{\eta}_n \in \mathcal{E}^1(\mathbf{R}^4 \setminus A)$ of u_n defined by the formula of §2. By the simple-connectivity of $\mathbf{R}^4 \setminus A$, this guage provides, as in §2, a corresponding Coulomb lift $\tilde{u}_n \in \mathcal{C}^\infty(\mathbf{R}^4 \setminus A, \mathbf{S}^3) \cap W^{1,3}$ with

$$\partial \mathbf{G}_{\tilde{u}_n} \ = \ \sum_{a \in A_n} \mathbf{m}_n(a)[[a]] \times [[\mathbf{S}^3]]$$

where $\mathbf{m}_n(a) = \deg\big(\tilde{u}_n | \partial \mathbf{B}_\epsilon(a)\big) = \mathrm{Hopf}\deg\big(u_n | \partial \mathbf{B}_\epsilon(a)\big)$ for all sufficiently small $\epsilon > 0$. As before, neither $\int |\nabla \tilde{u}_n|^3 \, dx$ nor $\mathbf{M}\big(\mathbf{G}_{\tilde{u}_n}\big)$ is necessarily uniformly bounded independent of n, but one still has all the integral estimates of Lemma 2.2 with u, $\tilde{u}$ replaced by u_n, $\tilde{u}_n$. With these, one verifies that $\tilde{u}_n$ converges weakly in $W^{1,\frac{5}{12}}$ (also strongly in $L^{1,\frac{5}{12}}$ and pointwise a.e.) to a Coulomb lift $\tilde{u}$ of u.

While we cannot, by Example 2.5, hope to find a uniformly mass-bounded minimal connection to cap off $\partial \mathbf{G}_{\tilde{u}_n}$, we can use a suitable level curve $\tilde{u}_n^{-1}\{y\}$.

To choose y, first note that the set Σ_n of critical values of u_n has $\mathcal{H}^3(\Sigma_n) = 0$. By Jensen's inequality, the coarea formula, and the above estimate, we also find that, for each $v \in \mathbf{S}^3$,

$$\begin{aligned}
\int_{\mathbf{S}^2} \int_{\mathbf{R}} \operatorname{card}\big(\tilde{u}_n^{-1}\{y\} \cap \pi_v^{-1}\{t\}\big)^{\frac{3}{4}} \, dt \, d\mathcal{H}^3 y \ &\leq \ \int_{\mathbf{R}} \Big(\int_{\mathbf{S}^2} \operatorname{card}\big(\tilde{u}_n^{-1}\{y\} \cap \pi_v^{-1}\{t\} \, d\mathcal{H}^2 y\big)^{\frac{3}{4}} \, dt \\
&\leq \ \int_{\mathbf{R}} \Big(\int_{\pi_v^{-1}\{t\}} |\tilde{u}_n^{\#} \omega_{\mathbf{S}^3}| \Big)^{\frac{3}{4}} \, dt \\
&= \ c \int_{\mathbf{R}} \Big(\int_{\pi_v^{-1}\{t\}} |\tilde{\eta}_n \wedge d\tilde{\eta}_n| \Big)^{\frac{3}{4}} \, dt \\
&\leq \ c \int_{\mathbf{R}} \big(\|\nabla u_n\|^4_{L^3(\pi_v^{-1}\{t\})} \big)^{\frac{3}{4}} \, dt \\
&= \ c \int_{\mathbf{B}^4} |\nabla u_n|^3 \, dx \ \leq \ c\,L \ .
\end{aligned}$$

In particular, integrating over $v \in \mathbf{S}^3$, we see from Fatou's Lemma and Fubini's Theorem that $\mathcal{H}^3(\Sigma) = 0$ where $\Sigma = \{y \in \mathbf{S}^3 \ : \ \mathcal{H}^3(W_y) > 0\}$ with

$$W_y \ = \ \{v \in \mathbf{S}^3 \ : \ \int_{\mathbf{R}} \liminf_{n\to\infty} \operatorname{card}\big(\tilde{u}_n^{-1}\{y\} \cap \pi_v^{-1}\{t\}\big)^{\frac{3}{4}} \, dt \ = \ \infty\} \ .$$

Finally, to get a uniform bound for $d_{\mathbf{e}}$ estimates, we can now, fix a single point $y \in \mathbf{S}^3 \setminus (\Sigma \cup \cup_{n=1}^{\infty} \Sigma_n)$ and pass to a subsequence so that

$$\sup_n \int_{\mathbf{R}} \mathrm{card}\left(\tilde{u}_n^{-1}\{y\} \cap \pi_{e_j}^{-1}\{t\}\right)^{\frac{3}{4}} dt \ \leq \ c\,L$$

for each $j \in \{1,2,3,4\}$.

Since $y \in \mathbf{S}^3$ is a regular value for each $\tilde{u}_n$, the set

$$R_n \ \equiv \ \tilde{u}_n^{-1}\{y\}$$

is a union of smooth curves that transversely intersect almost all of the hyperplanes $\pi_{e_j}^{-1}\{t\}$. The set R_n is readily oriented to become a 1 dimensional rectifiable current I_n with $\partial I_n = \sum_{a \in A_n} \mathbf{m}_n(a)[[a]]$ so that

$$\partial\big(I_n \times [[\mathbf{S}^3]]\big) \ = \ \partial G_{\tilde{u}_n\#}[[\mathbf{R}^4]] \ .$$

Then, in the language of scans,

$$\partial\big(G_{\tilde{u}_n\#} \ - \ T_n\big) \ = \ 0$$

where T_n is the corresponding vertical scan defined by $T_n(h) = \big(I_n \cap h\big) \times [[\mathbf{S}^3]]$ for a.e. $h \in H$.

We now discuss how to carry over essentially all the results of §5-§6 replacing the former sequence of scan cycles of smooth functions $G_{\tilde{u}_n\#}$ by the present sequence of *augmented scans* $G_{\tilde{u}_n\#} - T_n$. Since, for a.e. $t \in \mathbf{R}$,

$$\mathbf{M}\big(T_n[h(e_j,t)]\big) = 2\pi^2 \mathrm{card}\left(R_n \cap \pi_{e_j}^{-1}\{t\}\right)$$

we still have, from the new Lemma 2.2 estimate and the choice of y, the basic global bound

$$\sum_{j=1}^{4} \int \big[\mathbf{M}(G_{\tilde{u}_n\#} - T_n)\big(h(e_j,\tau) \llcorner \mathbf{B}_2\big)\,\big]^{\frac{3}{4}} d\tau \ \leq \ 4C_{\overline{\mathbf{B}}_2}(1 \ + \ L) \ + \ 32\pi^2 c\,L.$$

We need a new version of §5 in which the scan $G_{w\#}$ of a smooth map $w \in \mathcal{C}^\infty(\mathbf{R}^4, \mathbf{S}^3)$ is replaced by an augmented scan $G_{w\#} - T$ corresponding to a map $w \in \mathcal{C}^\infty(\mathbf{R}^4 \setminus A, \mathbf{S}^3) \cap W^{1,3}$ with A finite and a rectifiable vertical current T such that $\partial T = \partial G_{w\#}[[\mathbf{R}^4]]$. In the proof of Theorem 5.1, one now uses, for almost all $s, t \in \mathbf{R}$, the rectifiable current

$$T_{s,t} \ = \ \big(G_{w\#}[[\mathbf{R}^4]] - T\big) \llcorner p_v^{-1}\big(\overline{st}\big)$$

to connect $\big(G_{w\#} - T\big)h(s,v)$ to $\big(G_{w\#} - T\big)h(t,v)$. The remainder of the proof carries over to give a corresponding fractional maximal estimate for $G_{w\#}[[\mathbf{R}^4]] - T$.

In STEP I of the proof of the new version of Theorem 6.1, we now just consider directions $v \in \mathbf{S}^3$ outside the measure zero set W_y. In the application of the appendix Theorem 9.1, we simply take $f_n(t) = \big(G_{\tilde{u}_n\#} - T_n\big)h(t,v)$ and keep everything else the same. In STEP II, one considers, for a fixed $v \in \mathbf{S}^3 \setminus W_y$, subsequences of $\big(G_{\tilde{u}_n\#} - T_n\big)h(t,v)$ for almost all $t \in \mathbf{R}$. In addition to the measure zero sets Z, $\tilde{Z}$ defined as before, one also now avoids the set

$$\hat{Z} \;\equiv\; \pi_v(A_n) \cup \{t \in \mathbf{R} \;:\; \liminf_{n\to\infty} \operatorname{card}\big(R_n \cap \pi_v^{-1}\{t\}\big) = \infty\}\ ,$$

which has measure 0 by Fatou's Lemma and the fact that $v \notin W_y$. Then, for $t \in \mathbf{R} \setminus (Z \cup \tilde{Z} \cup \hat{Z})$, a $d_{\mathbf{e}}$ limit $P(t)$ of a subsequence of $\big(G_{\tilde{u}_n\#} - T_n\big)h(t,v)$ again has the form

$$P(t) \;=\; \mathbf{G}_{\tilde{u}|h(v,t)} \;+\; \sum_{a \in h(v,t)} \mathbf{m}_{v,t}(a)\big([[a]] \times [[\mathbf{S}^3]]\big)$$

because $P(t)$ is, by Fatou's Lemma, a weak limit of some uniformly mass bounded subsequence

$$G_{\tilde{u}_{n''}\#}h(t,v) \;-\; \big(I_{n''} \cap h(t,v)\big) \times [[\mathbf{S}^3]]$$

with $\tilde{u}_{n''}|h(t,v)$ smooth and uniformly 3-energy bounded and $I_{n''} \cap h(t,v)$ uniformly mass bounded. We also have the estimate

$$\int_{\mathbf{R}} \big(\sum_{a \in A_{v,t}} \mathbf{m}_{v,t}\big)^{3/4} dt \le C\big(1 + \int |\nabla u|^3\, dx\big)\ .$$

We again apply STEP I to obtain a subsequence $d_{\mathbf{e}}$ convergent at almost every coordinate hyperplane. Then one verifies the automatic convergence of this same subsequence on hyperplanes $h(v,t)$ for any other direction $v \in \mathbf{S}^3 \setminus W_y$ and a.e. $t \in \mathbf{R}$ by considering currents lying over the boundary of a small half-cube $\mathbf{Q}_r^v(a)$. For the total boundary, one now uses the relation

$$\begin{aligned}
&\big(G_{\tilde{u}_n\#}\partial[[\mathbf{Q}_r^v(a)]] \;+\; (I_n \cap \partial[[\mathbf{Q}_r^v(a)]]) \times [[\mathbf{S}^3]]\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big)\\
&= \; \partial q_{\#}\big(G_{\tilde{u}_n\#}[[\mathbf{Q}_r^v(a)]] \;-\; (I_n \cap [[\mathbf{Q}_r^v(a)]]) \times [[\mathbf{S}^3]]\big)\big(\omega_{\mathbf{S}^3}\big)\\
&= \; \partial\, 0\big(\omega_{\mathbf{S}^3}\big) \;=\; 0\ ,
\end{aligned}$$

which leads to

$$\begin{aligned}
&2\pi^2\mathbf{m}''(a) \;+\; \sigma\big(\mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\partial\mathbf{Q}_r^v(a)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;+\; P_r(a)\big(q^{\#}\omega_{\mathbf{S}^3}\big)\\
&= \; \lim_{n\to\infty} \big(G_{\tilde{u}_{n''}\#}\partial[[\mathbf{Q}_r^v(a)]] \;+\; (I_{n''} \cap \partial[[\mathbf{Q}_r^v(a)]]) \times [[\mathbf{S}^3]]\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big)\\
&= \; \lim_{n\to\infty} 0\\
&= \; \lim_{n\to\infty} \big(G_{\tilde{u}_{n'''}\#}\partial[[\mathbf{Q}_r^v(a)]] \;+\; (I_{n'''} \cap \partial[[\mathbf{Q}_r^v(a)]]) \times [[\mathbf{S}^3]]\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big)\\
&= \; 2\pi^2\mathbf{m}'''(a) \;+\; \sigma\big(\mathbf{G}_{\tilde{u}|h(v,t)} \llcorner p^{-1}\partial\mathbf{Q}_r^v(a)\big)\big(q^{\#}\omega_{\mathbf{S}^3}\big) \;+\; P_r(a)\big(q^{\#}\omega_{\mathbf{S}^3}\big),
\end{aligned}$$

so that again $\mathbf{m}''(a) = \mathbf{m}'''(a)$. Thus we have $d_{\mathbf{e}}$ convergence of a subsequence of $G_{\tilde{u}_{n'}\#}$ at a.e. $h \in H$ to a limiting scan $S = G_{\tilde{u}_{n'}\#} - T$ with T of the required form, and we again verify that S is a scan cycle. ■

Remark 8.2. It seems unlikely that the scan T obtained in the proof of §8.1 is 1 rectifiable in the sense of Theorem 7.2. One is tempted to replace the energy concentration set used in Lemma 7.1 by the *fractional mass slice concentration set*

$$F_\epsilon \;=\; \{x \in \mathbf{R}^4 \;:\; \lim_{r\to 0} \liminf_{n\to\infty} \frac{1}{r}\nu_n\big(\mathbf{B}_r(x)\big) > \epsilon\}$$

defined with the super-additive function

$$\nu_n(U) \;=\; \sum_{j=1}^{4} \Big(\int \operatorname{card}\big(U \cap R_n \cap \pi_{e_j}^{-1}\{t\}\big)^{\frac{3}{4}}\, dt\Big)^{\frac{4}{3}} \text{ for open } U \subset \mathbf{R}^4 \;.$$

However the new Lemma 4.1 then only gives an estimate of $\mathcal{H}^{4/3}(F_\epsilon)$ rather than of $\mathcal{H}^1(F_\epsilon)$. Following some of the proof of Theorem 7.2, one then only obtains an $\mathcal{H}^{4/3}$ measure estimate for the carrying set of the scan T. In future work we hope to consider a more efficient scan connection for the topological singularities.

§9. Appendix. Compactness from a Fractional Maximal Function Bound.

Let $0 < \alpha < 1$, X be a closed interval, Y be a metric space, and $\mathcal{N}$ is a nonnegative, lower semi-continuous function on Y such that $\{y \in Y \;:\; \mathcal{N}(y) \le R\}$ is sequentially compact for all $R > 0$. For any measurable $f : X \to Y$ and subinterval I of X, let $\mathcal{M}_I f$ denote the associated α-maximal function,

$$(\mathcal{M}_I f)(x) \;\equiv\; \operatorname{esssup}_{x\ne\tilde{x}\in I} \frac{\operatorname{dist}\big(f(x), f(\tilde{x})\big)}{|x-\tilde{x}|^\alpha}$$

for $x \in I$.

Theorem 9.1. *Suppose that for each $n = 1, 2, \ldots$, $f_n : X \to Y$ is a measurable map satisfying on each subinterval I of X, a (weak-type) measure estimate*

$$\sup_{\lambda>0} \lambda \,\big|\, \{x \in I \;:\; (\mathcal{M}_I f_n)^{1/\alpha}(x) > \lambda\} \,\big| \;\le\; \mu_n(I) \;, \tag{9.1}$$

for some nonnegative function μ_n of subintervals of X that satisfies the superadditivity

$$\mu_n(I) + \mu_n(J) \;\le\; \mu_n(K) \text{ for nonoverlapping subintervals I, J of } K \;.$$

If

$$L \;\equiv\; \sup_n \int_X \mathcal{N}\big(f_n(x)\big)\, dx \;<\infty \quad \text{and} \quad \sup_n \mu_n(X) \;<\; \infty \;,$$

then f_n contains a subsequence that converges pointwise almost everywhere. The limiting function $f : X \to Y$ satisfies similar bounds

$$\int_X \mathcal{N}\big(f(x)\big)\, dx \;\le\; L \;\text{ and }\; \lambda \,\big|\, \{x \in I \;:\; (\mathcal{M}_I f)^{1/\alpha}(x) > \lambda\} \,\big| \;\le\; \sup_n \mu_n(I) \;.$$

Proof. First we will show how to pass to a subsequence to replace (9.1) by an estimate

$$\limsup_{n\to\infty} \sup_{\lambda>0} \lambda \left| \{x \in I \ : \ (\mathcal{M}_I f_n)^{1/\alpha}(x) > \lambda\} \right| \ \le \ \mu(I) \tag{9.2}$$

involving one fixed superadditive μ. To do this, we first observe that, for each positive ϵ, the set

$$A_\epsilon \ = \ \{a \in X \ : \ \lim_{r\to 0} \liminf_{n\to\infty} \mu_n\bigl(X \cap [a-r, a+r] > \epsilon\bigr)\}$$

has $\operatorname{card} A_\epsilon \ \le \ \epsilon^{-1} \sup_n \mu_n(X)$ because we may, for any finite subset $a_1, a_2, \ldots, a_J$ of distinct points of A_ϵ, choose a small positive r and then n large so that $\{[a_j - r, a_j + r] \ : \ j = 1, 2, \ldots, J\}$ are disjoint and $\mu_n\bigl(X \cap [a - r_j, a_j + r]\bigr) > \epsilon$, hence,

$$J\epsilon \ \le \ \sum_{j=1}^{J} \mu_n\bigl(X \cap [a-r, a+r]\bigr) \ \le \ \mu_n\bigl(X\bigr) \ .$$

It follows that the concentration set

$$A \ = \ \{a \in \mathbf{R} \ : \ \lim_{r\to 0} \liminf_{n\to\infty} \mu_n\bigl(X \cap [a-r, a+r]\bigr) > 0\} \ = \ \cup_{k=1}^{\infty} A_{1/k}$$

is countable. By Cantor diagonalization, we may again pass to a subsequence (without changing notations) to insist that the limits

$$\mu\bigl([a,b]\bigr) \ \equiv \ \lim_{n\to\infty} \mu_n\bigl([a,b]\bigr)$$

exist for all endpoints a, b taken from the countable set $A \cup (\mathbf{Q} \cap X)$. Then μ is a monotone function of such intervals, and we may define, for an arbitrary closed interval $[s,t] \subset X$,

$$\mu\bigl([s,t]\bigr) \ = \ \sup\{\mu\bigl([a,b]\bigr) \ : \ s \le a \le b \le t, \ a, b \in A \cup (\mathbf{Q} \cap X)\} \ .$$

For any $\epsilon > 0$ and interval $I = [s,t]$, we may choose $a, b \in A \cup (\mathbf{Q} \cap X)$ with $s \le a \le b \le t$ so that, for n sufficiently large

$$\begin{cases} a \ = \ s & \text{in case } s \in A \cup (\mathbf{Q} \cap X) \\ \mu_n\bigl([s,a]\bigr) \ < \ \frac{\epsilon}{4} & \text{in case } s \notin A \cup (\mathbf{Q} \cap X) \end{cases}$$

and

$$\begin{cases} b \ = \ t & \text{in case } t \in A \cup (\mathbf{Q} \cap X) \\ \mu_n\bigl([b,t]\bigr) \ < \ \frac{\epsilon}{4} & \text{in case } t \notin A \cup (\mathbf{Q} \cap X). \end{cases}$$

In any case,

$$\mu_n\bigl(I \setminus [a,b]\bigr) \ < \ \frac{\epsilon}{2} \ ,$$

for all n sufficiently large, so that

$$\mu_n(I) \;<\; \mu_n\big([a,b]\big) \;+\; \frac{\epsilon}{2}$$

Also, for n sufficiently large,

$$\mu_n\big([a,b]\big) \;<\; \mu\big([a,b]\big) \;+\; \frac{\epsilon}{2}\;,$$

hence,

$$\mu_n(I) \;<\; \mu\big([a,b]\big) \;+\; \epsilon \;<\; \mu(I) \;+\; \epsilon\;.$$

Estimate (9.1) now gives (9.2) because

$$\limsup_{n\to\infty}\sup_{\lambda>0}\;\lambda\,\Big|\,\{x\in I\;:\;(\mathcal{M}_I f_n)^{\frac{4}{3}}(x)>\lambda\}\,\Big|\;\le\;\limsup_{n\to\infty}\mu_n(I)\;\le\;\mu(I)\;+\;2\epsilon,$$

and we may let $\epsilon\to 0$. Moreover, for nonoverlapping subintervals $I,\ J$ of $K\subset[-2,2]$ and $\epsilon>0$, we similarly choose closed subintervals $I'\subset I,\ J'\subset J,\ I'\cup J'\subset K'\subset K$ with endpoints in $A\cup\mathbf{Q}$, and then n sufficiently large so that

$$\mu(I)+\mu(J)\;\le\;\mu(I')+\mu(J')+2\epsilon\;\le\;\mu_n(I')+\mu_n(J')+4\epsilon\;\le\;\mu_n(K')+4\epsilon\;\le\;\mu(K)+5\epsilon\;.$$

Letting $\epsilon\to 0$ gives the desired super-additivity $\mu(I)+\mu(J)\le\mu(K)$.

Having established (9.2), we next observe that it suffices to show

there is a subsequence $f_{n'}$ that is pointwise a.e. <u>*Cauchy*</u> *convergent.*

In fact, for a.e. $x\in X$, the sequence $f_{n'}(x)$ will then have, in the *completion* $\hat{Y}$ of Y, a unique limit $f(x)$ a.e. Also, for a.e. $x\in X$, Fatou's Lemma will provide a subsequence n'' of n' (depending on x) so that

$$\sup_{n''\to\infty}\;\mathcal{N}\big(f_{n''}(x)\big)\;<\;\infty\;.$$

The compactness assumption then will give a subsequence n''' of n'' so that $f_{n'''}(x)$ converges to a point of Y. Thus, $f(x)$, being necessarily this limit point, will belong to Y. Moreover, then, for a.e. x,

$$\mathcal{M}_I f(x)\;\le\;\liminf_{n'\to\infty}\mathcal{M}_I f_{n'}(x)$$

because

$$\begin{aligned}\operatorname{dist}\big(f(x),f(\tilde{x})\big)\;&=\;\lim_{n'\to\infty}\operatorname{dist}\big(f_{n'}(x),f_{n'}(\tilde{x})\big)\\&\le\;\liminf_{n'\to\infty}\mathcal{M}_I f_{n'}(x)|x-\tilde{x}|^{\alpha}\;.\end{aligned}$$

The lower semi-continuity assumption on $\mathcal{N}$ and Fatou's Lemma gives

$$\int_X \mathcal{N}\big(f(x)\big)\,dx \;\le\; \int_X \liminf_{n'\to\infty} \mathcal{N}\big(f_{n'}(x)\big)\,dx \;\le\; \liminf_{n'\to\infty} \int_X \mathcal{N}\big(f_{n'}(x)\big)\,dx \;\le\; L\;.$$

For the measure estimate, we note that

$$\{x \in I \;:\; (\mathcal{M}_I f)^{1/\alpha}(x) > \lambda\}$$

is, except for a null set, contained in an increasing union of the sets

$$D_n \;=\; \cap_{m=n}^{\infty}\{x \in I \;:\; (\mathcal{M}_I f_m)^{1/\alpha}(x) > \lambda\}\;,$$

so that

$$\begin{aligned}\big|\{x \in I \;:\; (\mathcal{M}_I f)^{1/\alpha}(x) > \lambda\}\big| \;&=\; \lim_{n\to\infty} |D_n| \\ &\le\; \liminf_{n\to\infty} \big|\{x \in I \;:\; (\mathcal{M}_I f_n)^{1/\alpha}(x) > \lambda\}\big| \;\le\; \lambda^{-1}\mu(I)\;.\end{aligned}$$

We now construct the desired subsequence which is Cauchy convergent a.e. Starting with the sequence $n_0(j) = j$, we will choose, by inducton on k, a subsequence $n_k(j)$ of $n_{k-1}(j)$ and some countable family $\mathcal{I}_k$ of closed subintervals of X along with distinguished points $c_I \in I$ for each $I \in \mathcal{I}_k$ so that

$$Z_k \;=\; X \setminus \bigcup_{I\in\mathcal{I}_k} I$$

has measure 0, and, for every $I \in \mathcal{I}_k$,

$$f_{n_k(j)}(c_I) \textit{ is Cauchy convergent as } j\to\infty \quad\text{and}\quad \limsup_{j\to\infty} (\mathcal{M}_I f_{n_k(j)})^{1/\alpha}(c_I) \;\le\; \frac{1}{k|I|}\;.$$

This will do it because then the diagonal subsequence $f_{n_j(j)}$ will be Cauchy convergent at almost every point $x \in X$. In fact, for almost every $x \in X \setminus \cup_{k=0}^{\infty} Z_k$ and for $\epsilon > 0$, we will be able to choose:

first an integer $k > \big(\frac{4}{\epsilon}\big)^{1/\alpha}$,
second, an interval $I \in \mathcal{I}_k$ containing x, and
third, an integer $h \ge k$ so that

$$\operatorname{dist}\big(f_{n_k(i)}(c_I), f_{n_k(j)}(c_I)\big) < \;\epsilon/3 \quad\text{and}\quad (\mathcal{M}_I f_{n_k(j)})(c_I) \;<\; \frac{4}{3(k\,|I|)^{\alpha}}$$

for $i, j \ge h$. Then, since $n_i(i) = n_k(i')$ and $n_j(j) = n_k(j')$ for some $i', j' \ge h$,

$$\begin{aligned}\operatorname{dist}\big(f_{n_i(i)}(c_I), f_{n_j(j)}(c_I)\big) \;&<\; \epsilon/3 \\ \operatorname{dist}\big(f_{n_i(i)}(x), f_{n_i(i)}(c_I)\big) \;&\le\; (\mathcal{M}_I f_{n_i(i)})(c_I)|x - c_I|^{\alpha} \\ &\le\; (\mathcal{M}_I f_{n_k(i')})(c_I)(|I|)^{\alpha} \;<\; \frac{4|I|^{\alpha}}{3(k|I|)^{\alpha}} \;<\; \frac{\epsilon}{3}\;, \\ \operatorname{dist}\big(f_{n_j(j)}(x), f_{n_j(j)}(c_I)\big) \;&\le\; (\mathcal{M}_I f_{n_j(j)})(c_I)|x - c_I|^{\alpha} \;<\; \frac{\epsilon}{3}\;,\end{aligned}$$

hence,

$$\operatorname{dist}\big(f_{n_i(i)}(x), f_{n_j(j)}(x)\big) \ < \ \frac{\epsilon}{3} \ + \ \frac{\epsilon}{3} \ + \ \frac{\epsilon}{3} \ = \ \epsilon \ .$$

Fixing k, we will inductively choose subsequences $m_1(j)$ of $n_{k-1}(j)$, $m_2(j)$ of $m_1(j)$, $m_3(j)$ of $m_2(j)$, ... as well as subintervals $I_1, I_2, I_3, \ldots$ of X with points $c_i \in I_i$ so that

$$n_k(j) \ = \ m_j(j) \ , \ \mathcal{I}_k \ = \ \{I_1, I_2, \ldots\} \ , \ c_{I_i} = c_i$$

satisfy the desired conditions.

To do this, we first choose an integer

$$q \ > \ 2k\mu(X),$$

and let $\mathcal{I}$ be the decomposition of the interval X into $2q$ nonoverlapping subintervals of equal length. Since

$$\sum_{I \in \mathcal{I}} \mu(I) \ \le \ \mu(X) \ < \ \frac{q}{2k} \ ,$$

we may choose q "good" intervals $I_1, I_2, \ldots, I_q \in \mathcal{I}$ with

$$\mu(I_i) \ < \ \frac{1}{2k} \ .$$

Now, for each $i \ = \ 1, 2, ..., q$, we may use the weak-type bound with $I \ = \ I_i$ and $\lambda = 1/(k|I_i|)$ to see that each set

$$E_{m_0(j)} \ = \ \{x \in I_i \ : \ (\mathcal{M}_{I_i} f_{m_0(j)})^{1/\alpha}(x) > \frac{1}{k|I_i|}\}$$

has measure

$$|E_{m_0(j)}| \ \le \ k|I_i|\mu(I_i) \ < \ \frac{1}{2}|I_i|$$

for all j sufficiently large. By Fatou's Lemma,

$$\int_{I_i} \liminf_{j\to\infty} \left[\chi_{E_{m_0(j)}}(x) \ + \ \frac{|I_i|}{3L}\mathcal{N}(f_{m_0(j)}(x))\right] dx \ \le \ \frac{1}{2}|I_i| \ + \ \frac{|I_i|}{3L}L \ = \ \frac{5}{6}|I_i| \ .$$

Thus we may choose a point $c_i \in I_i$ and a subsequence $m_1(j)$ of $m_0(j)$ so that

$$\chi_{E_{m_1(j)}}(c_i) \ + \ \frac{|I_i|}{3L}\mathcal{N}(f_{m_1(j)}(c_i)) < \ 1$$

for all j. In particular, $c_i \notin E_{m_1(j)}$, hence,

$$(\mathcal{M}_{I_i} f_{m_1(j)})^{1/\alpha}(c_i) \ \le \ \frac{1}{k|I_i|} \ .$$

Also since $\mathcal{N}(f_{m_1(j)}(c_i))$ is bounded in j , we may use another subsequence to assume that

$$f_{m_i(j)}(c_i) \text{ is Cauchy convergent as } j \to \infty \ .$$

We are still left with the remaining q (possibly "bad") subintervals

$$\mathcal{I} \setminus \{I_1, \ldots, I_q\} = \{J_1, \ldots, J_q\} \ .$$

Scaling shows that we may repeat the above argument first with $X, n_{k-1}(j)$ replaced by $J_1, m_N(j)$ obtaining again q good subintervals $I_{q+1}, \ldots, I_{2q}$ of J_1 and then, inductively, points $c_{q+1} \in I_{q+1}, \ldots, c_{2q} \in I_{2q}$ and consecutive subsequences $m_{q+1}(j), \ldots, m_{2q}(j)$ so that

$$\mathcal{M}_{I_{q+i}} f^{1/\alpha}_{m_{q+i}(j)}(c_{I+i}) \ \leq \ \frac{1}{k|I_{q+i}|} \quad \text{and} \quad f_{m_{q+i}(j)}(c_{q+i}) \text{ is Cauchy convergent}$$

for $i = 1, \ldots, q$. Similarly, we extract q good subintervals with distinguished points from each of $J_2, \ldots, J_q$. Then we repeat with the remaining, possibly bad, subintervals of $J_1, J_2, \ldots, J_q$. Continuing, we finally obtain consecutive subsequences $m_i(j)$, intervals I_i, and points $c_i \in I_i$ so that $f_{m_i(j)}(c_i)$ is Cauchy convergent as $j \to \infty$.

It also follows that almost every $x \in X$ is eventually contained in some good subinterval I_i, that is,

$$\left|X \setminus \cup_{i=1}^{\infty} I_i\right| \ = \ 0$$

because, at each stage, the good subintervals cover at least $\frac{1}{2}$ of the interval being considered. For $j \geq i$, $n_k(j) = m_j(j)$ is a subsequence of $m_i(j)$, hence, $f_{n_k(j)}(c_i)$ is Cauchy convergent. Moreover, the estimate

$$\mathcal{M}_{I_i} f^{1/\alpha}_{n_k(j)}(c_i) \ \leq \ \frac{1}{k|I_i|}$$

now holds for each i and all $j \geq i$, and the proof is complete. ∎

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