Connecting Rational Homotopy Type Singularities.

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Abstract : *Let N be a compact simply connected smooth Riemannian manifold and p an arbitrary positive integer. For any map u from $\mathbb{R}^{p+1}$ into N whose gradient is in $L^p(\mathbb{R}^{p+1})$, the restriction of u to almost every p dimensional sphere in $\mathbb{R}^{p+1}$ defines an homotopy class in $\pi_p(N)$ ([Wh]). Evaluating a fixed element z of $Hom(\pi_p(N),\mathbb{R})$ on this homotopy class thus gives a map $\Phi_{z,u}$ from the space of "generic" p-spheres into $\mathbb{R}$. The main result of the paper is to show that, under the assumption that u can be approximated by a $W^{1,p}$ weakly convergent sequences of smooth maps in $C^\infty(\mathbb{R}^{p+1},N)$, there exists a rectifiable Poincaré dual of $\Phi_{z,u}$: a countable union Γ of C^1 curves in $\mathbb{R}^{p+1}$ together with a Hausdorff $\mathcal{H}^1-$measurable real multiplicity function θ and orientation $\vec{\Gamma}$ on Γ such that the intersection number between any "generic" sphere S and this Poincaré dual equals $\Phi_{z,u}(S)$. Moreover, we exhibit a nonnegative integer n_z, depending only on z, such that $\int_\Gamma |\theta|^{p(p+n_z)^{-1}} d\mathcal{H}^1 < \infty$. We give cases of N, p and z for which this rational number $p(p+n_z)^{-1}$ in the above integral is optimal. The construction of this Poincaré dual is based on 1 dimensional "bubbling" described by the notion of "scans" which was introduced in [HR1].*

I Introduction

Let (N,g) be a closed Riemannian manifold. With the help of Nash embedding theorem, we may assume that N is a submanifold, with the induced metric, of some Euclidian space $\mathbb{R}^k$. One then has, for any $m\in\mathbb{N}$ and $p\geq 1$, the space of Sobolev maps:

$$W^{1,p}(\mathbb{R}^m,N) \ = \ \{u\in W^{1,p}(\mathbb{R}^m,\mathbb{R}^k)\ :\ u(x)\in N \text{ for almost every } x\in\mathbb{R}^m\}\,.$$

An important issue regarding the description of these non-linear function spaces, which plays an increasing role in analysis, is the question of the density in $W^{1,p}(\mathbb{R}^m, N)$, for the $W^{1,p}$ norm, of smooth maps taking values into N.

In case $p > m$, Sobolev embedding shows that any map in $W^{1,p}(\mathbb{R}^m, N)$ is (Hölder) continuous. For such a continuous $W^{1,p}$ map u, it is not difficult to see, using standard smoothing in $W^{1,p}(\mathbb{R}^m, \mathbb{R}^k) \cap C^0$ and nearest-point projection to N, that u is strongly $W^{1,p}$ approximable by maps in $C^\infty(\mathbb{R}^m, N)$.

In case $p = m$, this continuity of a Sobolev map is no longer automatically true. Nevertheless, $C^\infty(\mathbb{R}^p, N)$ is still strongly dense in $W^{1,p}(\mathbb{R}^p, N)$ as noted by Schoen and Uhlenbeck [SU]. It follows similarly that any map $u \in W^{1,p}(S^p, N)$ admits a strong $W^{1,p}$ approximation by maps in $C^\infty(S^p, N)$. White [Wh] showed how this approximation gives a well-defined homotopy class in $\pi_p(N)$. Conversely, every homotopy class in $\pi_p(N)$ (which is, by definition, given by a continuous map) admits a smooth and hence $W^{1,p}$ representative.

In case $p < m$, the strong $W^{1,p}$ density of $C^\infty(S^p, N)$ in $W^{1,p}(\mathbb{R}^p, N)$ may fail (as seen in the example $x/|x| \in W^{1,2}(B^3_1, N)$ discussed in [SU]). The general problem of strong $W^{1,p}$ approximability was considered by Bethuel in [Be] (see also more recent works and updated results on the necessary and sufficient topological conditions by Hang and Lin in [HL1] and [HL2]). It was shown in particular in [Be] that smooth maps are *not* dense in $W^{1,p}(\mathbb{R}^m, N)$ whenever $\pi_{[p]}(N) \neq 0$ (where $[p]$ is the integer part of p).

In this paper we restrict to the case $m = [p] + 1$. For a map u in $W^{1,p}(\mathbb{R}^m, N)$, Fubini's Theorem, implies that, for each center $c \in \mathbb{R}^m$, and almost every radius $r > 0$, the restriction of u to the $[p]-$sphere $\partial B^m_r(c)$ belongs to $W^{1,p}(\partial B_r(c), N)$. Thus the map

$$u_{c,r} : S^p \longrightarrow N \ , \quad u_{c,r}(x) \ = \ u(c + rx)$$

gives, as discussed above, an element of $\pi_{[p]}(N)$ because $p \geq [p] = \dim \partial B_r(c)$. The map u is strongly $W^{1,p}$ approximable by smooth maps if and only if the homotopy class of such a restriction $u_{c,r}$ is zero for almost every (c, r). The goal of this paper is to describe, for an arbitrary map in $W^{1,p}(\mathbb{R}^{p+1}, N)$, "how big" is the obstruction of the strong approximability. The idea is to try to "connect" the *topological singularities* of u. Such a singularity is recognized by seeing that the homotopy class $[u_{c,r}]$ changes as the sphere $\partial B^m_r(c)$ moves across the singularity. We restrict to the obstruction coming from the *infinite non-torsion part* $\pi_{[p]}(N) \otimes \mathbb{R}$ of $\pi_{[p]}(N)$. Also to simplify the notations in the presentation, we henceforth assume that

$$p \ = \ [p] \quad .$$

Suppose z is an element in the vectorspace

$$(\pi_p(N) \otimes \mathbb{R})^* \ = \ \mathrm{Hom}\,(\pi_p(N)\,,\,\mathbb{R}) \quad ,$$

and $u \in W^{1,p}(\mathbb{R}^{p+1}, N)$. To study the z-topological singularities of u, we consider restriction to spheres with the map

$$\begin{aligned} \Phi_{z,u} : \quad \mathbb{R}^{p+1} \times \mathbb{R}+ \quad &\longrightarrow \quad z(\,\pi_p(N)\,) \subset \mathbb{R} \ , \\ (c, r) \quad &\mapsto \quad z(\,[u_{c,r}]\,) \ . \end{aligned}$$

This is defined for almost every (c, r) in $\mathbb{R}^{p+1} \times \mathbb{R}_+$ and is, as we shall see, Lebesgue measurable. Note that $\Phi_{z,u}(c, r) = 0$ in case u is continuous on the closed ball $\overline{B_r(c)}$ because then $u_{c,r}$ is homotopic to a constant.

Recall that any countable union Γ of C^1 embedded curves admits an $\mathcal{H}^1$ *measurable* orientation, that is, a unit vectorfield $\vec{\Gamma}$ so that, at $\mathcal{H}^1$ almost every point $x \in \Gamma$, $\Gamma(x)$ orients the approximate line for Γ at x (see [Fe],3.2.19). We keep denoting Γ as the set of points at which $\vec{\Gamma}$ exists. Moreover, for almost every (c, r) in $\mathbb{R}^{p+1} \times \mathbb{R}_+^*$, the sphere $\partial B_r(c)$ intersects Γ transversally (see Lemma V.1) ; that is,

$$\vec{\Gamma}(a) \cdot (a - c) \neq 0 \quad \text{for all} \quad a \in \Gamma \cap \partial B_r(c) \quad .$$

We can now state our main result.

Theorem I.1 *Let N be a compact simply connected Riemannian manifold, p be a positive integer, and z be an element of $Hom(\pi_p(N), \mathbb{R})$. Then there exist a nonnegative integer n_z and a positive constant C_z such that for any map u in $W^{1,p}(\mathbb{R}^{p+1}, N)$ which can be approximated weakly in $W^{1,p}(\mathbb{R}^{p+1}, N)$ by smooth maps, there exists a countable union Γ of C^1 curves with measurable orientation $\vec{\Gamma}$ and a nonnegative $\mathcal{H}^1$ measurable multiplicity function θ from Γ into $z(\,\pi_p(N)\,)$ such that*

$$\Phi_{z,u}(c, r) = z(\,[u_{c,r}]\,) \ = \sum_{a \in \Gamma \cap \partial B_r(c)} \vec{\Gamma}(a) \cdot (a - c) \ \theta(a) \tag{I.1}$$

for almost every $(c, r) \in \mathbb{R}^{p+1} \times \mathbb{R}_+$, $\mathcal{H}^1\{a \in \Gamma \,:\, \theta(a) \neq 0\} \ < \ \infty$, and

$$\int_\Gamma |\theta|^{\frac{p}{p+n_z}} \ d\mathcal{H}^1 \quad \leq \quad C_z \liminf_{n \to \infty} \int_{\mathbb{R}^{p+1}} |\nabla u_n|^p \, dx \quad < \quad \infty \tag{I.2}$$

for any sequence $u_n \in C^\infty(\mathbb{R}^{p+1}, N)$ converging $W^{1,p}$ weakly to u. $(\Gamma, \vec{\Gamma}, \theta)$ is called a rectifiable Poincaré dual to $\Phi_{z,u}$.

Remark I.1 *In the previous theorem, the assumption of $W^{1,p}$ weak approximability by maps in $C^\infty(\mathbb{R}^{p+1}, N)$ can be replaced by the assumption of $W^{1,p}$ weak approximability by maps u_n in $W^{1,p}(\mathbb{R}^{p+1}, N)$ satisfying $\Phi_{z,u_n} \equiv 0$.*

The reason why we call $(\Gamma, \vec{\Gamma}, \theta)$ a Poincaré dual to $\Phi_{z,u}$ is the following: In case u has only finitely many isolated singularities $c_1, \cdots, c_I$, each homotopy class $d_i = [u_{c_i,r}]$ is then independent of the choice of radius $r < \min_{j\neq i} |c_j - c_i|$. For any p cycle C with compact support in $M = \mathbb{R}^{p+1} \setminus \{c_1, \cdots, c_I\}$, $C = \partial B$ for some unique $(p+1)$ chain B of compact support in $\mathbb{R}^{p+1}$. The chain B has constant multiplicity in each component of $M \setminus \operatorname{spt} C$, and we suppose that n_j is the multiplicity of B at c_j. Then the map Φ given by

$$\Phi(C) \;=\; z\,(\sum_{i=1}^{I} n_i\, d_i\,)$$

gives a well-defined cohomology element in $H^p(M, \mathbb{R})$. It is easy to see that any choice of $(\Gamma, \vec{\Gamma}, \theta)$ satisfying (I.1) is a representative of the Poincaré dual in $H_1(M, \mathbb{R})$ of Φ. Recalling that maps having isolated point singularities are dense in $W^{1,p}(\mathbb{R}^{p+1}, N)$ (see [Be]), we see this notion of Poincaré dual can be interpreted as a limit of the classical one.

Theorem I.1 was first established in the particular case where $N = S^2$ in [BCL], [BBC], [GMS1], [ABL] (see the discussion in [GMS]). In that case, there is one generator z of $\mathrm{Hom}(\pi_p(S^p), \mathbb{R}) \simeq \mathbb{R}$ (the *topological degree*) and $n_z = 0$. These situations where $n_z = 0$ are a very special and allow the bubbled object $(\Gamma, \vec{\Gamma}, \theta)$ to be interpreted as a <u>current</u>. Being a limit of a mass-bounded sequence of rectifiable currents, it is also rectifiable by geometric measure theory (see [GMS]). Then in [Ri] and [HR1] the case where $N = S^2$ for arbitrary p was considered. In that case, for $p = 3$, $\mathrm{Hom}(\pi_3(S^2), \mathbb{R}) \simeq \mathbb{R}$ is also generated by one element z (the *Hopf degree*), but now $n_z = 1$, and any corresponding $(\Gamma, \vec{\Gamma}, \theta)$ <u>cannot</u>, by specific example [HR1],2.5, be interpreted as a current.

A critical general problem behind this work is the question:

For any homotopy invariant $z \in$Hom$(\pi_p(N), \mathbb{R})$ and $M > 0$, what is the minimum possible p energy $\int_{S^p} |\nabla u|^p d\mathcal{H}^p$ necessary for a map $u \in C^\infty(S^p, N)$ to have $z(\,[u]\,) \geq M$?

For $N = S^p$ and z being the topological degree, $n_z = 0$, and this minimum p energy is precisely $p^{p/2}\mathcal{H}^p(S^p) \cdot M$. On the other hand, for $N = S^2$ with $p = 3$, and z being the Hopf degree, $n_z = 1$, and the minimum 3 energy is asymptotically $C\, M^{\frac{3}{4}} = C\, M^{p(p+n_z)^{-1}}$ by [Ri]. There are other situations where we know that the integer n_z, as defined in III.2(iii) and II.5, is optimal for the inequality (I.2). Precisely we have the following result:

Proposition I.1 *Let N be a compact simply connected Riemannian manifold, p be a positive integer and z be an element of Hom$(\pi_p(N), \mathbb{R})$. Assume* *that the critical exponent $p(p+n_z)^{-1}$, with n_z defined in definition II.5 is optimal in the sense given by definition II.2. Then, for any $\beta > p(p+n_z)^{-1}$, there exists u in $W^{1,p}(\mathbb{R}^{p+1}, N)$, a weak limit of smooth maps, such that for any Poincaré dual of $\Phi_{z,u}$, $(\Gamma, \vec{\Gamma}, \theta)$, satisfying (I.1), one has*

$$\int_\Gamma |\,\theta|^\beta \, d\mathcal{H}^1 \;=\; \infty \quad .$$

¿From section 2 we know, for instance, that the optimality assumption of this proposition is fulfilled for N being a sphere or a connected sum of $\mathbb{CP}^2$ and $S^2 \times S^2$ and arbitrary $z \neq 0$. We believe that this should be true for a large class of N (including in particular every 4-dimensional simply connected manifold). The proof of Proposition I.1 is based on the construction corresponding to the one presented in the example 2.5 of [HR1] which deals with the case $N = S^2$ and $p = 3$.

Finally we make the following observation. If $p(p+n_z)^{-1}$ is optimal in the sense given by definition II.2, then the following converse of Theorem I.1 holds : let u be an arbitrary map in $W^{1,p}(\mathbb{R}^{p+1}, N)$ admitting a rectifiable Poincaré dual $(\Gamma, \vec{\Gamma}, \theta)$ satisfying (I.1) such that

$$\int_\Gamma |\theta|^{\frac{p}{p+n_z}} \, d\mathcal{H}^1 \quad < \quad \infty \quad ,$$

Then there exists a sequence of maps u_n in $W^{1,p}(\mathbb{R}^{p+1}, N)$ satisfying $\Phi_{z,u_n} \equiv 0$ converging weakly in $W^{1,p}$ to u. The proof of this assertion is quite immediate if Γ is made of finitely many C^1 curves, but requires an approximation theorem similar to the one in [ABO], section 5 for dealing with the general case.

The goal of the paper is to prove Theorem I.1. We spend some time in Section II recalling facts and establishing new tools regarding the Novikov integral representation of Sullivan's rational homotopy groups that we need to prove our main result. Generalizing known formulas for the topological degree or for the Hopf degree, we derive, for any $z \in \mathrm{Hom}(\pi_p(N), \mathbb{R})$ and $u \in C^\infty(S^p, N)$ an integral expression

$$z(\,[u]\,) \quad = \quad \int_{S^p} u^{K_z}$$

where the p form u^{K_z} is constructed from u pull-backs of closed forms on N by operations of wedge product and explicit (and analytically estimable) "d^{-1} integrations" using certain Gauss integrals. The combinatorial form of these operations is described by the notion of a "tree-graph" associated with z, which is defined and illustrated by several specific examples, in II.3. In Section III we discuss the z-type bubbling for a $W^{1,p}$ weakly convergent sequence $u_n \in C^\infty(S^p, N) \rightharpoonup u \in W^{1,p}(S^p, N)$. In particular, a subsequence of the p forms $u_n^{K_z}$ converge as Radon measures to a sum $u^{K_z} + \sum_{i=1}^{I} m_i \delta_{a_i}$ where the $m_i \delta_{a_i}$ are the "bubbles". Finally in Section IV, we turn to maps on R^{p+1}, again considering $W^{1,p}$ weakly convergent sequences of smooth maps and assemble the bubbles in a limiting "scan". We prove, for a subsequence, the uniqueness and rectifiability of this scan, and, by integration, obtain Theorem I.1.

II Gauss Forms and Integral Representations of Rational Homotopy Groups.

In this part we shall exhibit Gauss forms associated to the Novikov linear forms of the rational homotopy groups from a smooth compact simply connected manifold N. To that aim we need to review D.Sullivan [Sul] and S.Novikov results in [Nov1], [Nov2] and [Nov3].

II.1 Integral Representation of Elements from Hom$(\pi_p(N), \mathbb{R})$.

II.1.1 Minimal Models and Geometric Realizations.

A **differential graded algebra** A over $\mathbb{R}$ is an $\mathbb{R}$-graded vector space of the form $A = \oplus_{i \geq 0} A^i$ together with a skew-commutative law

$$a \cdot b = (-1)^{rg\,a\ rg\,b}\ b \cdot a \quad ,$$

and an antiderivation of degree 1 satisfying

$$d(a \cdot b) = da \cdot b + (-1)^{rg\,a} a \cdot db \quad .$$

It is **free** when it possesses no other relation than this skew-commutative law and the associativity rule. Let $V = \text{span}\{x_1, \cdots, x_k\}$ be a graded $\mathbb{R}-$vector space, each x_i having a degree in $\mathbb{N}$, and let $\wedge(x_1 \cdots x_k)$ denote the free graded commutative algebra generated by $x_1, \cdots, x_k$. If, for instance, $x_1, \cdots, x_q$ are of even degree and $x_{q+1}, \cdots, x_n$ are odd, then, ignoring the grading, $\wedge(x_1 \cdots x_k)$ identifies with $S(\otimes_{i=1}^q x_i) \otimes \wedge_{j=1}^{n-q} x_j$. Here $\wedge_{j=1}^{n-q} x_j$ denotes the exterior algebra of $\otimes_{j=1}^{n-q} x_j$ while $S(B)$ is the symmetric algebra of $B:\ S(B) = B/(x \otimes y - y \otimes x)$. An element $a \in A$ is said to be **decomposable** if it is a sum of the product of two elements in $A^* = \oplus_{i>0} A^i$. A differential graded algebra

$\mathcal{M}$ is called a **minimal model** for another differential graded algebra A if $\mathcal{M}$ satisfies the following three conditions:

i) $\mathcal{M}$ is **free**. This means that there exists a graded $\mathbb{R}-$vector space $V = \oplus_{i\geq 1} V^i$ such that $\mathcal{M} = S(V^{even}) \otimes \wedge V^{odd}$

ii) There is a morphism of D.G.A.'s $\Psi : \mathcal{M} \rightarrow A$, called a **geometric realization** of $\mathcal{M}$, which induces an isomorphism in cohomology.

iii) The exterior differential of a generator is either 0 or decomposable.

Since $V^0 = \{0\}$, $\mathcal{M}^0 = \mathbb{R}$. Observe that iii) means that $dV^p \in \mathcal{M}^+ \cdot \mathcal{M}^+$ where $\mathcal{M}^+$ is the maximal idea $\mathcal{M}^+ = \oplus_{i\geq 1}\mathcal{M}^i$. A minimal model $\mathcal{M}$ is said to be **simply connected** if $\mathcal{M}^1 = 0$. A minimal model $\mathcal{M} = S(V^{even}) \otimes \wedge V^{odd}$ is also said to be **nilpotent** if each space V^i is finite dimensional. A basic result is the following:

Proposition II.1 *[Sul] For any compact simply connected manifold N, the exterior algebra of differential forms on N, $A = \wedge^* N$, admits a nilpotent simply connected minimal model $\mathcal{M}_N$.*

A proof of Proposition II.1 can for instance be found in [BT] pages 230-231 or [GM] page 116-117. The uniqueness of $\mathcal{M}_N$ (modulo isomorphism of D.G.A.'s) and the uniqueness of the associated geometric realization (modulo homotopy of morphisms of D.G.A.'s) is given in [GM] Theorem 10.9. For any integer $p > 1$, we have the following important identification of linear forms on $\pi_p(N) \otimes \mathbb{R}$.

Theorem II.1 *[Sul] Let $\mathcal{M}_N = S(V^{even}) \otimes \wedge V^{odd}$ be the minimal model for the compact simply connected manifold N, the space $Hom(\pi_p(N), \mathbb{R})$ is isomorphic to V^p the vector space spanned by the generators of degree p in $\mathcal{M}_N$ (or indecomposable elements of degree p in $\mathcal{M}_N$).*

Proofs of this theorem can be found in [Sul], [GM]...etc. In [Sul] page 312 an expression of this duality between V^p and $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ involving some "integral expression" is explained briefly : let u be a map from S^p into N representing a class in $\pi_p(N)$. By pull-back, u^* induces a D.G.A. morphism between $\wedge^* N$ and $\wedge^* S^p$. Given two geometric realizations Ψ_N between $\mathcal{M}_N$ and $\wedge^* N$ and Ψ_{S^p} between $\mathcal{M}_{S^p}$ and $\wedge^* S^p$, one can prove that u^* lifts into a D.G.A. morphism $\hat{u}$ between $\mathcal{M}_N$ and $\mathcal{M}_{S^p}$ such that the following diagram is commutative *modulo homotopy* (see [GM] chapter XIV).

$$\begin{array}{ccc} \mathcal{M}_N & \xrightarrow{\hat{u}} & \mathcal{M}_{S^p} \\ \downarrow \Psi_N & & \downarrow \Psi_{S^p} \\ \wedge^* N & \xrightarrow{u^*} & \wedge^* S^p \end{array}$$

The space generated by the generators of degree p in $\mathcal{M}_{S^p}$ is isomorphic to $\mathbb{R}$. (There is exactly one generator x - see the computation of $\mathcal{M}_{S^p}$ in [GM] and this isomorphism is given by integrating $\Psi_{S^p}(x)$ on S^p.) Therefore, $\hat{u}$ restricted to the space V^p generated by the generators of degree p in $\mathcal{M}_N = S(V^{even}) \otimes \wedge V^{odd}$ is a linear form : $\int_{S^p} \Psi_{S^p} \circ \hat{u} : V^p \to \mathbb{R}$. It is not difficult to check that it only depends on the homotopy class of u. The dual of the map

$$\pi_p(N) \quad \longrightarrow \quad \mathrm{Hom}(V^p, \mathbb{R}) \ ,$$

$$u \quad \mapsto \quad \int_{S^p} \Psi_{S^p} \circ \hat{u} \ \Big| \ V^p$$

is the isomorphism between V^p and $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ given by Theorem II.1 (see [GM] chapter XIV).

Remark II.1 *Note that this isomorphism between V^p and $Hom(\pi_p(N), \mathbb{R})$ depends on a choice of the geometric realization Ψ_N. If z is an element in V^p, we will keep denoting by z the image of z in $Hom(\pi_p(N), \mathbb{R})$ by this isomorphism when there is no ambiguity about which geometric realization we are using.*

Given a geometric realization $\Psi_N : \mathcal{M}_N \to \wedge^* N$, it is tempting to identify the correspondance between V^p and $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ in a more tractable way - the construction of $\hat{u}$ from u^* which holds up to homotopy in D.G.A.'s has to be made more explicit (see for instance this construction for $[u] \in \pi_3(N)$ pages 159-161 of [GM]). We aim to get a procedure to construct some more concrete expression of $\int_{S^p} \Psi_{S^p} \circ \hat{u}(z)$ for the elements z in V^p involving only u and smooth differential forms in N, like for instance the well known integral expression of the topological degree between S^p and S^p

$$[u] \in \pi_p(S^p) \longrightarrow \int_{S^p} u^* \omega$$

where ω generates $H^p(S^p)$ or the Hopf degree between S^{4p-1} and S^{2p}

$$[u] \in \pi_{4p-1}(S^{2p}) \longrightarrow \int_{S^{4p-1}} \eta \wedge u^* \omega$$

where ω generates $H^{2p}(S^{2p})$ and $d\eta = \omega$.

[Nov1], [Nov2], and [Nov3] contain a relatively simple procedure to compute the linear form $\int_{S^p} \Psi_{S^p} \circ \hat{u}$ on V^p. In the next section we recall that procedure.

II.2 d-extensions of Minimal Models and the Hopf-Novikov Integral Representation of Elements in $\mathbf{Hom}(\pi_p(N), \mathbb{R})$.

Starting from a given geometric realization Ψ_N of the minimal model $\mathcal{M}_N$, we construct the following free extension of $\mathcal{M}_N$. Let $x_{2,1}, \cdots, x_{2,p_2}$ be the generators of degree 2 (i.e. $V^2 = \text{Span}\{x_{2,1}, \cdots, x_{2,p_2}\}$). We call $C_2(\mathcal{M}_N)$ the algebra $\mathcal{M}_N$ to which we add p_2 free generators of degree 1 : $y_{1,1}, \cdots, y_{1,p_2}$ satisfying

$$d(y_{1,i}) = x_{2,i} \quad \text{for all } i \quad .$$

Thus

$$C_2(\mathcal{M}_N)) = \mathcal{M}_N[y_{1,1}, \cdots, y_{1,p_2}] \quad .$$

This has the effect to kill the H^2 of $\mathcal{M}_N$, moreover, we have

$$H^3(C_2(\mathcal{M}_N)) \simeq V^3 \quad .$$

Indeed, given $x_{3,j}$ a generator of degree 3 of $\mathcal{M}_N$, $dx_{3,j}$ is a linear combination of wedges of degree 2 :

$$dx_{3,j} = \sum_{k<l} \alpha_j^{kl}\ x_{2,k} \wedge x_{2,l} = \sum_{k<l} \alpha_i^{kl}\ d(y_{1,k} \wedge x_{2,l}) \quad .$$

It is straightforward to check that the family $z_{3,i} = x_{3,i} - \sum_{k<l} \alpha_i^{kl}\ y_{1,k} \wedge x_{2,l}$ generates $H^3(C_2(\mathcal{M}_N))$, and so we add p_3 free generators $y_{2,i}$ so that $dy_{2,i} = z_{3,i}$. We then go further in this construction until reaching the $d-$extension of order $p-1$ of $\mathcal{M}_N$: $C_{p-1}(\mathcal{M}_N)$. This procedure goes as follows : for $q < p-1$ $H^q(C_{q-1}(\mathcal{M}_N))$ is generated by the family of elements of the form $z_{q,i} = x_{q,i} + t_{q,i}$ for $i = 1, \cdots, p_q$, satisfying $dz_{q,i} = 0$, where the $x_{q,i}$ are the generators of degree q of $\mathcal{M}_N$ and $t_{q,i}$ are elements of degree q in the ideal $I^{q-1}(C_{q-1}(\mathcal{M}_N))$ generated by the elements of degree strictly less than q in $C_{q-1}(\mathcal{M}_N)$ and we pass from $C_{q-1}(\mathcal{M}_N)$ to $C_q(\mathcal{M}_N)$ by adding p_q free generators $y_{q-1,1}, \cdots, y_{q-1,p_q}$ satisfying

$$dy_{q-1,i} = z_{q,i} = x_{q,i} + t_{q,i} \quad .$$

Consider then the ideal generated by the elements of degree less or equal to q in $C_{q-1}(\mathcal{M}_N)$. It is a free graded algebra

$$\begin{aligned} &I^q(C_{q-1}(\mathcal{M}_N)) \\ &\quad = \wedge(y_{1,1} \cdots y_{1,p_2}, z_{2,1} \cdots z_{2,p_2}, \cdots, y_{q-1,1} \cdots y_{q-1,p_q}, z_{q,1} \cdots, z_{q,p_q}) \end{aligned}$$

generated by elements $y_{i-1,j}$ and $z_{i,j}$ for $i = 2, \cdots, q$ and $j = 1, \cdots, p_q$ and where

$$
\begin{aligned}
&deg\ y_{i-1,j} = i-1 \quad \text{ and } \quad deg\ z_{i,j} = i \\
&d\,y_{i-1,j} = z_{i,j}
\end{aligned}
\tag{II.1}
$$

It is then easy to verify that such a free algebra has a trivial cohomology

$$
H^*(I^q(C_{q-1}(\mathcal{M}_N))) = \{0\} \tag{II.2}
$$

Indeed, consider $a \in I^q(C_{q-1}(\mathcal{M}_N))$ such that $da = 0$ and take $i_0 \in [1, q]$ and $j_0 \in [1, p_q]$ such that a contains y_{i_0-1,j_0} or z_{i_0,j_0} in its decomposition in this free algebra. Assuming for instance i_0 is even, one has

$$
a = \sum_k \; y_{i_0-1,j_0} \wedge z^k_{i_0,j_0} \wedge A_k + \; z^{k+1}_{i_0,j_0} \wedge B_k + R \quad ,
$$

where A_k, B_k and R contain no y_{i_0-1,j_0} or z_{i_0,j_0} in their decomposition in linear combinations of products of generators y and z. Since $da = 0$ one has

$$
0 = \sum_k \; z_{i_0,j_0} \wedge z^k_{i_0,j_0} \wedge A_k + \sum_k \; y_{i_0-1,j_0} \wedge z^k_{i_0,j_0} \wedge d\,A_k + \sum_k \; z^{k+1}_{i_0,j_0} \wedge d\,B_k + dR \tag{II.3}
$$

Because of (II.1) it is clear that $d\,A_k$, $d\,B_k$ and $d\,R$ contain no y_{i_0-1,j_0} or z_{i_0,j_0} in their decompositions in linear combinations of products of generators y and z. Thus since the algebra $I^q(C_{q-1}(\mathcal{M}_N))$ is free, we have uniqueness in decompositions, and we get from (II.3) that

$$
A_k = d\,B_k \quad .
$$

Therefore, we see that

$$
a = \sum_k \; d\left(y_{i_0-1,j_0} \wedge z^k_{i_0,j_0} \wedge d\,B_k + \; z^{k+1}_{i_0,j_0} \wedge B_k\right) + R
$$

We may iterate this fact for R this time. After finitely many steps, we finally get that a is exact. Thus (II.2) is showed. Considering now one generator $x_{q+1,i}$ of degree $q+1$ in $C_q(\mathcal{M}_N)$, since $x_{q+1,i}$ is in $\mathcal{M}_N$, $d\,x_{q+1,i}$ is decomposable in $\mathcal{M}_N$ which means in particular that $d\,x_{q+1,i}$ is in $I^q(C_{q-1}(\mathcal{M}_N))$. Because of (II.2), there exists $t_{q+1,i}$ in $I^q(C_{q-1}(\mathcal{M}_N))$ such that $d(x_{q+1,i} + t_{q+1,i}) = 0$. It is moreover clear that $x_{q+1,i} + t_{q+1,i}$ is not an exact form of $C_q(\mathcal{M}_N)$. Thus, $H^{q+1}(C_q(\mathcal{M}_N)) \simeq V^q$, and we have proved by induction the following lemma.

Lemma II.1 *With the above notations and* p *being a positive integer, the following spaces are isomorphic*

$$H^p(C_{p-1}(\mathcal{M}_N)) \simeq V^p \simeq Hom(\pi_p(N),\mathbb{R}) \quad . \tag{II.4}$$

Going back now to the question of finding a procedure for getting explicit expressions of the integral representations $\int_{S^p} \Psi_{S^p} \circ \hat{u}(z)$ for arbitrary $[u] \in \pi_p(N)$ and arbitrary $z \in V^p$, we proceed as follows. We first construct a d-continuation $\tilde{u}$ of u^* between $C_{p-1}(\mathcal{M}_N)$ and $\wedge^* S^p$. Contrary to the case of $\hat{u}$ where this lifting existed only modulo homotopies of D.G.A.'s, there is here a procedure to get $\tilde{u}$ which goes by induction as follows. First we construct $\tilde{u}$ between $C_1(\mathcal{M}_N) = \mathcal{M}_N$ and $\wedge^* S^p$ by taking $\tilde{u}(x) = u^*\Psi_N(x)$. Suppose $p > 2$. In order to construct $\tilde{u}$ between $C_2(\mathcal{M}_N)$ and $\wedge^* S^p$ we just have to define the images of the $y_{1,j}$ by $\tilde{u}$ and, in order to have a morphism of D.G.A.'s, they have to verify in particular $d\,\tilde{u}(y_{1,j}) = \tilde{u}(x_{2,j}) = u^*\Psi_N(x_{2,j})$. We look for a specific operation d^{-1} on $\wedge^k S^p$ for $0 < k < p$. It must satisfy

$$d(d^{-1}\alpha) = \alpha$$

for every closed form $\alpha \in \wedge^k S^p$. We may, using the standard metric on S^p, take the "Coulomb Gauge"

$$d^{-1} = d^*\Delta^{-1}$$

where Δ is the Hodge-Laplacian $dd^* + d^*d$ and d^* is the Hodge adjoint differential $d^* = (-1)^{p(k+1)+1} * d *$. Here Δ is invertible because $H^k(S^p) = 0$. As we will see below, other operations d^{-1} can also be very useful. We will often consider the one given by (II.11) which corresponds to the Coulomb Gauge but with respect to the flat metric on $\mathbb{R}^p$ after pull-back by the inverse of the stereographic projection.

Now fixing such an operation d^{-1}, we take

$$\tilde{u}(y_{1,j}) = d^{-1}\tilde{u}(x_{2,j}) \quad .$$

The construction of $\tilde{u}$ then goes further, following the inductive construction we made for $C_q(\mathcal{M}_N)$ by taking for $y_{q,j}$ $(q+1 < p)$

$$\tilde{u}(y_{q,j}) = d^{-1}\tilde{u}(z_{q+1,j}) = d^{-1}\tilde{u}(x_{q+1,j} + t_{q+1,j})$$

$$= d^{-1}u^*\Psi_N(x_{q+1,j}) + d^{-1}\tilde{u}(t_{q+1,j}) \quad .$$

Once $\tilde{u}$ is completely constructed, it is then straightforward to verify that

$$\int_{S^p} \Psi_{S^p} \circ \hat{u}(z) = \int_{S^p} \tilde{u}([z]\,)$$

where $[z]$ is the class in $H^p(C_{p-1}(\mathcal{M}_N))$ corresponding to $z \in V^p$ via the isomorphism (II.4) constructed by induction and the differential form $\tilde{u}(\,[z]\,)$ has been constructed by induction also as described just above. We have then an explicit procedure to construct the integral representation of the elements in $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ starting from a geometric realization Ψ_N of the minimal model $\mathcal{M}_N$. Following the procedure, the forms $\tilde{u}(\,[z]\,)$ can be described with the help of graphs. Suppose that, for each i, the degree i forms $\omega_{i,j}$ give a basis for the degree i part of the the geometric realization Ψ_N of the minimal model $\mathcal{M}_N$. Thus,

$$\mathrm{Span}_{i,j}\{\omega_{i,j}\} = \Psi_N(\mathcal{M}_N) \subset \wedge^* N \quad .$$

It is straightforward to observe that $\tilde{u}(\,[z]\,)$ is a finite linear combination of p forms obtained as follows.

Each p form is obtained by first constructing a connected and simply connected *tree-graph* $K = K_z$ as shown in figure II.2. The tree-graph contains

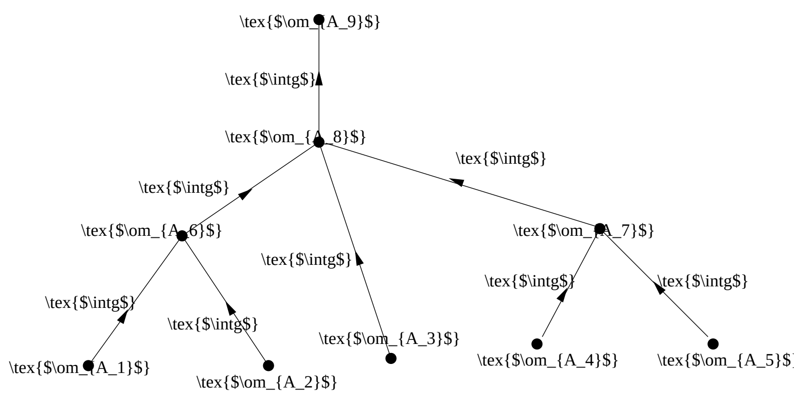

Figure 1: An example of tree graph arising in computing $\mathrm{Hom}(\pi_p(N), \mathbb{R})$.

finitely many vertices and finitely many segments connecting these vertices. Two vertices are connected by at most one segment, and the graph is assumed to be simply connected, that is, it contains no closed path. To each vertex A is assigned a closed element ω_A of the ideal generated by $\Psi_N(\mathcal{M}_N)$. Each segment is oriented such that at each vertex, except one, there is exactly one segment leaving. The exceptional vertex is the summit or "end" of the graph where all the attached segments are arriving. Each segment corresponds to the integration procedure d^{-1} in the orientation direction and the connection of a segment to a vertex corresponds to taking a wedge product. Such a tree-graph is called **(p dimensional) simply connected tree-graph of forms**.

There is the single important p form

$$u^K$$

associated to the tree graph K for the given map $u : S^p \longrightarrow N$ and element z in $\mathrm{Hom}(\pi_p(N), \mathbb{R})$. It is obtained as follows : starting from the ends of the branches, we pick the ω_A at each end, each independently from the other, and introduce the forms $u^*\omega_A$ in $\wedge^* S^p$. Then from these ends one goes one segment backward in the graph by integrating the u^*A that were assigned to each end (i.e. one considers $\eta_A = d^{-1}u^*\omega_A$) and one wedges the resulting form η_A with the pull-back by u of the form $\omega_{A'}$ sitting at the vertex A' of the graph we have reached. One goes further in this algorithm until reaching the summit of the tree-graph and the p form obtained there is u^K. For instance the form u^K given by the graph II.2 is

$$u^K = u^*\omega_{i_9,j_9} \wedge d^{-1}\left[u^*\omega_{i_8,j_8} \wedge d^{-1}k_1 \wedge d^{-1}(u^*\omega_{i_3,j_3}) \wedge d^{-1}k_2\right]$$

where

$$k_1 = d^{-1}\left[u^*\omega_{i_6,j_6} \wedge d^{-1}(u^*\omega_{i_1,j_1}) \wedge d^{-1}(u^*\omega_{i_2,j_2})\right] \quad ,$$

and

$$k_2 = d^{-1}\left[u^*\omega_{i_7,j_7} \wedge d^{-1}(u^*\omega_{i_4,j_4}) \wedge d^{-1}(u^*\omega_{i_5,j_5})\right] \quad .$$

The graph has to be read from the left to the right : if K^1 and K^2 are two subgraphs connecting a node A and if K denotes the graph made of this two subgraphs union the node A and the two segment starting respectively from the summit of K_1 and the summit of K_2 and if K_1 is at the left of K_2, u^K is obtained by respecting the left-right order : we have $u^K = u^*\omega_A \wedge d^{-1}(u^{K_1}) \wedge d^{-1}(u^{K_2})$.

Similarly, one has a form u^L corresponding to any sub-tree-graph L of K whose summit is some vertex of K. In general

$$\deg u^L = [\sum_{\text{vertices of } L} deg\, \omega_A] \quad - \quad n^K \tag{II.5}$$

where

$$n^K \;=\; \text{number of segments of } L \;= (\text{number of vertices of } L) - 1 \;,$$

and $\deg u^L \;\leq\; p\; = \deg u^K$, with equality if and only if $L = K$. We have established the following:

Proposition II.2 *To a compact simply connected manifold N, an element z in $(\pi_p(N) \otimes \mathbb{R})^*$ and any geometric realization Ψ_N of the minimal model* *of N, one assigns, using the notation above, a formal linear combination of simply connected tree-graphs $K = \sum_i \lambda_i K_i$ ($\lambda_i \in \mathbb{R}$) such that for any class $[u]$ in $\pi_p(N)$, represented by a map $u \in C^\infty(S^p, N)$, one has*

$$z(\,[u]\,) = \int_{S^p} \Psi_{S^p} \circ \hat{u}(z) = \int_{S^p} \tilde{u}(\,[z]\,) = \int_{S^p} u^K$$

where $u^K = \sum_i \lambda_i u^{K_i}$. Starting from Ψ_N and z, the formal linear combination of tree-graphs $K = \sum_i \lambda_i K_i$ is given by the algorithm described above in this subsection.

Remark II.2 *For a Sobolev map $u \in W^{1,p}(S^p, N)$, the p form u^K is defined $\mathcal{H}^p$ almost everywhere on S^p and is $\mathcal{H}^p$ integrable, and the equation*

$$z(\,[u]\,) \;=\; \int_{S^p} u^K$$

is still valid.

This is immediate from [Wh] and Proposition II.2 because $\int_{S^p} u_n^K \to \int_{S^p} u^K$ whenever $u_n \in C^\infty(S^p, N) \to u$ strongly in $W^{1,p}$. (With only weak $W^{1,p}$ convergence, there may be additional "bubbled" limiting terms, as discussed below in Section III.)

II.3 Examples

We give here examples of application of the algorithm above to express elements of $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ in terms of formal linear combinations of tree-graphs.

Example 1 : $N = \mathbb{CP}^2$. We first construct the minimal model of $\mathbb{CP}^2$ and a geometric realization of it. Let ω be the Kähler form on $\mathbb{CP}^2$. It is easy to check that $\mathcal{M}_{\mathbb{CP}^2}$ is generated by two elements α and β of degree respectively 2 and 5 satisfying

$$d\beta = \alpha^3 \;, \quad \Psi_{\mathbb{CP}^2}(\alpha) = \omega \;, \quad \text{and} \quad \Psi_{\mathbb{CP}^2}(\beta) = 0 \quad .$$

Therefore only $\pi_2(\mathbb{CP}^2)$ and $\pi_5(\mathbb{CP}^2)$ have a non-torsion part. We have respectively

$$C_1(\mathcal{M}_{\mathbb{CP}^2}) = \mathcal{M}_{\mathbb{CP}^2} \quad \text{and} \quad C_4(\mathcal{M}_{\mathbb{CP}^2}) = S(\alpha) \otimes \wedge[a, \beta]$$

where a is of degree 1, and satisfies $da = \alpha$. So we have that $H^2(C_1(\mathcal{M}_{\mathbb{CP}^2})) \simeq V^2$ is generated by α and $H^5(C_4(\mathcal{M}_{\mathbb{CP}^2})) \simeq V^5$ is generated by $\beta - a\wedge\alpha^2$. The

Figure 2: the two tree-graphs arising in computing $\mathrm{Hom}(\pi_2(\mathbb{CP}^2), \mathbb{R})$ and $\mathrm{Hom}(\pi_5(\mathbb{CP}^2), \mathbb{R})$.

tree-graphs associated to these two elements are respectively, for α, a vertex alone with ω assigned to it and, for β, two vertices connected by one segment going from ω to ω^2 (see figure II.3) The corresponding integral expressions are

$$\alpha(\,[u]\,) = \int_{S^2} u^*\omega \quad \text{and} \quad \beta(\,[u]\,) = \int_{S^5} u^*\omega^2 \wedge d^{-1}(u^*\omega) \quad .$$

Example 2 : $N = S^2 \times S^2$.

Let $\xi_1, \xi_2 \,:\, \mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$ being the projections of the three first (resp. three last) coordinates, and denote $\omega_i = \xi_i^*\omega$ where ω is a given generator of $H^2(S^2)$. Easy computations give $\mathcal{M}_{S^2\times S^2} = S(\alpha_1, \alpha_2) \otimes \wedge[\beta_1, \beta_2]$ where the α_i are of degree 2, whereas the β_j are of degree 3, and the following holds

$$d\beta_1 = \alpha_1^2, \quad d\beta_2 = \alpha_2^2.$$

We can chose ω_1 and ω_2 such that $\omega_1 \wedge \omega_1 \equiv 0$ and $\omega_2 \wedge \omega_2 \equiv 0$. Therefore, we have the following geometric realization :

$$\Psi_{S^2\times S^2}(\alpha_i) = \omega_i \quad \Psi_{S^2\times S^2}(\beta_i) = 0 \quad .$$

Only $\pi_2(S^2 \times S^2)$ and $\pi_3(S^2 \times S^2)$ have non-torsion parts. One has

$$C_1(\mathcal{M}_{S^2\times S^2}) = \mathcal{M}_{S^2\times S^2} \quad \text{and} \quad C_2(\mathcal{M}_{S^2\times S^2}) = S(\alpha_1, \alpha_2) \otimes \wedge[\beta_1, \beta_2, a_1, a_2]$$

where $da_i = \alpha_i$. So $H^2(C_1(\mathcal{M}_{S^2\times S^2})) \simeq V^2$ is generated by α_1 and α_2 and $H^3(C_2(\mathcal{M}_{S^2\times S^2})) \simeq V^3$ is generated by $\beta_1 - a_1\ \alpha_1$ and $\beta_2 - a_2\ \alpha_2$. The tree-graphs associated to these elements are, for α_1 (resp. α_2) one vertex to which ω_1 (resp. ω_2) is assigned, and, for β_1, (resp. β_2) two vertices connected by one segment going from ω_1 to ω_1 (resp. ω_2 to ω_2). The corresponding integrals are

$$\alpha_i(\,[u]\,) = \int_{S^2} u^*\omega_i \quad \text{and} \quad \beta_i(\,[u]\,) = \int_{S^3} u^*\omega \wedge d^{-1}(u^*\omega) \quad .$$

Example 3 : $N = (S^2 \times S^2)\#\mathbb{CP}^2$.

N is the connected sum of the two 4-manifolds we studied in examples 1 and 2. We shall denote $\mathcal{M}_N^4$ the ideal in $\mathcal{M}_N$ generated by the elements of degree less or equal to 5. We shall only compute the integral expressions of the elements in $\pi_p(\mathbb{CP}^1 \times \mathbb{CP}^1)\#\mathbb{CP}^2)^*$ for $p \leq 4$. After some computations one gets that

$$\mathcal{M}_N^4 = S(\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23}) \otimes \wedge[\beta_{11}, \beta_{22}, \beta_{13}, \beta_{23}, \beta_{123}]$$

where the following relations hold

i) for arbitrary i and j such that β_{ij} exists, one has

$$d\beta_{ij} = \alpha_i\alpha_j$$

and

$$d\beta_{123} = \alpha_1 \wedge \alpha_2 - \alpha_3^2$$

ii)

$$d\gamma_1 = \alpha_3\beta_{11} - \alpha_1\beta_{13}$$

$$d\gamma_2 = \alpha_3\beta_{22} - \alpha_2\beta_{23}$$

$$d\gamma_3 = \alpha_1\beta_{23} - \alpha_2\beta_{13}$$

$$d\gamma_{13} = \alpha_1\beta_{123} - \alpha_2\beta_{11} + \alpha_3\beta_{13}$$

$$d\gamma_{23} = \alpha_2\beta_{123} - \alpha_1\beta_{22} + \alpha_3\beta_{23}$$

One has

$$C_2(\mathcal{M}_N^4) = \mathcal{M}_N^4[a_1, a_2, a_3] \quad ,$$

where $d\,a_i = \alpha_i$ and $H^3(C_2(\mathcal{M}_N^5)) \simeq V^3$ is generated by $\beta_{ij} - a_i\alpha_j$ and $\beta_{123} - a_1\alpha_2 + a_3\alpha_3$. One has

$$C_3(\mathcal{M}_N^4) = C_2(\mathcal{M}_N^4)[b_{11}, b_{22}, b_{13}, b_{23}, b_{123}]$$

where $d\,b_{ij} = \beta_{ij} - a_i\alpha_j$ and $d\,b_{123} = \beta_{123} - a_1\alpha_2 + a_3\alpha_3$. $H^4(C_3(\mathcal{M}^4_N)) \simeq V^4$ is generated by

$$\gamma_1 - \alpha_3 b_{11} + \alpha_1 b_{13}$$

$$\gamma_2 - \alpha_3 b_{22} + \alpha_2 b_{23}$$

$$\gamma_3 - \alpha_1 b_{23} + \alpha_2 b_{13} - a_1 a_2 \alpha_3$$

$$\gamma_{13} - \alpha_1 b_{123} + \alpha_2 b_{11} - \alpha_3 b_{13} + a_1 a_3 \alpha_3$$

$$\gamma_{23} - \alpha_2 b_{123} + \alpha_1 b_{22} - \alpha_3 b_{23} + a_2 a_3 \alpha_3 \quad .$$

Denoting $\Psi_N(\alpha_i) = \omega_i$. It is not difficult to see that we can choose ω_1, ω_2 and ω_3 representative in $\wedge^2 N$ of a basis of $H^2(N)$ (generating $H^*(N)$) satisfying

$$\omega_1 \wedge \omega_1 \equiv 0 \quad , \quad \omega_2 \wedge \omega_2 \equiv 0 \quad ,$$

$$\omega_1 \wedge \omega_3 \equiv 0 \quad , \quad \omega_2 \wedge \omega_3 \equiv 0$$

The goal is to simplify the geometric realization Ψ_N. Let η_{123} be a form such that

$$d\eta_{123} = \omega_1 \wedge \omega_2 - \omega^3$$

The geometric realization Ψ_N restricted to $\mathcal{M}^4_N$ is then defined by

$$\Psi_N(\alpha_i) = \omega_i \quad , \Psi_N(\beta_{ij}) = 0 \quad , \quad \Psi_N(\beta_{123}) = \eta_{123}$$

$$\Psi_N(\gamma_i) = 0 \quad \text{and} \quad \Psi_N(\gamma_{ij}) = 0$$

for every β_{ij}, γ_i and γ_{ij} defined above.

Given a smooth map u from S^4 into N, the $d-$continuation of u^* between $C_3(\mathcal{M}^4_N)$ into $\wedge^* S^4$ is defined by

$$\tilde{u}(\cdot) = u^*(\Psi_N(\cdot)) \qquad \text{on } \mathcal{M}^4_N$$

$$\tilde{u}(a_i) = d^{-1}(u^*\omega_i) \quad , \quad \tilde{u}(b_{ij}) = -d^{-1}(d^{-1}(u^*\omega_i) \wedge u^*\omega_j)$$

$$\tilde{u}(b_{123}) = d^{-1}\left(u^*\eta_{123} - d^{-1}(u^*\omega_1) \wedge u^*\omega_2 - d^{-1}(u^*\omega_3) \wedge u^*\omega_3\right)$$

Thus the following forms are generating $(\pi_4((S^2 \times S^2)\#\mathbb{CP}^2) \otimes \mathbb{R})^*$: For all

$u \in C^1(S^4, (S^2 \times S^2)\#\mathbb{CP}^2)$,

$$\gamma_1(\,[u]\,) = \int_{S^4} u^*\omega_1 \wedge d^{-1}(u^*\omega_1) \wedge d^{-1}(u^*\omega_3)$$

$$\gamma_2(\,[u]\,) = \int_{S^4} u^*\omega_2 \wedge d^{-1}(u^*\omega_2) \wedge d^{-1}(u^*\omega_3)$$

$$\gamma_3(\,[u]\,) = \int_{S^4} u^*\omega_3 \wedge d^{-1}(u^*\omega_1) \wedge d^{-1}(u^*\omega_2)$$

$$\gamma_{13}(\,[u]\,) = \int_{S^4} -u^*\eta_{123} \wedge d^{-1}(u^*\omega_2) + u^*\omega_3 \wedge d^{-1}(u^*\omega_1) \wedge d^{-1}(u^*\omega_2)$$

$$+ \int_{S^4} u^*\omega_3 \wedge d^{-1}(u^*\omega_3) \wedge d^{-1}(u^*\omega_2) + u^*\omega_1 \wedge d^{-1}(u^*\omega_1) \wedge d^{-1}(u^*\omega_2)$$

$$\gamma_{23}(\,[u]\,) = \int_{S^4} -u^*\eta_{123} \wedge d^{-1}(u^*\omega_1) + u^*\omega_3 \wedge d^{-1}(u^*\omega_2) \wedge d^{-1}(u^*\omega_1)$$

$$+ \int_{S^4} u^*\omega_3 \wedge d^{-1}(u^*\omega_3) \wedge d^{-1}(u^*\omega_1) + u^*\omega_2 \wedge d^{-1}(u^*\omega_2) \wedge d^{-1}(u^*\omega_1)$$

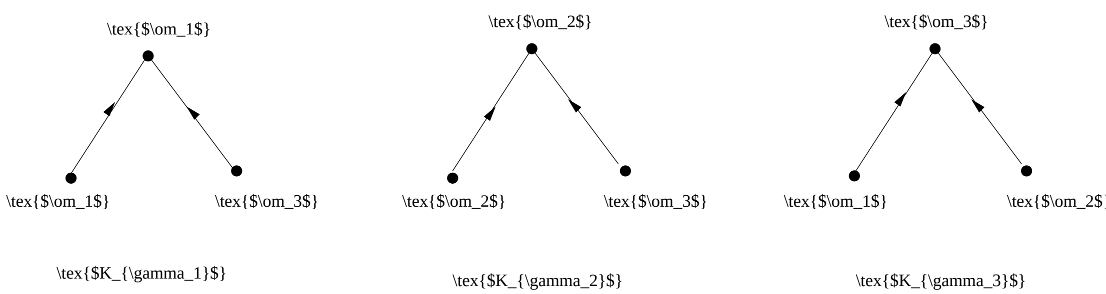

Figure 3 : The three first tree-graphs K_{γ_1},

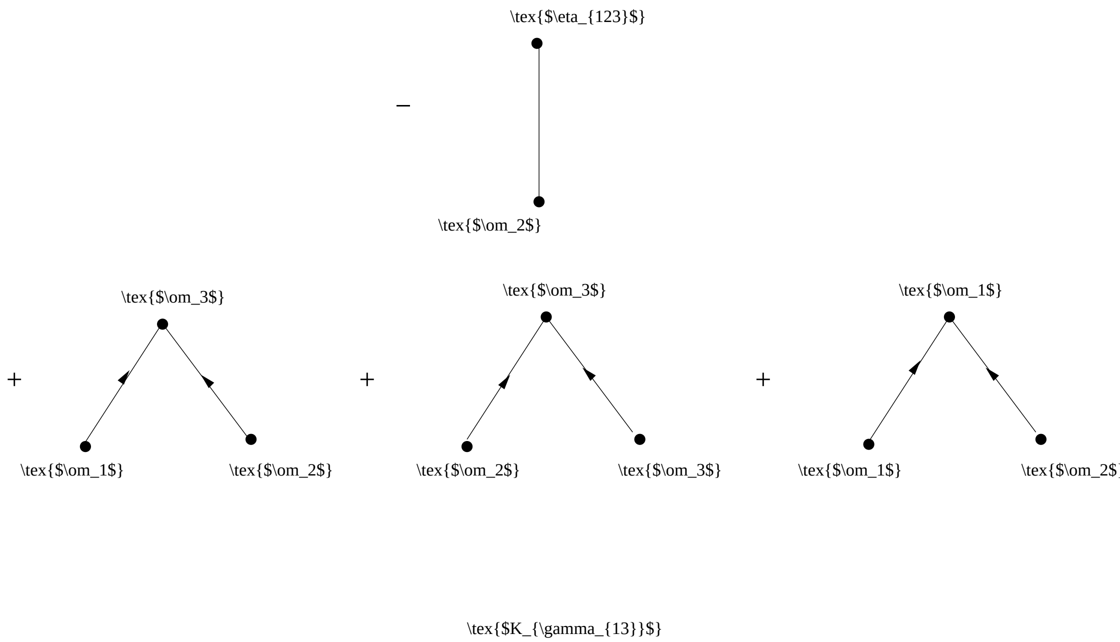

Figure 4 : The linear combination of tree-graph $K_{\gamma_{13}}$ arising in computing $\mathrm{Hom}(\pi_4(S^2 \times S^2 \# \mathbb{CP}^2), \mathbb{R})$.

Remark II.3 *Observe that we can restrict to graphs having only* <u>*closed*</u> *forms.*

Indeed, let $\mathcal{M}_N^k$ be the minimal model at the stage k (i.e. the ideal generated by the elements of degree less or equal to k). Suppose an element ξ of degree $k+1$ is introduced in the minimal model in order to kill some closed polynomial expression of degree $P(x_1, \cdots, x_l)$ of elements from $\mathcal{M}_N^k$, and ξ is not exact in $\mathcal{M}_N^k$ but $\Psi_N(\xi)$ is exact in N, we can decide that, for any $n \in \mathbb{N}$,

$$\tilde{u}(\xi) = d^{-1}(u^* \Psi_N(P(x_1, \cdots, x_l))$$

This modifies the graph as shown in the following example : replace for instance $u^*\eta_{123} = \tilde{u}(\beta_{123})$ by

$$d^{-1}(u^*(\omega_1\omega_2 - \omega_3^2)) \quad .$$

¿From the graph point of view this corresponds to the change described in the following figure

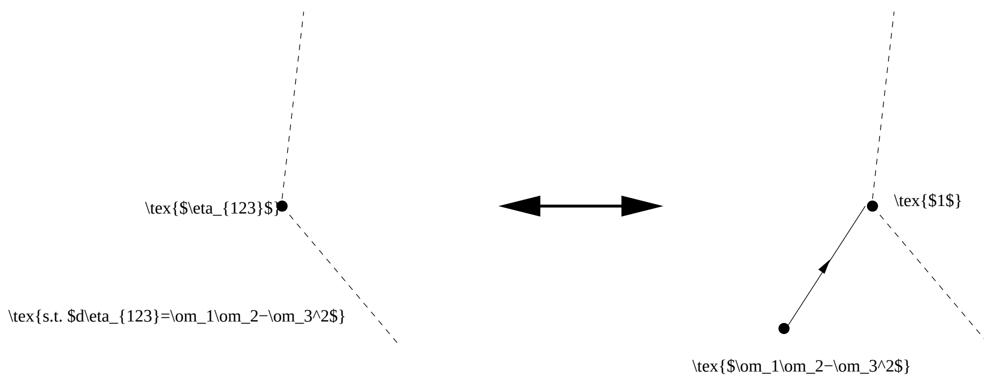

Figure 5: Replacing non closed forms in tree-graphs by closed ones.

II.4 Gauss Forms Associated to Elements in $\mathbf{Hom}(\pi_p(N), \mathbb{R})$.

To make analytic estimates, we need to have an *explicit* expression for evaluating an element of $\mathrm{Hom}(\pi_p(N), \mathbb{R})$. In this subsection we will define one specific integration operation d^{-1} by introducing certain Gauss forms associated with the tree-graphs described in II.2.

The Gauss forms are easier to describe explicitly with formulas in $\mathbb{R}^p$ instead of S^p. In this section we will consider, in place of $C^\infty(\wedge^q S^p)$, the subspace $\wedge^q_{slow}\mathbb{R}^p$ of smooth q forms ω in $\wedge^q\mathbb{R}^p$ satisfying

$$(1+|x|^2)^{k+q}|\nabla\omega|(x) \ \leq \ C_{\omega,k} \quad .$$

In particular, if

$$\pi \ : \ S^p \setminus \{(0,\cdots,0,1)\} \ \longrightarrow \ \mathbb{R}^p$$

denotes stereographic projection, then the pull-back by π^{-1} of any smooth q form on S^p is in $\wedge^q_{slow}\mathbb{R}^p$.

Let G be the Green's function for the Laplacian on $\mathbb{R}^p$:

$$G(x) = C_p|x|^{2-p} \text{ for } p>2 \quad \text{and} \quad G(x) = C_2\log|x| \text{ for } p=2 \ ,$$

where $C_p = (n-2)^{-1}|S^{p-1}|^{-1}$ and $C_2 = -(2\pi)^{-1}$. Given a $q-$form ω in $\wedge^q_{slow}\mathbb{R}^p$, $\omega = \sum_I \omega_I \ dx_I$ where $I = (i_1,\cdots,i_p)$ runs over all q tuples of $1,\cdots,p$ such that $i_1 < i_2 < \cdots < i_q$,we define the operator

$$d^{-1}\omega \ = \ d^*\Delta^{-1}\omega \ = \ d^*\sum_I G\star\omega_I \ dx_I \quad ,$$

where $dx_I = dx_{i_1} \cdots dx_{i_q}$, where $*$ is the Hodge operator for the flat metric on $\mathbb{R}^p$ and $\star$ the convolution operator. Observe that for $\omega \in \wedge^q_{slow}\mathbb{R}^p$, the convolution $\omega_I \star G$ is well-defined. If ω is closed, it is clear that

$$d\left(d^{-1}\omega\right) = \omega \quad .$$

We claim that there exists a form $\mathcal{G}^p_q$ in $\wedge^{q-1}\mathbb{R}^p_x \wedge \wedge^{p-q}\mathbb{R}^p_y$ such that for any ω in $\wedge^q_{slow}\mathbb{R}^p$

$$\left[d^*\Delta^{-1}\omega\right](x) = \int_{y\in\mathbb{R}^p} \omega(y) \wedge \mathcal{G}^p_q(x,y) \quad . \tag{II.6}$$

Indeed we have by definition

$$\left[d^*\Delta^{-1}\omega\right](x) = (-1)^{q(p-q)} * \sum_I \left[d\left(G \star \omega_I\right) \wedge *dx_I\right] \quad .$$

Letting I^c denote the $(p-q)-$tuple made of the complement of I ordered so that $dx_{I^c} = *dx_I$ (i.e. $dx_I \wedge dx_{I^c} = \omega_{\mathbb{R}^p} = dx_1 \cdots dx_p$), we then have,

$$\begin{aligned}
d^*\Delta^{-1}\omega(x) &= C_p(-1)^{q(p-q)} \sum_I * \int_{y\in\mathbb{R}^p} \omega_I(y)\; d\frac{1}{|x-y|^{p-2}} \wedge dx_{I^c} \wedge \omega_{\mathbb{R}^p}(y) \\
&= C_p(-1)^{q(p-q)}\,(2-p) * \sum_i \sum_I \int_{\mathbb{R}^p} \omega_I(y)\; \frac{x_i - y_i}{|x-y|^p}\; dx_i \wedge dx_{I^c} \wedge \omega_{\mathbb{R}^p}(y)
\end{aligned}$$

Also denoting $dx_{I_k} = dx_{i_1} \cdots dx_{i_{k-1}} dx_{i_{k+1}} \cdots dx_{i_q}$, we see that

$$*\left(dx_{i_k} \wedge dx_{I^c}\right) = (-1)^{k-1}(-1)^{(q-1)(p-q)} dx_{I_k} \quad .$$

With this notation one has

$$\begin{aligned}
&d^*\Delta^{-1}\omega(x) \\
&= C_p(-1)^{(p-q)}\,(2-p) \sum_I \int_{y\in\mathbb{R}^p} \omega_I(y)\; \omega_{\mathbb{R}^p}(y) \wedge \sum_{k=1}^{q} (-1)^{k-1} \frac{x_{i_k} - y_{i_k}}{|x-y|^p} dx_{I_k} \\
&= C_p(-1)^{(p-q)}\,(2-p) \int_{y\in\mathbb{R}^p} \omega(y) \wedge \sum_J \sum_{k=1}^{q} (-1)^{k-1} \frac{x_{j_k} - y_{j_k}}{|x-y|^p} dy_{J^c} \wedge dx_{J_k}
\end{aligned}$$

Thus the form

$$\mathcal{G}^p_q(x,y) \;=\; |S^{p-1}|^{-1}(-1)^{(p-q)} \sum_J \sum_{k=1}^{q} (-1)^{k-1} \frac{x_{j_k} - y_{j_k}}{|x-y|^p} dy_{J^c} \wedge dx_{J_k} \tag{II.7}$$

solves (II.6).

We introduce now the notion of a Gauss form associated to a simply connected tree-graph of closed forms. Let N be compact simply connected manifold. Consider Ψ_N a geometric realization of the minimal model $\mathcal{M}_N$ from N and consider K a simply connected tree-graphs of forms from the ideal generated by $\Psi_N(\mathcal{M}_N)$. Let A_i be the vertices of the graph, and let ω_{A_i} denote the closed form assigned to A_i. Let n_K be the number of segments in the graph. Denote also p_i the degree of ω_{A_i}. To each vertex we assign 2 variables x^i in $\mathbb{R}^p$ and y^i in N. To each vertex A_i we assign the sub-tree-graph K_i of K whose summit is A_i and made of the segments and vertices "below" A_i, that is the part of the graph connected to A_i as one follows the positively oriented paths ending at the summit of K see figure 6.

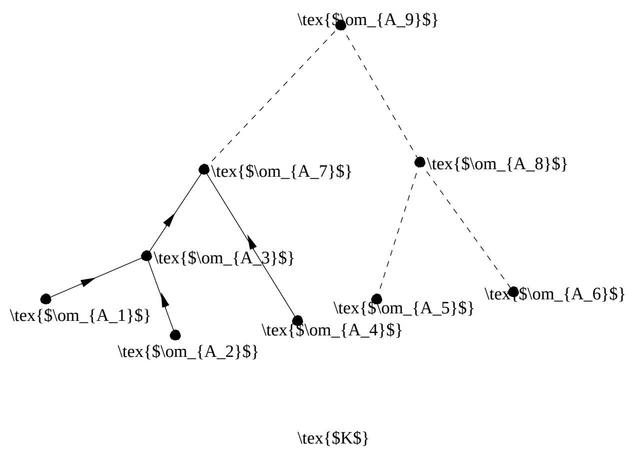

Figure 6 : Plain lines correspond to the sub-tree-graph K^7 with summit A_7.

We denote by n_i the degree of the form obtained from this graph :

$$n_i = [\sum_{A \text{ vertex in } K_i} deg\, \omega_A] - n_{K_i}$$

where n_{K_i} is again the notation for the number of segments in K_i. If A_i is a "starting vertex" with no other vertex below in the graph, then K_i is just made of A_i and $n_i = p_i$. We denote $\mathcal{I}$ the set of pairs of indices (i_1, i_2) such that A_{i_1} and A_{i_2} are connected by a segment in K going from A_{i_2} to A_{i_1}. The total form of the tree-graph is a form denoted $\omega^K \wedge \mathcal{G}^K$ in $\wedge_i \wedge^{p_i} N \wedge_{(i_1,i_2)\in\mathcal{I}} \wedge^{n_{i_2}-1}\mathbb{R}^p \wedge \wedge^p \mathbb{R}^{p-n_{i_2}}$ where ω^K and $\mathcal{G}^K$ are defined as follows

$$\omega^K = \wedge_i \omega_i(y^i). \tag{II.8}$$

and $\mathcal{G}^K$, the Gauss form associated to the simply connected tree-graph of forms K is defined by

$$\mathcal{G}^K = \wedge_{(i_1,i_2)\in\mathcal{I}} \mathcal{G}^p_{n_{i_2}}(x_{i_1}, x_{i_2}) \quad . \tag{II.9}$$

With any map $u \in C^\infty(S^p, N)$ we associate the map U_K from $(\mathbb{R}^n)^{n_K+1}$ into N^{n_K+1} defined by

$$U_K(x_1, x_2, \cdots, x_{n_K+1}) = \Big((u\circ\pi^{-1})(x_1), (u\circ\pi^{-1})(x_2), \cdots, (u\circ\pi^{-1})(x_{n_K+1})\Big) \ .$$

Recall that n_K, the number of segments in a simply connected tree-graph, is the number of nodes -1. Considering the "coordinates" $(y^1, \cdots, y^{n_K+1})$ on $(N)^{n_k+1}$, it is then easy to verify that

$$\int_{S^p} u^K \ = \ \int_{x_1} \cdots \int_{x_{n_K+1}} U_K^*(\omega^K) \wedge \mathcal{G}^K \quad . \tag{II.10}$$

where the integration operation d^{-1} of a form α in $\wedge^k S^p$ is given by

$$d^{-1}\alpha = \pi^*\left(\int_{y\in\mathbb{R}^p} (\pi^{-1})^*\alpha(y) \wedge \mathcal{G}^p_k(x,y)\right) \tag{II.11}$$

Considering now a class $z \in (\pi_p(N) \otimes \mathbb{R})^*$. Suppose $K = \sum_l \lambda_l K^l$ is the formal linear combination of tree-graphs of closed forms associated to z (for a given geometric geometric realization Ψ_N on $\mathcal{M}_N$), and u^{K^l} are the associated p forms, constructed in the previous subsection. With the p form

$$u^K \quad = \quad \sum_l \lambda_l u^{K^l} \quad ,$$

we now have, using the above notations,

$$z(\,[u]\,) \ = \ \int_{S^p} u^K \ = \ \sum_l \lambda_l \int_{(\mathbb{R}^p)^{n_{K^l}+1}} U^*_{K^l}(\omega^{K^l}) \wedge \mathcal{G}^{K^l} \quad . \tag{II.12}$$

We have thus succeeded in expressing the action of any element in $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ on $\pi_p(N)$ as linear combinations of pull-backs by u of closed forms depending only on the class we chose in $\mathrm{Hom}(\pi_p(N), \mathbb{R})$ (modulo of course a choice of geometric realization). This was the main goal of this section. This generalizes the integral expression of the topological degree as the integral of the pull-back of a form. This will be extremely useful for analysis purposes through the rest of the paper.

We establish now a slightly different form of the integral expression of $z(\,[u]\,)$ which is of particular geometrical interest. We first observe :

Lemma II.2 *Let $\Pi_{q,p}$ denote the canonical projection of $\wedge^{p-1}(\mathbb{R}^p_x \times \mathbb{R}^p_y)$ onto $\wedge^{q-1}\mathbb{R}^p_x \wedge^{p-q} \mathbb{R}^p_y$. The following identity holds*

$$\Pi_{p,q}\left[\frac{x-y}{|x-y|}^{\,*}\left(\sum_{k=1}^{p}(-1)^{k-1}X_k\,(dX)^k\right)\right] = (-1)^{(q-1)(p-q)}|S^{p-1}|^{-1}\ \mathcal{G}^p_q(x,y) \tag{II.13}$$

where $(dX)^k$ *is the* $p-1$ *form in* $\mathbb{R}^p$ *given by* $dX_1 \cdots dX_{k-1} dX_{k+1} \cdots dX_p$ *and* $\mathcal{G}^p_q(x,y)$ *is the form given by (II.7).*

Proof of Lemma II.2. A classical computation gives

$$\frac{x-y}{|x-y|}^* \left(\sum_{k=1}^{p} (-1)^{k-1} X_k \, (dX)^k \right) = \sum_{k=1}^{p} (-1)^{k-1} \frac{x_k - y_k}{|x-y|^p} (d(x-y))^k \quad , \tag{II.14}$$

where $(d(x-y))^k = d(x_1-y_1)\cdots d(x_{k-1}-y_{k-1}) d(x_{k+1}-y_{k+1}) \cdots d(x_p - y_p)$. With the previous notations, $dx_{I^c} \wedge dx_{I_k} = (-1)^{k-1}(-1)^{(q-1)(p-q)} * dx_{i_k}$, and therefore it is clear that

$$\begin{aligned} &\Pi_{p,q} \left[\sum_{k=1}^{p} (-1)^{k-1} \frac{x_k - y_k}{|x-y|^p} (d(x-y))^k \right] \\ &= (-1)^{(q-1)(p-q)} \sum_J \sum_{k=1}^{q} (-1)^{k-1} \frac{x_{j_k} - y_{j_k}}{|x-y|^p} dy_{J^c} \wedge dx_{J_k} \quad . \end{aligned} \tag{II.15}$$

The desired equality (II.13) then follows from (II.22). ■

We now prove the following result:

Lemma II.3 *Let* K *be a simply connected tree-graph of forms in* $\wedge^* N$ *of dimension* p*. Let* $u \in C^\infty(S^p, N)$*. Then the following formula holds*

$$\begin{aligned} \int_{S^p} u^K &= \int_{x_1} \cdots \int_{x_{n_K}} U_K^*(\omega^K) \wedge \mathcal{G}^K \\ &= |S^{p-1}|^{-n_K} (-1)^{m_K} \int_{x_1} \cdots \int_{x_{n_K}} U_K^*(\omega^K) \wedge_{(i_1,i_2)\in\mathcal{I}} \frac{x_{i_1} - x_{i_2}}{|x_{i_1} - x_{i_2}|}^* \Omega_{S^{p-1}} \quad , \end{aligned} \tag{II.16}$$

where $\Omega_{S^{n-1}} = \sum_{k=1}^n (-1)^{k-1} X_k \, (dX)^k$ *and* m_K *is some integer depending on* K*.*

Proof of Lemma II.3. This can be proved by induction on the number of segments in the graph. Consider an end node i_2 in the graph (which has no segment pointing to it), and let i_1 be the node to which i_2 is connected. Let ω_{i_1} and ω_{i_2}, of degree n_{i_1} and n_{i_2}, be the forms assigned to each of these two nodes. By the previous lemma and dimensional reasons, it is clear that

$$
\begin{aligned}
&|S^{p-1}| \int_{x_{i_2}} (u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2}) \wedge \mathcal{G}^p_{n_{i_2}}(x_{i_1},x_{i_2}) \\
&= (-1)^{(n_{i_2}-1)(p-n_{i_2})} \int_{x_{i_2}} (u\circ\pi^{-1})^*\omega_2(x_{i_2}) \wedge \Pi_{p,n_{i_2}} \left[\frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}^* \Omega_{S^{p-1}} \right] \\
&= (-1)^{(n_{i_2}-1)(p-n_{i_2})} \int_{x_{i_2}} (u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2}) \wedge \frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}^* \Omega_{S^{p-1}}
\end{aligned}
\tag{II.17}
$$

We then modify the graph by removing the node i_2 along with the branch starting from it and by changing ω_{i_1} into the form $\omega'_{i_1}(x_{i_1})$ given by

$$
(-1)^{(n_{i_2}-1)(p-n_{i_2})}|S^{p-1}|^{-1}\omega_{i_1}(x_{i_1}) \wedge \int_{x_{i_2}} (u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2}) \wedge \frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}^* \Omega_{S^{p-1}} \quad .
$$

We then apply to this new graph, having one less node, the induction assumption and Lemma II.3 is proved. ■

Remark II.4 *The construction of the tree-graph of forms can be described in terms of linking and "angular forms" (see [BT]).*

To see this, again let π denote the stereographic projection from S^p into $\mathbb{R}^p$, singular at the north pole, and introduce the **linking map** or **segment map**

$$
\begin{aligned}
L \ : \ S^p \times S^p \ &\longrightarrow \ S^{p-1} \\
(x,y) \ &\longmapsto \ \frac{\pi(x)-\pi(y)}{|\pi(x)-\pi(y)|} \quad .
\end{aligned}
\tag{II.18}
$$

Let $\Omega_{S^{p-1}}$ be a $p-1-$form on S^{p-1} satisfying $\int_{S^{p-1}} \Omega_{S^{p-1}} = 1$. It is interesting to observe that

$$
d(L^*\Omega) \ = \ T_\Delta \ - \ T_{\{Nord\}\times S^p} \ - \ T_{S^p\times\{Nord\}}
$$

where Δ denotes the diagonal $\{(x,y)\in S^p\times S^p \ : \ x=y\}$,$Nord=(1,\cdots,0,1)$ is the north pole of S^p, and T_Δ, $T_{\{Nord\}\times S^p}$ and $T_{S^p\times\{Nord\}}$ are the currents of integration along respectively Δ, $\{Nord\}\times S^p$ and $S^p\times\{Nord\}$. Then if one restricts to forms in $\oplus_{k=1}^p \wedge^k S^p \wedge^{p+1-k} S^p$, $L^*\Omega$ plays the role of an "angular form" (see [BT]) of the diagonal Δ in $S^p\times S^p$ (observe that it was not possible to find $\beta\in\wedge^{p-1}(S^p\times S^p)$, even singular, such that $d\beta = T_\Delta$ in $\mathcal{D}'\wedge^p(S^p\times S^p)$). Thus in our representation by tree-graphs of forms of elements of $\mathrm{Hom}(\pi_p(N),\mathbb{R})$, two connections to one vertex by segments of two sub-tree-graphs corresponds to a connection of the two corresponding forms by wedging them together with the angular form of the diagonal Δ in $S^p \times S^p$. This implies in particular that if we replace everywhere $L^*\Omega$ by any form whose restriction to $\oplus_{k=1}^p \wedge^k S^p \wedge^{p+1-k} S^p$ is the angular form of the diagonal Δ in $S^p \times S^p$ (for instance $L_\Psi^*\Omega$, where Ψ is an arbitrary diffeomorphism of S^p and $L_\Psi(x,y) = L(\Psi(x),\Psi(y))$), the computation of $\int_{S^p} u^K$ is unchanged.

II.5 Critical Exponents Associated to Elements in Hom$(\pi_p(N),\mathbb{R})$.

II.5.1 Definition of the Critical Exponent.

Lemma II.4 *Given a simply connected tree-graph of forms K, then, there exists a constant C_K such that, for any map $u \in C^\infty(S^p, N)$, the following inequality holds*

$$\left|\int_{S^p} u^K\right|^{\frac{p}{p+n_K}} \le C_K \int_{S^p} |\nabla u|^p \, d\mathcal{H}^p \tag{II.19}$$

where n_K is the number of segments in K.

Definition II.1 *In a formal linear combination $K = \sum_l \lambda_l K^l$ of tree-graphs of closed forms, consider the total number of segments in all the K^l. The* **critical exponent** *of a generator $z \in Hom(\pi_p(N),\mathbb{R})$ is the number*

$$\frac{p}{p+n_z}$$

where n_z is the minimum of such total number of segments among geometric realizations Ψ_N of the minimal model $\mathcal{M}_N$ and among all formal linear combinations K of tree-graphs of closed forms associated to z via Ψ_N. If K is a linear combination of tree-graphs of closed forms representing z with minimum total number of segments, we say that K is **optimal** *for z*

Remark II.5 *The constraint of restricting to tree-graphs of closed forms is important. By allowing tree-graphs of non closed forms the maximal number of branches could decrease (see Remark II.3) but won't be the one we are defining.*

Proof of Lemma II.4. The standard Sobolev inequality gives that for any q form ω on S^p with $1 < q < p$, there exists a constant C_q such that

$$\|d^*\Delta^{-1}\omega\|_{\frac{p}{q-1}} \;\le\; C_q\|\omega\|_{\frac{p}{q}} \quad . \tag{II.20}$$

Given a sub-tree-graph K_i one first easily proves, by induction on the number of branches of K^i, that

$$\|u^{K_i}\|_{\frac{p}{q_i}} \ \leq \ \prod_{j\in J} \|u^*\omega_j\|_{\frac{p}{p_j}} \quad , \tag{II.21}$$

where J is the set of indices of vertices in K_i, ω_j is the element of the ideal generated by $\Psi_N(\mathcal{M}_N)$ at the vertex j, p_j is it's degree, and $q_i = \sum_{j\in J} p_j - n_{K_i}$ (where, as before, n_{K_i} is the number of segments in K_i). For $K_i = K$, (II.21) implies

$$\|u^K\|_1 \ \leq \ C_K \prod_i \|\nabla u\|_p^{p_i} \quad . \tag{II.22}$$

Combining now (II.5) and (II.22), we obtain (II.19) and Lemma II.4 is proved.

■

II.5.2 Optimality of the Critical Exponent.

An important question is to know whether the critical exponent of a generator z of the minimal model is optimal or not : That is, given a sequence of maps u_k from S^p into N such that

$$\lim_{k\to\infty} [z](\,[u_k]\,) \ = \ \sum_l \lambda_l \int_{S^p} u_k^{K^l} \ = \ a_k \to \ \infty \quad , \tag{II.23}$$

and denoting

$$E_k \ = \ \inf\left\{\int_{S^p} |\nabla u|^p \, d\mathcal{H}^p \quad : \quad [z](\,[u]\,) = a_k\right\} \quad ,$$

do we have

$$\lim_{k\to\infty} \frac{\log E_k}{\log a_k} \ = \ \frac{p}{p+n_z} \quad ? \tag{II.24}$$

If $z(\,[u_k]\,) \neq 0$, then there exists a positive constant $\varepsilon_{N,p}$, depending only on p and N, such that

$$\int_{S^p} |\nabla u|^p \, d\mathcal{H}^p \quad \geq \quad \varepsilon_{N,p}$$

(see [Wh]). Combining this fact and Lemma II.4, we have the existence of $C_{z,N}$ depending only on z and the metric on N such that

$$z(\,[u]\,)^{\frac{p}{p+n_z}} \ \leq \ C_{z,N} \int_{S^p} |\nabla u|^p \, d\mathcal{H}^p \ .$$

for all $u \in W^{1,p}(S^p, N)$. Thus the inequality

$$\liminf_{k\to\infty} \frac{\log E_k}{\log a_k} \geq \frac{p}{p+n_z} \tag{II.25}$$

is clear. The difficulty is whether a reverse inequality holds or not. A similar question was raised in [Gr1] and [Gr2].

Definition II.2 *For a fixed nonzero element z in $Hom(\pi_p(N), \mathbb{R})$, we say that the critical exponent $p(p+n_z)^{-1}$ is optimal if (II.24) holds for some sequence $[u_k]$ in $\pi_p(N)$ satisfying (II.23).* ■

It was proved in [Ri] that this always holds if for instance N is a sphere (if $N \simeq S^q$, then, for $p = q$, $n_z = 0$ and, for $p = 2q-1$ and q even, $n_z = 1$). Answering this question in general seems to be an interesting difficult open problem. We do not try to make a systematic presentation of this problem in the present work, but just illustrate this question by considering more specific N (related to the examples we exposed in the previous subsection) for which the critical exponents are always optimal. Precisely we have.

Proposition II.3 *Let N be a 4-dimensional Riemannian manifold diffeomorphic to the connected sum $\#\mu\,\mathbb{CP}^2\#\nu\,\overline{\mathbb{CP}^2}\#\xi(\,S^2\times S^2)$ where μ, ν and ξ are 3 arbitrary natural numbers, ($\#m\cdots$ denotes the connected sum of m copies of $\cdots$ and $\overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ with the opposite orientation to the standard one). Then for any $p\in\mathbb{N}$ and nonzero $z\in Hom(\pi_p(N),\mathbb{R})$ the critical exponent $p(p+n_z)^{-1}$ is optimal.*

Proof of Proposition II.3. Denote $\omega_1\cdots\omega_\mu \in H^2(N)$ the Poincaré duals of each of the standard $\mathbb{C}P^1$ embedded in each of the $\mathbb{CP}^2$ in the connected sum $\#\mu\mathbb{CP}^2\#\nu\overline{\mathbb{CP}^2}\#\xi(S^2\times S^2)$. Similarly, denote $\overline{\omega}_1\cdots\overline{\omega}_\nu \in H^2(N)$ the Poincaré duals of each of the standard $\overline{\mathbb{CP}^1}$ embedded in each of the $\overline{\mathbb{CP}^2}$ in the connected sum $\#\mu\mathbb{CP}^2\#\nu\overline{\mathbb{CP}^2}\#\xi S^2\times S^2$. And finally denote $\alpha_1\cdots\alpha_\xi$ and $\beta_1\cdots\beta_\xi$ the Poincaré duals respectively of each of the $S^2\times\{Nord\}$ and each of the $\{Nord\}\times S^2$ in the connected sum. For any $k\in\mathbb{Z}$, we first construct a map F_k from $N\simeq\#\mu\mathbb{CP}^2\#\nu\overline{\mathbb{CP}^2}\#\xi(S^2\times S^2)$ into itself such that

$$\begin{array}{c} F_k^*\omega_h = k\omega_h \quad \text{for } h=1,\cdots,\mu \quad , \quad F_k^*\overline{\omega}_i = k\,\overline{\omega}_i \quad \text{for } i=1,\cdots,\nu \\ \text{and } F_k^*\alpha_j = k\ \alpha_j \quad \text{and} \quad F_d^*\beta_j = d\ \beta_j \quad \text{for} j=1,\cdots,\xi \quad , \end{array} \tag{II.26}$$

and

$$\|\nabla F_k\|_\infty \ \leq C\sqrt{k} \tag{II.27}$$

with C independent of k. First, the existence of F_k, in the case where $\mu = 0$, $\nu = 0$ and $\xi = 1$, is quite elementary to establish : it is not difficult (see [Ri]) to construct a family of maps ϕ_k from S^2 into itself such that

$$\deg \phi_k \ = \ k \quad \text{ and } \quad \|\nabla^j \phi_k\|_\infty \ \leq \ C_j \sqrt{k^j} \quad ,$$

where C_j is independent of k. Then observe that $F_k(x, y) = (\phi_d(x), \phi_d(y))$ is a solution to (II.26) and (II.27) in the case $N = S^2 \times S^2$.

We now construct F_k solving (II.26) and (II.27) in the case $N = \mathbb{CP}^2$. We split first $\mathbb{CP}^2$ into two parts $N^1 \simeq E$ where E is diffeomorphic to the *Hopf disk bundle* over S^2 which is the D^2 bundle whose principal bundle is the S^1 *Hopf bundle* which is diffeomorphic to S^3 (we then have $\partial E \simeq S^3$) and $N^2 \simeq B^4$. We first construct F_k from N^1 into N^1. Let H be the Hopf fibration from $\partial E \simeq S^3$ into S^2 and let ϕ_k be the map described above. We claim that we can lift ϕ_k to a map $\tilde{\phi}_k \ : \ S^3 \to S^3$ (i.e $H \circ \tilde{\phi}_k = \phi_k \circ H$) satisfying

$$\|\nabla \tilde{\phi}_k\|_\infty \ \leq C \sqrt{k} \quad , \tag{II.28}$$

where C is independent of k. We follow the idea in [HR1]. Let (e_1^*, e_2^*, e_3^*) be the orthonormal coframe of $\wedge^1 S^3$ given by the Lie Group action on S^3 starting from $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ at $(1, 0, 0, 0)$. Classical computations give $2^{-1} de_i^* = e_{i+1}^* \wedge e_{i-1}^*$ where we use indexation in $\mathbb{Z}_3$. We get the *Coulomb Hopf lift* (see [HR1]) of ϕ_k in the following way : there exists $\tilde{\phi} = \tilde{\phi}_k$ satisfying (forgetting the subscript k)

$$\begin{aligned}
\tilde{\phi}^* e_1^*(x) \quad &= \ (dH_{\tilde{\phi}(x)} \cdot e_1; d\phi_{H(x)} \cdot (dH_x \cdot e_1))\, e_1^*(x) \\
&\quad + (dH_{\tilde{\phi}(x)} \cdot e_1; d\phi_{H(x)} \cdot (dH_x \cdot e_2))\, e_2^*(x) \\
\tilde{\phi}^* e_2^*(x) \quad &= \ (dH_{\tilde{\phi}(x)} \cdot e_2; d\phi_{H(x)} \cdot (dH_x \cdot e_2)) e_2^*(x) \\
&\quad + (dH_{\tilde{\phi}(x)} \cdot e_2; d\phi_{H(x)} \cdot (dH_x \cdot e_1))\, e_1^*(x) \\
\tilde{\phi}^* e_3^*(x) \quad &= \ \eta(x)
\end{aligned}$$

where (e_1, e_2, e_3) is the dual basis to (e_1^*, e_2^*, e_3^*), $(\cdot; \cdot)$ is the scalar product on S^2 and η is the 1 form on S^3 solving the following elliptic problem

$$\left\{ \begin{array}{l} d\eta \ = \ \dfrac{1}{2} H^* \phi^* \omega_{S^2} \\[2ex] d^*\eta \ = \ 0 \quad , \end{array} \right.$$

where ω_{S^2} is the volume form on S^2. We observe that the operator

$$L: \ C^2(\Omega, R^4\backslash B_{\frac{1}{2}}) \longrightarrow C^0(\Omega, R^4) \quad ,$$

$$L(u) \ = \ \left(d^*(u^*e_1^*), d^*(u^*e_2^*), d^*(u^*e_3^*), d^*\left(u^*\frac{\partial}{\partial r}\right)\right)$$

where $\Omega = B_2^4 \setminus B_{\frac{1}{2}}^4$, is elliptic. Using a classical interpolation result (see [BBH]), we get that for any subdomain Ω' of Ω one has

$$\|\nabla u\|^2_{L^\infty(\Omega')} \ \leq \ C\|u\|_{L^\infty(\Omega)}\|Lu\|_{L^\infty(\Omega)} + C\|u\|^2_{L^\infty(\Omega)}$$

Applying this result to $u = \tilde{\phi}_k \circ (\frac{x}{|x|})$, we obtain that

$$\begin{aligned} \|\nabla\tilde{\phi}_k\|^2_\infty \ &\leq C\|\tilde{\phi}_k\|_\infty \left[\|\nabla^2\phi_k\|_\infty + \|\nabla\tilde{\phi}_k\|_\infty\|\nabla\phi_k\|_\infty\right] + \|\tilde{\phi}_k\|^2_\infty \\ &\leq \ Cd + \sqrt{k} \ \|\nabla\tilde{\phi}_k\|_\infty \end{aligned}$$

Thus we have found a family of liftings of the ϕ_k that satisfies (II.28). Now we can extend $\tilde{\phi}_k$ to a map from E into E by homogeneity : We take the flat metric on each fiber of the disc bundle E over S^2 and we take $F_k(x) = |x|_1\tilde{\phi}_k(x/|x|_1)$ where $|x|_1$ is the distance of x to the zero section $\simeq S^2$ of the disk bundle and where we are using the linearity on the D^2 fibers . It is clear that F_k so defined on N_1 satisfies $\|\nabla F\|_{L^\infty(N_1)} \leq C\sqrt{k}$. On $N_2 \simeq B^4$ we define F_k from $N_2 \simeq B^4$ into $N_2 \simeq B^4$ by $F_k(x) = |x|_2\tilde{\phi}_k(x/|x|_2)$ where this time $|x|_2$ is the distance to the center 0 of B^4 and where we are using the linearity on $B^4 \subset \mathbb{R}^4$. By gluing the two pieces N_1 and N_2 together we get a family of maps F_k from $\mathbb{CP}^2$ into $\mathbb{CP}^2$ satisfying (II.26) and (II.27). Thus we have then F_k in the cases $(\mu,\nu,\xi) = (0,0,1)$ $(\mu,\nu,\xi) = (0,1,0)$ and $(\mu,\nu,\xi) = (1,0,0)$. We get F_k for general (μ,ν,ξ) by simple iterative gluing of the previous ones.

Consider now a class z in $(\pi_p(N)\otimes\mathbb{R})^*$. Since $H^*(N)$ is generated by the classes ω_i, $\overline{\omega}_i$ α_i and β_i, each form at each node of every tree graph K_l arising in the linear combination K representing z is a non exact 2 form representing one of the class above or $4-$ forms, wedge of two nonexact two forms of this family. We restrict to tree-graphs which are optimal in the sense of the

definition II.1. Let $u \ : \ S^p \rightarrow N$ be such that $z(\,[u]\,) \neq 0$. Consider now $\sum_l \lambda_l(F_k\circ u)^{K^l}$. It is a polynomial in $\sqrt{k}$ of the form

$$z(\,[u]\,) \ = \ \sum_l k^{\frac{n_l+p}{2}}\lambda_l\int_{S^p} u^{K^l} \ ,$$

where we have used the identity (II.5). Since K is optimal $n_l \leq n_z$. Assume that for every u such that $z(\,[u]\,) \neq 0$ the coefficient in front of $k^{\frac{n_z+p}{2}}$ is always

0, then, in representing z, we could remove all trees that contain n_z node, and we will prove that the minimal number of nodes for representing z would be strictly less than n_z which would be a contradiction. Thus we may choose u such that the coefficient in front of $k^{\frac{n_z+p}{2}}$ is non zero, and we have

$$A_k \;=\; z(\,[F_k \circ u]\,) \;=\; a d^{\frac{n_z+p}{2}} + P(\sqrt{d}) \quad , \tag{II.29}$$

where $a \neq 0$ and $\deg P \;<\; n_z + p$. Observe that we have

$$\begin{aligned} E_k \;=\; \inf\left\{\int_{S^p} |\nabla u|^p \; z(\,[u]\,)\, d\mathcal{H}^p \;=\; A_k\right\} \quad &\le\; \textstyle\int_{S^p} |\nabla F_k \circ u|^p \\ &\le\; C_u \|\nabla F_k\|_\infty^p \;\le\; C_u k^{\frac{p}{2}} \;, \end{aligned}$$

which, combined with (II.29), implies that

$$\limsup_{k\to\infty} \frac{\log E_k}{\log A_k} \;\le\; \frac{p}{n_z + p}$$

¿From this inequality and (II.25), we deduce that

$$\lim_{k\to\infty} \frac{\log E_k}{\log A_k} \;=\; \frac{p}{n_z + p}$$

so that $p\,(n_z + p)^{-1}$ is optimal. Proposition II.3 is proved. ∎

II.6 Rigidity Property of Linear Combinations of Tree-graphs and Interpretation of Homotopy Integrals as Linking Numbers.

One question we did not address yet is the invariance of the isomorphism $V^p \simeq \mathrm{Hom}(\pi_p(N), \mathbb{R})$ under small deformation of the geometric realization Ψ_N. In the previous subsections we have constructed the formal linear combination of simply connected tree-graphs of forms $K = \sum_l \lambda_l K^l$ which have the **homotopy property** : For any u

$$\int_{S^p} u^K \quad \textit{remains unchanged under homotopic deformation of } u \;,$$

and which therefore correspond to an element in $\mathrm{Hom}(\pi_p(N), \mathbb{R})$. This is why these integrals are also called *homotopy integrals*. Now we address the question of finding the formal linear combination of simply connected tree-graphs of forms of $\Psi_N(\mathcal{M}_N)$ which have the **rigidity property** : For every smooth u from S^p into N, $\int_{S^p} u^K$ is unchanged under a deformation of Ψ_N by adding an exact form to every closed generator of $\Psi_N(\mathcal{M}_N)$ (see [Nov1], [Nov2] and [Nov3]). For instance, for $\pi_3(S^2)$ the linear form

$$\int_{S^3} u^*\omega \wedge d^*\Delta^{-1}(u^*\omega')$$

where $\int_{S^2}\omega = 1$ and $\int_{S^2}\omega' = 1$ remains unchanged as one adds to ω and ω' arbitrary exact 2 forms $d\alpha$ and $d\alpha'$. This means that this tree graph (made of 2 vertices connected by 1 segment and where one generator of $H^2(S^2)$ is assigned at each vertex) has the rigidity property. This problem of the rigidity property under deformation of geometric realization of minimal models was first raised in [Nov1], and sufficient conditions for this rigidity property to hold are given in [Nov2], [Nov3], [Mi1] and [Mi2].

We say that an element in $H^*(N)$ is in general position if the corresponding Poincaré dual is an integer multiplicity combination of simplices (i.e. their integral on cycles in $H_*(N,\mathbb{Z})$ are in $\mathbb{Z}$). The goal of this subsection is to prove the following result.

Proposition II.4 *Let N be a compact simply connected manifold, and Ψ_N be a geometric realization of the minimal model $\mathcal{M}_N$. Suppose z is an element of $Hom(\pi_p(N),\mathbb{R})$ admitting a representation via Ψ_N by a linear combination in $\mathbb{Z}$ of simply connected tree-graphs K_l of closed forms that we take in general position. Assume also that K has the rigidity property and finally that in each K^l, every pair of closed form connected by a segment have Poincaré duals that can be represented by disjoint closed polyhedral chains. Then for any map u from S^p into N, one has*

$$\int_{S^p} u^K \in \mathbb{Z}$$

Remark II.6 *We can always choose a basis of $H^*(N)$ which is in general position.*

In fact, first take free generators of $H_*(N,\mathbb{Z})$. They form a basis in $H_*(N,\mathbb{R})$ (see for instance [GMS] 5.4.1) and the Poincaré duals of these classes form a basis of $H^*(N)$ in general position. We can then proceed to the construction of a geometric realization Ψ_N of the minimal model of N starting from these classes. In order to apply Proposition II.4, it remains to check both the rigidity property of K representing z and whether each pair of forms connected by a segment in the tree graph can be realized by disjoint cycles. Observe, as an illustration of Proposition II.4, that all these conditions are fulfilled by each example we gave above ($\pi_p(\mathbb{C}P^2)$, $\pi_p(\mathbb{C}P^1\times\mathbb{C}P^1)$

and $\pi_p(\mathbb{CP}^2\#(\mathbb{CP}^1\times\mathbb{CP}^1))$ for $p=2,3,4$ except for $K_{\gamma_{13}}$ and $K\gamma_{23}$ arising while computing $\pi_4(\mathbb{CP}^2\#(\mathbb{CP}^1\times\mathbb{CP}^1))$ where our geometric interpretation of $\int_{S^p}u^K$ is no longer valid.

Proof of Proposition II.4. Let K^l be a tree-graph of closed forms arising in the linear combination of tree-graphs of forms $K=\sum_l\lambda_l K^l$ in the representation of z. Let $\omega_1\cdots\omega_{n_{K^1}}$ be the closed forms at the nodes of K^l. Let C_i be the closed polyhedral chains representing the Poincaré duals of the ω_i such that if $(i_1,i_2)\in\mathcal{I}$ then $C_{i_1}\cap C_{i_2}=\emptyset$ (recall that $\mathcal{I}$ is the set of pairs of indices whose corresponding nodes are connected by a segment in the tree-graph). Assume, to simplify the presentation, that the C_i are smooth submanifolds of N of dimension $p_i=n-\deg\omega_i$ (for general polyhedral chains the approach below requires a more technical presentation that we skip). Let N_i be an open tubular neighborhood of C_i (diffeomorphic to the normal bundle of C_i in N) that we choose sufficiently thin in order to guaranty that $\forall(i_1,i_2)\in\mathcal{I}$, $N_{i_1}\cap N_{i_2}=\emptyset$. We denote by π_i the orthogonal projection from N_i into C_i. We replace ω_i by a (cohomologically equivalent to ω_i) representative of the Thom form of N_i supported in N_i and whose integral along each $n-p_i-$plane perpendicular to C_i gives 1. We keep denoting ω_i this new representative of $[\omega_i]$ and since K has the rigidity property $\int_{S^p}u^K$ is not altered by this change of geometric realization. Let S_i be a $\mathbb{R}^{q_i}$ vector bundle over C_i whose sum with N_i gives a trivial bundle $N_i\oplus S_i\simeq C_i\times\mathbb{R}^{n-p_i+q_i}$ and let $\tilde{\omega}_i$ be a representative of the Thom class of S_i. Let $\Omega_i=u^{-1}(N_i)$ and denote by E_i the pull-back bundle of S_i by $\pi_i\circ u$ over Ω_i

$$E_i=(\pi_i\circ u)^{-1}S_i\quad.$$

Denote by Π_i the projection map from E_i into N_i and, by ϕ_i the canonical bundle map from E_i into S_i lifting $\pi_i\circ u$ and realizing an isomorphism from any fiber of E_i into the image fiber by $\pi_i\circ u$. Finally we denote by Φ_i the following map

$$\begin{aligned}\Phi_i:\ E_i &\longrightarrow S_i\oplus N_i\simeq C_i\times\mathbb{R}^{n-p_i+q_i}\\ x &\longrightarrow \phi_i(x)+u(\Pi_i(x))\end{aligned}$$

Following [BT] we denote $(\Pi_i)_*$ the integration on E_i along the fibers which assign a $\wedge^{k-q_i}N_i$ form to any $\wedge^k E_i$ form. Using the projection formula (Proposition 6.15) in [BT], we have for any q form $\alpha\in C^\infty(\wedge^q N_i)$

$$(\Pi_i)_*(\Phi_i^*\omega_i\wedge\tilde{\omega}_i\wedge\Pi^*\alpha)\ =\ u^*\omega_i\wedge\alpha\quad.$$

This implies in particular for $q=p-\deg\omega_i$ that

$$\int_{E_i}\Phi_i^*(\omega_i\wedge\tilde{\omega}_i)\wedge\Pi_i^*\alpha\ =\ \int_{S^p}u^*\omega_i\wedge\alpha \tag{II.30}$$

We assume that each N_i avoids the north pole and denote by L the **linking map** defined by (II.18). Furthermore let $\Omega_{S^{p-1}}$ be a $p-1$−form on S^{p-1} satisfying $\int_{S^{p-1}} \Omega_{S^{p-1}} = 1$. Following (II.16), we have that

$$
\begin{aligned}
&(-1)^{m_K} \int_{S^p} u^{K^l} \\
&= \int_{S^p} \cdots \int_{S^p} \wedge_{i=1}^{n_{K^l}+1} u^*\omega_i(x_i) \wedge_{(i_1,i_2)\in\mathcal{I}} L^*\Omega_{S^{p-1}}(x_{i_1}, x_{i_2}) \quad .
\end{aligned}
\tag{II.31}
$$

Combining (II.30) and (II.31), we then have

$$
\begin{aligned}
&(-1)^{m_K} \int_{S^p} u^{K^l} \\
&= \int_{E_1} \cdots \int_{E_{n_K+1}} \wedge_{i=1}^{n_{K^l}+1} \Phi_i^*(\omega_i \wedge \tilde{\omega}_i)(z_i) \wedge_{I\in\mathcal{I}} \Pi_I^* L^*\Omega_{S^{p-1}}(z_{i_1}, z_{i_2}) \quad ,
\end{aligned}
$$

where $\Pi_I(z_{i_1}, z_{i_2}) = (\Pi_{i_1}(z_{i_1}), \Pi_{i_2}(z_{i_2}))$. Since the N_i are disjoint and also disjoint from the north pole of S^p, $L\circ\Pi_I \in C^\infty(E_{i_1}\times E_{i_2}, S^{p-1})$ for all $I \in \mathcal{I}$. Thus we have, for all $I \in \mathcal{I}$

$$
d\left(L^*\Omega_{S^{p-1}}(z_{i_1}, z_{i_2})\right)) \;=\; 0 \qquad \text{in } E_{i_1}\times E_{i_2}
\tag{II.32}
$$

Let $\Xi_i \;:\; S_i \oplus N_i \to C_i \times \mathbb{R}^{n-p_i+q_i}$ be a bundle isomorphism and let P_i be the canonical projection from $C_i \times \mathbb{R}^{n-p_i+q_i}$ into $\mathbb{R}^{n-p_i+q_i}$ which assigns X to (x, X). Using (II.32) since $\omega_i \wedge \tilde{\omega}_i$ is cohomologous to

$$
A_i \;=\; \Xi_i^* P_i^*(f_i(X) dX_1 \wedge \cdots \wedge dX_{n-p_i+q_i}) \quad ,
$$

where f_i is the characteristic function of the unit ball $B_1^{n-p_i+q_i}$ divided by it's volume, we have

$$
\begin{aligned}
&(-1)^{m_K} \int_{S^p} u^{K^l} \\
&= \int_{E_1} \cdots \int_{E_{n_K+1}} \wedge_{i=1}^{n_{K^l}+1} \Phi_i^* A_i(z_i) \wedge_{I\in\mathcal{I}} \Pi_I^* L^*\Omega_{S^{p-1}}(z_{i_1}, z_{i_2}) \,.
\end{aligned}
$$

Using Federer's coarea formula, we have, denoting $r_i = n - p_i - q_i$,

$$
\begin{aligned}
&(-1)^{m_{K^l}} \prod_{i=1}^{n_{K^l}+1} |B_1^{r_i}| \int_{S^p} u^{K^l} \\
&= \int_{B_1^{r_1}} \cdots \int_{B_1^{r_{n_K+1}}} \wedge_{i=1}^{n_K+1} d\xi_i \int_{\mathsf{Q}_{i=1}^{n_K+1} (P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i)} \wedge_{I\in\mathcal{I}} \Pi_I^* L^*\Omega_{S^{p-1}}(z_{i_1}, z_{i_2}) \,.
\end{aligned}
\tag{II.33}
$$

For a regular value $(\xi_1,\cdots,\xi_{n_K+1})$ of $\prod_{i=1}^{n_K+1}(P_i\circ\Xi_i\circ\Phi_i)$, we introduce the map

$$
\begin{array}{rrcl}
V_{\xi_1,\cdots,\xi_{n_K+1}}: & \displaystyle\prod_{i=1}^{n_K+1}(P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i) & \longrightarrow & (S^{p-1})^{n_K} \\
 & (x_1,\cdots,x_{n_K+1}) & \longrightarrow & \displaystyle\prod_{I\in\mathcal{I}}(L\circ\Pi_I)(x_{i_1},x_{i_2})
\end{array}
$$

A short computation using (II.5) shows that

$$
\dim\ \prod_{i=1}^{n_K+1}(P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i)=\dim\ (S^{p-1})^{n_K}\quad .
$$

A standard deformation argument shows also that the topological degree M of $V_{\xi_1,\cdots,\xi_{n_K+1}}$ is independent of $(\xi_1,\cdots,\xi_{n_K+1})$. Combining this fact together with (II.33) and the integral expression of the topological degree, we have shown that, modulo a sign, $\int_{S^p}u^{K^l}$ equals this topological degree. Proposition II.4 follows. ■

Remark II.7 *An alternative proof of Proposition II.4 may be obtained, following [BT] pages 230-234, by interpreting each elements of the integrand as a geometric operation.*

Here one interprets $u^*\omega_i$ as the integration operation on $u^{-1}(C_i)$, where C_i is the Poincaré dual to ω_i in N, one interprets the d^{-1} operation as taking a chain bounding a boundary and the wedge $\wedge$ as the intersection operation - see (6.31) page 69 in [BT]- (see such an approach also in [Ri] Proposition 2.2).

III $\pi_p(N)\otimes\mathbb{R}-$Type Bubbling for Sequences of Maps in $W^{1,p}(S^p,N)$.

Given a class $z\in$Hom$(\pi_p(N),\mathbb{R})$ and a sequence u_n in $W^{1,p}(S^p,N)$ which weakly converges to a map u in $W^{1,p}(S^p,N)$, the goal of this section is to show that one can extract a subsequence $u_{n'}$ from u_n such that

$$
\lim_{n\rightarrow\infty}z(\,[u_{n'}]\,)\ =\ z(\,[u]\,)\ +\ \sum_{i=1}^{I}z(\,[w_i]\,)
$$

where w_j are disjoint "bubbles" from u_n (i.e. maps in $W^{1,p}(\mathbb{R}^p,N)$, weak limits of the dilated map $u_n(r_n^jx+a_j^n)$ where a_j^n converges to a limiting point

a_j in S^p and r_n^j tends to zero, moreover for two distinct j and j' both r_n^j and $r_n^{j'}$ cannot be comparable to $|a_n^j - a_n^{j'}|$). We will in fact formulate that convergence using a modification of the "Graph" approach of Giaquinta, Modica and Soucek [GMS] and by introducing in our context the Cartesian Currents.

III.1 The Cartesian Current Associated to a Map u in $W^{1,p}(S^p, N)$ and a p dimensional simply connected tree-graph K of closed forms from N.

Let N be a compact simply connected Riemannian manifold, K be a p dimensional simply connected tree-graph of closed forms, and u be a map in $W^{1,p}(S^p, N)$. Recall that the number n_K of branches equals the number of nodes -1. We first introduce the following definition.

Definition III.3 *To any map u in $W^{1,p}(S^p, N)$ we assign a map $\mathcal{U}_K$ from $(S^p)^{n_K+1}$ into $N^{n_K+1} \times (S^{p-1})^{n_K}$ defined by*

$$\mathcal{U}_K(x_1, \cdots, x_{n_K+1}) = \left(u(x_1), \cdots, u(x_{n_K+1}), \cdots, \frac{\pi(x_i) - \pi(x_j)}{|\pi(x_i) - \pi(x_j)|}, \cdots \right)$$

where the pairs (i, j) run over the set of pairs $\mathcal{I}_K$ for which the nodes i and j are connected, in the graph K, by a segment going from j to i. Such a map will be called the **tree-graph map** *associated to u and K.*

Observe then that with this notation we have, using Lemma II.3,

$$\begin{aligned} \int_{S^p} u^K &= \int_{x_1} \cdots \int_{x_{n_K+1}} U_K^*(\omega^K) \wedge \mathcal{G}^K \\ &= \int_{(S^p)^{n_K+1}} \mathcal{U}_K^* \left(\omega^K \wedge_{J \in \mathcal{J}} \Omega(X_J) \right) \quad . \end{aligned}$$

where, for any $J = (j_1, j_2) \in \mathcal{J}$, X^J is a variable in S^{p-1} and $\Omega(X^J)$ denotes the standard unit-volume form on $S^{p-1} \subset \mathbb{R}^p$ given by

$$\Omega(X^J) = |S^{p-1}|^{-1} \sum_{k=1}^{p} X_k^J \ (-1)^{k-1} \ dX_1^J \cdots dX_{k-1}^J dX_{k+1}^J \cdots dX_p^J \quad .$$

In the product space $(S^p)^{n_K+1} \times N^{n_K+1} \times (S^{p-1})^{n_K}$ we have the $p\,(n_K + 1)$ integer-multiplicity rectifiable current $Graph(\mathcal{U}_K)$ defined by integration on the graph of $\mathcal{U}_K$: For any smooth $p\,(n_K+1)-$form Ψ in $(S^p)^{n_K+1}\times N^{n_K+1}\times (S^{p-1})^{n_K}$,

$$Graph(\mathcal{U}_K)(\Psi) \;=\; \int_{(S^p)^{n_K+1}} \mathcal{V}_K^*\Psi \quad \text{where } \mathcal{V}_K(x)=(x,\mathcal{U}_K(x))\ , \qquad \text{(III.1)}$$

The following proposition shows that $Graph(\mathcal{U}_K)$ is indeed an integer rectifiable current whose boundary also has finite mass (and hence is also rectifiable [Fe],4.2.16.)

Proposition III.1 *Under the notation above we have the existence of a constant C_K independent of $u\in W^{1,p}(S^p,N)$ such that*

$$\mathbb{M}\left(Graph(\mathcal{U}_K)\right) \;\le\; C_K\left[1+\|\nabla u\|_n^{p\,(n_K+1)}+\|\nabla u\|_p^{p+n_K}\right] \quad . \qquad \text{(III.2)}$$

Moreover there exists a constant C_K such that

$$\mathbb{M}\left(\partial Graph(\mathcal{U}_K)\right) \;\le\; C_K\left[1+\|\nabla u\|_p^{p\,n_K}+\|\nabla u\|_p^{p+n_K-1}\right] \quad . \qquad \text{(III.3)}$$

Proof of Proposition III.1. Any smooth $p\,(n_K+1)-$form Ψ in the product space $(S^p)^{n_K+1}\times N^{n_K+1}\times (S^{p-1})^{n_K}$ can be written as a finite sum of smooth simple forms. Corresponding to a choice of nonnegative integers $(m_1,\cdots,m_{n_K+1},p_1,\cdots,p_{n_K+1},q_1,\cdots,q_{n_K})$ such that

$$\sum_{i=1}^{n_K+1} m_i \;+\; \sum_{j=1}^{n_K+1} p_j \;+\; \sum_{k=1}^{n_K} q_k \;=\; p\,(n_K+1)$$

and $m_i\le p$, $p_j\le \dim N$ and $q_k\le p-1$, we may let Ψ' be the canonical projection of Ψ on $\wedge^{m_1}S^p\cdots\wedge^{p_{n_K+1}}S^p\wedge^{p_1}N\cdots\wedge^{p_{n_K+1}}N\wedge^{q_1}S^{p-1}\cdots\wedge^{q_{n_K}}S^{p-1}$. Then Ψ' is the product of a function $f\in C^\infty(S^p,\mathbb{R})$ and a simple form

$$\alpha_1(x_1)\wedge\cdots\wedge\alpha_{n_K+1}(x_{n_K+1})\wedge\beta_1(y_1)\wedge\cdots\wedge\beta_{n_K+1}(y_{n_K+1})\wedge\gamma_1(y_1)\wedge\cdots\wedge\gamma_{n_K}(z_{n_K}).$$

We have

$$\begin{aligned}
&|(Graph(\mathcal{U}_K)(\Psi)|\\
&= \|f\|_\infty\left|\int_{(S^p)^{n_K+1}} \wedge_{i=1}^{n_K+1}\alpha_i(x_i)\wedge_{j=1}^{n_K+1}u^*\beta_j(y_j)\right.\\
&\qquad\qquad\left.\wedge_{J=(j_1,j_2)\in\mathcal{I}_K}\pi^*\frac{x_{j_1}-x_{j_2}}{|x_{j_1}-x_{j_2}|}^*\gamma_J(z_j)\right|\\
&\le C_K\|\Psi'\|_\infty\int_{(S^p)^{n_K+1}}\prod_{j=1}^{n_K+1}|\nabla u|^{p_j}(y_j)\prod_{J=(j_1,j_2)\in\mathcal{I}_K}\frac{1}{|\pi(x_{j_1})-\pi(x_{j_2})|^{q_J}}
\end{aligned} \qquad \text{(III.4)}$$

We prove now by induction on the number of nodes n_K+1 in the tree-graph K that the integral

$$\int_{(S^p)^{n_K+1}} \prod_{i=1}^{n_K+1} |f_i|(x_i) \prod_{J=(j_1,j_2)\in\mathcal{I}_K} \frac{1}{|\pi(x_{j_1})-\pi(x_{j_2})|^{q_J}} \tag{III.5}$$

is bounded by

$$C_K \prod_{i=1}^{n_K+1} \|f_i\|_{\frac{p}{p_i}} \quad . \tag{III.6}$$

To facilitate calculation, consider the sphere measure $\mu=\pi_\#(\mathcal{H}^p\llcorner S^p)$ on $\mathbb{R}^p$ so that

$$\|f\circ\pi^{-1}\|_{L^p(\mu)} \ = \ \|f\|_{L^p}$$

for $f\in L^p(S^p)$. Let's take an <u>end node</u> i_2 connected to the node i_1 so that $J=(i_1,i_2)\in\mathcal{I}_K$. Classical estimates on Riesz potentials (see [St]) give

$$\left\|\int_{x_{i_2}\in\mathbb{R}^p} \frac{|f_{i_2}|(\pi^{-1}(x_{i_2}))}{|x_{i_1}-x_{i_2}|^{q_J}}\right\|_{L^{\frac{p}{p_{i_2}+q_J-n}}(\mu)} \le C_K\|f_{i_2}\circ\pi^{-1}\|_{L^{\frac{p}{p_{i_2}}}(\mu)} = C_K\|f_{i_2}\|_{\frac{p}{p_{i_2}}} \ . \tag{III.7}$$

Therefore

$$\left\|f_{i_1}(\pi^{-1}(x_{i_1}))\int_{x_{i_2}\in\mathbb{R}^p} \frac{|f_{i_2}|(\pi^{-1}(x_{i_2}))}{|x_{i_1}-x_{i_2}|^{q_J}}\right\|_{L^{\frac{p}{p_{i_1}+p_{i_2}+q_J-p}}(\mu)} \le \ C_K\|f_{i_1}\|_{\frac{p}{p_{i_1}}} \ \|f_{i_2}\|_{\frac{p}{p_{i_2}}} \ . \tag{III.8}$$

Replacing then $f_{i_1}\circ\pi^{-1}$ by

$$f'_{i_1}(\pi^{-1}(x_{i_1})) \ = \ f_{i_1}(\pi^{-1}(x_{i_1}))\int_{x_{i_2}} |f_{i_2}|(\pi^{-1}(x_{i_2}))|x_{i_1}-x_{i_2}|^{-q_J} \ ,$$

replacing p_{i_1} by $p'_{i_1}=p_{i_1}+p_{i_2}+q_J-p$ and removing the node i_2 from the graph K, we are in the position to apply our induction assumption to this new graph and this permits to bound (III.5) by (III.6). Applying this fact to the inequality (III.4) we obtain (III.2).

We establish now (III.3). Suppose Φ be a smooth compactly supported $p\,(n_K+1)-1$ form in $(S^p)^{n_K+1}\times N^{n_K+1}\times(S^{p-1})^{n_K}$. This time we may consider the projection Φ' in

$$\wedge^{m_1}S^p\cdots\wedge^{m_{n_K+1}}S^p\wedge^{p_1}N\cdots\wedge^{p_{n_K+1}}N\wedge^{q_1}S^{p-1}\cdots\wedge^{q_{n_K}}S^{p-1}$$

where $m_1,\cdots,m_{n_k+1},p_1,\cdots,p_{n_K+1},q_1,\cdots,q_{n_K}$ are nonnegative integers such that

$$\sum_{i=1}^{n_K} m_i \ + \ \sum_{j=1}^{n_K+1} p_j \ + \ \sum_{k=1}^{n_K} q_k \ = \ p\,(n_K+1)-1$$

with $m_i \leq p$, $p_j \leq \dim N$, and $q_k \leq p-1$. Again assume that Φ' is simple.

We also observe that for any smooth form γ in $\wedge^q S^{p-1}$, the following identity holds, in the sense of distributions, if $q < p-1$:

$$d\pi^* \left(\frac{x-y}{|x-y|}^* \gamma \right) = \pi^* \frac{x-y}{|x-y|}^* (d\gamma) \qquad \text{in } \mathcal{D}'(\wedge^q(\mathbb{R}^p \times \mathbb{R}^p)) \tag{III.9}$$

whereas if $\gamma \in C^\infty(\wedge^{p-1} S^{p-1})$, a short computation shows that, for any form ϕ in $C_0^\infty(\wedge^p(S^p \times S^p))$,

$$\int_{S^p \times S^p} d\phi \wedge \pi^* \frac{x-y}{|x-y|}^* \gamma \ = \ (-1)^p \int_{S^{p-1}} \delta \wedge \pi^* \int_{\mathbb{R}^p} \Delta^* \phi \tag{III.10}$$

where Δ is the diagonal map assigning (x,x) to x. Thus, because of (III.9) and (III.10) the d commutes with U_K^* modulo the operation which corresponds to sum all pull backs of forms obtained by removing one segment $J_0 = (j_1, j_2)$ in K for which $q_J = n-1$ and by fusing the nodes j_1 and j_2 : precisely the form

$$\beta_{j_1}(y_{j_1}) \wedge \beta_{j_2}(y_{j_2}) \wedge \pi^* \frac{x_{j_2} - x_{j_1}}{|x_{j_2} - x_{j_1}|}^* \gamma_{J_0} \wedge_{\{j,\ J=(j,j_2) \in \mathcal{I}_K\}} \pi^* \frac{x_{j_2} - x_j}{|x_{j_2} - x_j|}^* \gamma_J$$

is changed into

$$\beta_{j_1}(y_{j_1}) \wedge \beta_{j_2}(y_{j_1}) \wedge \int_{S^{p-1}} \gamma_{J_0} \wedge_{\{j,\ J=(j,j_2) \in \mathcal{I}_K\}} \pi^* \frac{x_{j_1} - x_j}{|x_{j_1} - x_j|}^* \gamma_J \quad .$$

We then obtain a linear combination of new tree-graphs and applying the result (III.2) we obtain (III.3) and Proposition III.1 is proved.

III.2 $\pi_p(N) \otimes \mathbb{R}-$Type Bubbling.

In this section we will consider the behavior of a sequence of p dimensional tree-graph forms corresponding to a $W^{1,p}$ weakly convergent sequence of maps in $C^\infty(S^p, N)$ and a fixed element of $\mathrm{Hom}(\pi_p(N), \mathbb{R})$. We first note that a strict sub-tree-graph L will have dimension $q < p$, and consider the behavior of the corresponding q forms in the following:

Lemma III.1 *Let u_n be a sequence weakly converging to u in $W^{1,p}(S^p, N)$ and L be a q dimensional simply connected tree-graph of forms with $q < p$. Then*

$$u_n^L \rightharpoonup u^L \qquad \text{in } L^{\frac{p}{q}}(S^p, \wedge^q S^p) \quad .$$

Proof of Lemma III.1.

We prove this lemma by induction on the number n_L+1 of nodes in L. If there is only one node, this result can be found in [GMS]. Let $\omega \in \mathcal{D}^r(N)$ be, as in section II.2, the form at the summit of L. There exists then a family of tree-graphs $L_1, \cdots, L_m$ whose dimensions sum to $q-r+m$ such that

$$u_n^L(x) = u_n^*\omega(x) \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x-x_i}{|x-x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)$$

¿From the proof of Proposition III.1 we have that

$$\|u_n^L\|_{\frac{p}{q}} \ \leq \ C_L \|\nabla u\|_p^{q+n_L}$$

where n_L+1 is the number of nodes in L. Extracting a subsequence if necessary, we can always assume that u_n^L converges weakly in $L^{\frac{p}{q}}$ and the goal is to show that this limit is u^L. Assuming that N is isometrically embedded in $\mathbb{R}^k$, we can write ω as the pull-back under the inclusion map of N of a form in $\mathbb{R}^k$

$$\omega \ = \ \sum_{J\in\mathcal{J}} a_J \ dy_J \ \in \ \mathcal{D}^r(\mathbb{R}^k) \quad .$$

(Here $\mathcal{J}$ denotes the collection of increasing $r-$tuples in $\{1,\cdots,k\}$ and $dy_J = dy_{j_1} \wedge \cdots \wedge dy_{j_r}$ for $J \in \mathcal{J}$.)

First, ignoring the coefficient functions $a_J(u_n)$ occurring in $u_n^*\omega$, we study convergence of the q forms

$$du_n^{j_1} \wedge \cdots \wedge du_n^{j_r} \wedge_{i=1}^m \int_{x_i\in S^p} \pi^* \frac{x-x_i}{|x-x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i) \ . \qquad \text{(III.11)}$$

Note that we may rewrite III.11 as the sum of two terms

$$\begin{aligned} & d\left[u_n^{j_1}\, du_n^{j_2} \wedge \cdots \wedge du_n^{j_r} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x-x_i}{|x-x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)\right] \\ & +(-1)^r\, u_n^{j_1}\, du_n^{j_2} \wedge \cdots \wedge du_n^{j_d} \wedge d\left[\wedge_{i=1}^m \int_{S^p} \pi^* \frac{x-x_i}{|x-x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)\right] \ . \end{aligned} \qquad \text{(III.12)}$$

For fixed $n_L \geq 1$, we will also use an induction on r. In case $r=1$, the bracketed expression in the first term is simply

$$u_n^{j_1} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x-x_i}{|x-x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i) \ .$$

Here the functions $u_n^{j_1}$, being uniformly bounded in $W^{1,p}(S^p)$, converge, by Sobolev embedding, strongly in L^m, for all $m<\infty$, to u^{j_1}. The remaining wedge product is, by III.7 and III.8, weakly convergent in some $L^{m'}$ space with $m' < \infty$ to the corresponding wedge product with $u^{L_i}(x_i)$ replacing $u_n^{L_i}(x_i)$. Multiplying and applying d, we conclude that the first term in the r.h.s. of (III.12) with $r = 1$ converges, in the sense of distributions, to

$$d\left[u^{j_1} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i)\right] .$$

In case $r \in \{2, 3, \cdots\}$, the induction assumption on r implies that

$$u_n^{j_1}\ du_n^{j_2} \wedge \cdots \wedge du_n^{j_d} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)$$

converges weakly in $L^{\frac{p}{q-1}}$ sense to

$$u^{j_1}\ du^{j_2} \wedge \cdots \wedge du^{j_d} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i)$$

So again, the first term in the r.h.s. of (III.12) converges in the sense of distributions to the corresponding term with with u_n replaced by u.

The second term of the r.h.s. of (III.12) equals, modulo a multiplication by a constant,

$$\sum_{l=1}^m (-1)^{s_l} a_J(u_n)\, u_n^{j_1}\ du_n^{j_2} \wedge \cdots \wedge du_n^{j_r} \wedge u_n^{L_l} \wedge_{i \neq l} \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i) .$$

where s_l is some integer depending on l. Each of the terms of this sum correspond to a $u_n^{L'_l}$ where the number of nodes in L'_l has been decreased by 1 and now equals $n_L + 1 - 1 = n_L$. We can thus apply our induction assumption on the number of nodes and deduce that each of this $u_n^{L'_l}$ converges weakly in $L^{\frac{p}{q}}$ to the corresponding $u^{L'_l}$.

Thus we have showed by induction on r that

$$du_n^{j_1} \wedge du_n^{j_2} \wedge \cdots \wedge du_n^{j_r} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)$$

converges weakly in $L^{\frac{p}{q}}$ to

$$du^{j_1} \wedge du^{j_2} \wedge \cdots \wedge du^{j_r} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i)$$

Finally we consider the coefficients $a_J(u_n)$. Since a_J is smooth and bounded on the compact submanifold N, Sobolev embedding implies that the sequence $a_J(u_n)$ converges strongly in every L^m space to $a_J(u)$, and therefore (III.11) converges in the distribution sense to

$$a_J(u)\, du^{j_1} \cdots \wedge du^{j_d} \wedge_{i=1}^m \int_{S^p} \pi^* \frac{x - x_i}{|x - x_i|}^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i) \quad .$$

Since it is bounded in $L^{\frac{p}{q}}$, this convergence also holds weakly in $L^{\frac{p}{q}}$ and Lemma III.1 is proved. ■

Now we are ready to study the behavior of the smooth p forms u_n^K on S^p associated with $W^{1,p}$ weakly convergent sequence of maps $u_n \in C^\infty(S^p, N)$ and a linear combination $K = \sum_l \lambda_l K_l$ of tree-graphs constructed in section II. The weak convergence of the mappings u_n implies by (II.19) the boundedness in L^1 of the sequence of p forms u_n^K or equivalently of the dual functions $* u_n^K \in C^\infty(S^p)$. Thus the corresponding sequence of *signed measures* on S^p

$$(* u_n^K)\, \mathcal{H}^p \llcorner S^p \tag{III.13}$$

has a subsequence $(* u_{n_k}^K)\, \mathcal{H}^p \llcorner S^p$ convergent to a signed Radon measures ν on S^p. Here, for notational simplicity, we indicate this by simply saying

$$u_{n_k}^K \rightharpoonup \nu \quad \text{weakly as Radon measures .}$$

The goal of this subsection is to prove the following proposition.

Proposition III.2 *Let z be an element of $Hom(\pi_p(N), \mathbb{R})$, and $K = K_z = \sum_l \lambda_l K_l$ be a formal linear combination of tree-graphs of closed forms on N associated to z for a given choice of geometric realization Ψ_N (as described in part II). For any sequence $u_n \in C^\infty(S^p, N)$ which $W^{1,p}$ weakly converges to a map $u \in W^{1,p}(S^p, N)$, there exist a subsequence u_{n_k}, finitely many points $a_1, \cdots, a_I$ in S^p, and finitely many maps $w_1, \cdots, w_I$ from S^p to N with each $z(\,[w_i]\,) \neq 0$ such that the following assertions hold :*

i) The differential p forms $u_{n_k}^K$ converge weakly as Radon measures

$$u_{n_k}^K \rightharpoonup u^K + \sum_{i=1}^I z(\,[w_i]\,)\, \delta_{a_i} \quad . \tag{III.14}$$

ii) For k sufficiently large, the following identity holds

$$z(\,[u_{n_k}]\,) = z(\,[u]\,) + \sum_{i=1}^I z(\,[w_i]\,) \quad . \tag{III.15}$$

iii) The following inequality holds in the sense of measures

$$|\nabla u|^p \mathcal{H}^p \llcorner S^p + C_K \sum_{i=1}^{I} |z(\,[w_i]\,)|^{\frac{p}{p+n_K}}\ \delta_{a_i}\ \leq\ \liminf_{k\to\infty} |\nabla u_{n_k}|^p \mathcal{H}^p \llcorner S^p \tag{III.16}$$

where C_K is a positive constant depending only on K and n_K is the total number of segments among all K^l in K.

iv) Given any diffeomorphism Ψ of S^p, we have

$$(u_{n_k}\circ\Psi)^K\ \rightharpoonup\ (u\circ\Psi)^K\ +\ \sum_{i=1}^{I} z(\,[w_i]\,)\ \delta_{\Psi^{-1}(a_i)}\quad . \tag{III.17}$$

v) There exists a constant $\varepsilon_{p,N}>0$ depending only on p and N such that

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{S^p\cap B_r(a_i)} |\nabla u_{n_k}|^p\, d\mathcal{H}^p\ \geq\ \varepsilon_{p,N} \tag{III.18}$$

whenever $z(\,[w_i]\,)\neq 0$ for some $i=1,\cdots,I$.

vi) Let B be an open subdomain of S^p, and let v_n be another weakly converging sequence in $W^{1,p}(S^p,N)$ such that $v_n = u_n$ on B for all $n\in\mathbb{N}$. Then, as Radon measures,

$$(u^K_{n_k} - v^K_{n_k})\llcorner B\ \rightharpoonup\ (u^K - v^K)\llcorner B\quad , \tag{III.19}$$

where v denotes the weak limit of v_n and $\llcorner B$ is the restriction operator to the subdomain B.

Remark III.1 *Using the language of currents we introduced above in (III.1), i) says that, for any function f in $C^0(S^p)$ the sequence*

$$(Graph\,\mathcal{U}^{n_k}_{K_l})\Big(f(x_1)\,\omega^{K_l}(y_1,\cdots,y_{n_{K_l}+1})\wedge_{J\in\mathcal{I}_{K_l}}\Omega_{S^{p-1}}(X_J)\Big)$$

converges to

$$(Graph\,\mathcal{U}_{K_l})\Big(f(x_1)\,\omega^{K_l}(y_1,\cdots,y_{n_{K_l}+1})\wedge_{J\in\mathcal{I}_{K_l}}\Omega_{S^{p-1}}(X_J)\Big)+\sum_{i=1}^{I}\ f(a_i)\ z(\,[v_i]\,)$$

where x_l always denotes the variable assigned to the summit of the graph K_l.

Proof of Proposition III.2. In this proof, we will, for simplicity, not change notations when we pass to subsequences. In particular, we first pass to a subsequence to assume that

$$|\nabla u_n|^p \,\mathcal{H}^p \llcorner S^p \ \rightharpoonup\ \mu \qquad \text{and} \qquad u_n^K \ \rightharpoonup\ \nu$$

converge weakly as Radon measures on S^p. There exists, by B. White's result in [Wh], a positive number $\varepsilon_{p,N}$ such that any map $u \in W^{1,p}(S^p, N)$ satisfying

$$\int_{S^p} |\nabla u|^p \, d\mathcal{H}^p \ \leq\ \varepsilon_{p,N} \tag{III.20}$$

is homotopically trivial. Thus there are only finitely many points $a_1, \cdots, a_I$ in S^p satisfying

$$\mu(\{a_1\}) \ >\ \varepsilon_{p,N}\ ,\ \cdots\ ,\ \mu(\{a_I\}) \ >\ \varepsilon_{p,N}\ .$$

We will first verify that

$$\nu \llcorner (S^p \setminus \{a_1, \cdots, a_I\}) \ =\ u^K \quad . \tag{III.21}$$

To do this, it suffices by the Besicovitch covering lemma, to find, for each point $x_0 \in S^p \setminus \{a_1, \cdots, a_I\}$ and each positive δ, a positive numbers $r < \delta$ so that

$$\lim_{n\to\infty} \int_{S^p \cap B_r(x_0)} u_n^K \ =\ \int_{S^p \cap B_r(x_0)} u^K \quad .$$

We may assume, for simplicity of the presentation, that x_0 does not coincide with the pole sent to infinity by stereographic projection. The idea will be to chose r so that $\int_{S^p \cap \partial B_r(x_0)} |\nabla u_n|^p$ is small, and then use the equation

$$\int_{S^p \cap B_r(x_0)} u_n^K \ =\ \int_{S^p} \hat{u}_n^K \ -\ \int_{S^p \setminus B_r(x_0)} \hat{u}_n^K \quad , \tag{III.22}$$

with a certain extension $\hat{u}_n \in W^{1,p}(S^p, N)$ of the restriction $u_n|\ [S^p \cap B_r(x_0)]$. We will show below how to choose $\hat{u}_n$ and the corresponding limit $\hat{u}$ on $S^p \setminus B_r(x_0)$ to have strong $W^{1,p}$ convergence there. This will take care of the convergence of the second term in (III.22). The first term will involve the topological quantities, namely $z(\,[\hat{u}_n]\,)$ and $z(\,[\hat{u}]\,)$, that will all vanish provided the total p energies of the $\hat{u}_n$ are all less than $\varepsilon_{p,N}$.

To choose r and find a suitable extension $\hat{u}_n$, first let ε be a small positive number, to be determined later, and chose a positive $s = s(\delta, \varepsilon)$ small enough so that

$$s \ <\ \min\{\delta,\, |x_0 - a_1|, \cdots, |x_0 - a_I|\} \quad \text{and} \quad \int_{S^p \cap B_s(x_0)} \mu \ \leq\ \frac{\varepsilon}{2}\ .$$

Passing to a subsequence, we may, by Fatou's lemma and Fubini's theorem, choose $r \in (\frac{t}{2}, t)$ so that

$$\int_{S^p \cap \partial B_r(x_0)} |\nabla u_n|^p \, d\mathcal{H}^{p-1} \ \leq \ r\varepsilon$$

for all n large. This implies that

$$\|u_n\|_{W^{1-1/p,p}(S^p \cap \partial B_r(x_0), \mathbb{R}^k)} \ \leq \ C\varepsilon$$

(where $\mathbb{R}^k$ is an ambient space in which N is isometrically embedded). Since $\dim[S^p \cap \partial B_r(x_0)] = p - 1 < p$, there exist elements ξ_n in N such that

$$\|u_n - \xi_n\|_{L^\infty(S^p \cap \partial B_r(x_0))} \ \leq \ C\varepsilon \tag{III.23}$$

where C is independent of s, ε and u_n. By (III.23), we can form a Whitney C^1 extension $\tilde{u}_n$ of $u_n |\, S^p \cap \partial B_r(x_0)$ to $S^p \setminus B_r(x_0)$. Letting π_N denote the nearest-point projection from a tubular neighborhood of N onto N, we see that the map $\hat{u} : S^p \longrightarrow N$ defined by

$$\hat{u}_n = u \quad on \quad S^p \cap B_r(x_0) \quad ,$$

$$\hat{u}_n = \pi_N \circ \tilde{u}_n \quad on \quad S^p \setminus B_r(x_0)$$

satisfies the small energy bound

$$\int_{S^p} |\nabla \hat{u}_n|^p \, d\mathcal{H}^p \ \leq \ C\varepsilon \quad .$$

(The fact that constants are independent of r comes from the scaling invariance of the $p-$energy in $\mathbb{R}^p$).

It is clear, since u_n converges weakly to u, that $\hat{u}_n$ converges to $\hat{u}$ and

$$\int_{S^p} |\nabla \hat{u}|^p \, d\mathcal{H}^p \ \leq \ C\varepsilon \quad .$$

Moreover, our choice of r so that $\int_{\partial B_r(x_0)} |\nabla u_n|^p$ is uniformly bounded (with respect to n) and our choice of extension guarantee that $\hat{u}_n |\, (S^p \setminus B_r(x_0))$, converges <u>strongly</u> in $W^{1,p}(S^p \setminus B_r(x_0), N)$. Let $\hat{u}_n^K$ be the form obtained by replacing u_n by $\hat{u}_n$ in each of the forms $U^*_{K_l}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$. By also insisting that $C\varepsilon < \varepsilon_{p,N}$, we then have that $z(\,[\hat{u}_n]\,) = 0$ so that

$$0 \ = \ \int_{S^p} \hat{u}_n^K \ = \ \sum_l \lambda_l \int_{x_1 \in \mathbb{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbb{R}^p} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \quad , \tag{III.24}$$

and similarly

$$0 = \int_{S^p} \hat{u}^K \;=\; \sum_l \lambda_l \int_{x_1 \in \mathbb{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbb{R}^p} \hat{U}^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \quad . \tag{III.25}$$

Next denoting

$$\Omega_0 = \pi[S^p \cap B_r(x_0)] \quad \text{and} \quad \Omega_1 = \mathbb{R}^p \setminus \pi[S^p \cap B_r(x_0)] \quad ,$$

we write, for every tree-graph K_l arising in K,

$$\begin{aligned} \int_{S^p} \hat{u}_n^{K_l} \;&=\; \int_{x_1 \in \mathbb{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbb{R}^p} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \\ &= \sum_{(i_1,\cdots,i_{n_{K_l}+1}) \in \mathfrak{I}_l} \int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \ , \end{aligned} \tag{III.26}$$

where $\mathfrak{I}_l$ denotes the set of $n_{K_l}-$tuples of ordered numbers taken from the set $\{0,1\}$. Thus the integral we are interested in, namely, $\int_{S^p \cap B_r(x_0)} u_n^K$, equals the single term in the sum given by the $n_{K_l}-$tuple $(0,\cdots,0)$. For each of the remaining terms, $i_k = 1$ for some $k \in \{1,\cdots,n_{K_l+1}\}$, and we claim that we have the convergence

$$\begin{aligned} \lim_{n\to\infty} &\int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \\ &= \int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \hat{U}^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \quad . \end{aligned} \tag{III.27}$$

To verify this, we may assume that $i_1 = 1$. We observe that the modification of the orientation of segments in the graph only leads to a possible change of sign in the integrand $\hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$. Therefore we can always assume that the node for which the variable is integrated in Ω_1 is the summit of our graph. We have chosen indexation so that this is the variable x_1. Let ω_1 be the form at the summit of the tree-graph K_l. We now can write

$$\begin{aligned} &\int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \quad = \\ &\int_{x_1 \in \pi(S^p \setminus B_r(x_0))} (\hat{u}_n \circ \pi^{-1})^* \omega_1(x_1) \wedge_k \int_{x_{j_k}} \frac{x_1 - x_{j_k}}{|x_1 - x_{j_k}|}^* \Omega_{S^{p-1}} \wedge \hat{u}_n^{K_{l,k}}(\pi^{-1}(x_{j_k})) \end{aligned}$$

where j_k are all the nodes connected to the summit and $K_{l,k}$ are the tree-graph issued from these nodes. Denote $d_1 > 0$ the degree of the form ω_1.

From Lemma III.1 we know that, for every k, $\hat{u}_n^{K_{l,k}}$ converges weakly in $L^{\frac{p}{s_k}}$ for some $s_k > 0$ and the s_k satisfy $\sum_k (s_k - 1) = p - d_1$ (where we are using Proposition III.1). Therefore, in the distribution sense,

$$\int_{x_{j_k}} \frac{x_1 - x_{j_k}}{|x_1 - x_{j_k}|}^* \Omega_{S^{p-1}} \wedge (\hat{u}_n^{K_{l,k}} \circ \pi^{-1}) \ \rightharpoonup \ \int_{x_{j_k}} \frac{x_1 - x_{j_k}}{|x_1 - x_{j_k}|}^* \Omega_{S^{p-1}} \wedge (\hat{u}^{K_{l,k}} \circ \pi^{-1}) \ .$$

Using classical estimates on Riesz integrals (see [St]), we obtain that

$$\int_{x_{i_k}} \frac{x_1 - x_{i_k}}{|x_1 - x_{i_k}|}^* \Omega_{S^{p-1}} \wedge \hat{u}_n^{K_{l,k}}(\pi^{-1}(x_{i_k}))$$

is uniformly bounded in $L^{\frac{p}{s_k - 1}}$. Therefore it converges weakly in $L^{\frac{p}{s_k-1}}$ to the limit $\int_{x_{j_k}} \frac{x_1 - x_{j_k}}{|x_1 - x_{j_k}|}^* \Omega_{S^{p-1}} \wedge \hat{u}^{K_{l,k}} \circ \pi^{-1}$. Since $\hat{u}_n^* \omega_1$ converges strongly to $\hat{u}^* \omega_1$ on $\Omega_2 = \mathbb{R}^p \setminus B_r(x_0)$, (III.27) is proved. Therefore, combining (III.24), (III.25), (III.26) and (III.27) we obtain that

$$\begin{aligned} &\lim_{n\to\infty} \sum_l \lambda_l \int_{x_1 \in B_r(x_0)} \cdots \int_{x_{n_{K_l}+1} \in B_r(x_0)} U^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \\ &\quad = \sum_l \lambda_l \int_{x_1 \in B_r(x_0)} \cdots \int_{x_{n_{K_l}+1} \in B_r(x_0)} U^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \end{aligned} \qquad \text{(III.28)}$$

Considering now the integral of $u_n^K = \sum_l \lambda_l u_n^{K_l}$ on $B_r(x_0)$. We make the following elementary remark : Let f_n be some sequence of functions weakly converging in $L^{\frac{p}{s}}(R^p \setminus B_r(x_0))$, for $2 \le s \le p-1$, to a limit f. Then

$$\int_{\mathbb{R}^p \setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} \ f_n(y)\, dy$$

converges strongly in $L^q_{loc}(B_r(x_0))$ for any $q > 0$ and converges weakly in $L^{\frac{p}{s-1}}(B_r(x_0))$ to $\int_{\mathbb{R}^p \setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} \ f(y)$. Moreover, it is not difficult to check that $\| \int_{\mathbb{R}^p \setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} \ f_n(y) \|_{L^q(B_r(x_0))}$ is uniformly bounded for some $q > \frac{p}{s-1}$. Therefore, we deduce that there exists some $q > \frac{p}{s-1}$ such that

$$\int_{\mathbb{R}^p \setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} \, f_n(y) dy \to \int_{\mathbb{R}^p \setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} \, f(y)\, dy \ \text{ in } L^q(B_r(x_0))$$

Considering a decomposition of the domain $\mathbb{R}^p$ into $B_r(x_0)$ and $\mathbb{R}^p \setminus B_r(x_0)$ in computing the integral $\int_{\mathbb{R}^p} \hat{u}_n^{K_l} = \int_{x_1 \in \mathbb{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbb{R}^p} \hat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$, only the convergence of the term

$$\int_{x_1 \in B_r(x_0)} \cdots \int_{x_{n_{K_l}+1} \in B_r(x_0)} U^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$$

is a-priori problematic. But using the result of our efforts above in (III.28) we can deduce that

$$\int_{B_r(x_0)} u_n^K \longrightarrow \int_{B_r(x_0)} u^K$$

This holds for any x_0 in $S^p \setminus \{a_1, \cdots, a_I\}$ and any r small enough, therefore (III.21) holds true and we have that

$$u_n^K \rightharpoonup u^K + \sum_{i=1}^{I} m_i \delta_{a_i} \quad \text{as Radon measures}$$

for some real numbers $m_1, \cdots, m_I$. A careful and classical concentration compactness blow-up study at each a_i, using similar arguments as above, shows that each number m_i may be identified as a sum of $z(\,[v_k]\,)$ where v_k are maps from S^p into N. One readily finds a single map $w_i : S^p \to S^p$ with $[w_i] = \sum_k [v_k]$. This ends the proof of (III.14) and (III.15).

The proof above shows that $\underline{\text{in } B_r(a_i)}$ the part of $u_n^{K_l}(x_1)$ given by

$$\sum_{(i_2,\cdots,i_{n_{K_l}+1})\in\mathfrak{I}_l} \int_{x_2\in\Omega_{i_2}} \cdots \int_{x_{n_{K_l}+1}\in\Omega_{i_{n_{K_l}+1}}} U^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$$

where $\mathfrak{I}_l$ is the set of n_{K_l}-tuples of numbers in $\{1,2\}$ such that at least one i_k is equal to 2, converges to

$$\sum_{(i_2,\cdots,i_{n_{K_l}+1})\in\mathfrak{I}_l} \int_{x_2\in\Omega_{i_2}} \cdots \int_{x_{n_{K_l}+1}\in\Omega_{i_{n_{K_l}+1}}} U^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$$

in $B_r(a_i)$. Only the local term

$$V_{r,a_i} \;=\; \int_{x_2\in B_r(a_i)} \cdots \int_{x_{n_{K_l}+1}\in B_r(a_i)} U^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$$

is concentrating. Using (II.19) (i.e. Lemma II.4), we have

$$|\int_{B_r(a_i)} V_{r,a_i}\,|^{\frac{n}{n+n_{K_l}}} \;\le\; C_{K_l} \int_{B_r} |\nabla u_n|^p \, d\mathcal{H}^p \quad .$$

Combining this facts with the results of B.White quoted at the beginning of the proof of the proposition which says that

$$\lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(a_i)} |\nabla u_n|^p \, d\mathcal{H}^p \quad \ge \quad \varepsilon_{p,N}$$

as long as $z[w_i] \neq 0$, we get (III.16). In order to establish (III.17) it suffices to combine the proof of (III.14) with the last part of Remark II.4. (III.19) is also a direct consequence of the previous observations. Proposition III.2 is then proved. ■

IV Scans of p Dimensional Tree-graph Forms for Maps of R^{p+1}.

IV.1 Notations and definitions.

For the remainder of the paper we will be discussing $W^{1,p}$ mappings

$$u \; : \; \mathbb{R}^{p+1} \quad \longrightarrow \quad N$$

by considering their restrictions to p spheres.

Letting $\mathbb{R}^{p+1} \times \mathbb{R}_+$ parameterize the space of all p dimensional spheres in $\mathbb{R}^{p+1}$ with $(c, r) \mapsto \partial B_r(c)$, we observe that, for almost all $(c, r) \in \mathbb{R}^{p+1} \times \mathbb{R}_+$, the *restriction* $u \,|\, \partial B_r(c) \in W^{1,p}(\partial B_r(c), S^p)$. We will be applying all the work of the previous chapters to the corresponding $W^{1,p}$ map on the unit sphere,

$$u_{c,r} \; : \; S^p \; \longrightarrow \; S^p \; , \quad u_{c,r}(x) \; = \; u(c + rx) \text{ for } x \in \partial B_r(c) \; .$$

A **scan** $\mathcal{S}$ in $\mathbb{R}^{p+1}$ will be simply a measurable map from the space of p spheres to the space $\mathcal{M}(S^p)$ of Radon measures on S^p,

$$\mathcal{S} \; : \; \mathbb{R}^{p+1} \times \mathbb{R}_+ \quad \longrightarrow \quad \mathcal{M}(S^p) \quad .$$

As described in (III.13), we may use, for any integrable p form ω on S^p, the same symbol ω to denote the corresponding absolutely continuous measure $(*\,\omega)\, \mathcal{H}^p \llcorner S^p \in \mathcal{M}(S^p)$. Thus, a linear combination K of p dimensional tree-graphs on S^p and a mapping $u \in C^\infty(\mathbb{R}^{p+1}, N)$ induce the **K-scan of the mapping** u ,

$$(c, r) \quad \longmapsto \quad u^K_{c,r} \quad ,$$

where $u^K_{c,r}$ is the p form constructed in II.2 using the Gauss integrals of II.4. Even for nonsmooth $u \in W^{1,p}(\mathbb{R}^{p+1}, N)$, where $u_{c,r} \in W^{1,p}(S^p, N)$ for almost all (c, r), similar constructions and integrations give a K-scan of u.

We next consider the behavior of the K-scans of mappings in a $W^{1,p}$ weakly convergent sequence.

IV.2 Convergence of Scans

The goal of this subsection is to prove the following proposition.

Proposition IV.1 *Let $K = K_z$ be a linear combination of simply connected tree-graphs of closed forms associated to a class z $\in$Hom$(\pi_p(N), \mathbb{R})$. For* *any sequence* $u_n \in C^\infty(\mathbb{R}^{p+1}, S^p)$ *converging* $W^{1,p}$ *weakly to some limit* $u \in W^{1,p}(\mathbb{R}^{p+1}, S^p)$*, there exist a subsequence* $u_{n'}$ *and a scan* $\mathcal{S}$ *of the form*

$$\mathcal{S}(c,r) \quad = \quad u^K_{c,r} \; + \; \sum_{i=1}^{I(c,r)} m_i(c,r)\, \delta_{a_i(c,r)} \tag{IV.1}$$

so that, for every $c \in \mathbb{R}^{p+1}$ *and almost every* $r \in \mathbb{R}$*, each subsequence of* $u_{n'}$ *contains a subsequence* $u_{n''}$ *so that*

$$(u_{n''})^K_{c,r} \quad \rightharpoonup \quad \mathcal{S}(c,r)$$

weakly as Radon measures. Here the quantities, $I(c,r) \in \mathbb{N}$*,* $m_i(c,r) \in \mathbb{R}_+ \cap z(\Pi_p(n))$*, and* $a_i(c,r) \in S^p$*, are all measurable in* (c,r)*.*

Proof of Proposition IV.1.

Let $\{(c_1,s_1),(c_2,s_2),\cdots\}$ be a countable dense subset of $\mathbb{R}^{p+1} \times \mathbb{R}_+$. By Fatou's lemma and Fubini's theorem,

$$\int_{r\in[s_1-1,s_1+1]} \liminf_{n\to\infty} \int_{\partial B_r(c_1)} |\nabla u_n|^p d\mathcal{H}^p dr \;\leq\; \sup_n \int_{\mathbb{R}^{p+1}} |\nabla u_n|^p \, dx \;<\; \infty \; ,$$

and we may choose a number $r_1 \in [s_1-1, s_1+1]$ and a subsequence $\alpha_1(n)$ so that

$$\sup_n \int_{\partial B_{r_1}(c_1)} |\nabla u_{\alpha_1(n)}|^p \, d\mathcal{H}^p \quad < \quad \infty \; .$$

Similarly we inductively find, for $k = 2, 3, \cdots$, numbers $r_k \in [s_k - \frac{1}{k}, s_k + \frac{1}{k}]$ and a subsequence $\alpha_k(n)$ of $\alpha_{k-1}(n)$ so that

$$\sup_n \int_{\partial B_{r_k}(c_k)} |\nabla u_{\alpha_k(n)}|^p \, d\mathcal{H}^p \quad < \quad \infty \; .$$

Then $\{(c_1,r_1),(c_2,r_2),\cdots\}$ is also dense in $\mathbb{R}^{p+1} \times \mathbb{R}_+$ and the diagonal sequence $\alpha_n(n)$ gives

$$\sup_n \int_{\partial B_{r_k}(c_k)} |\nabla u_{\alpha_n(n)}|^p \, d\mathcal{H}^p \quad < \quad \infty \; .$$

for all $k = 1, 2, \cdots$. By Proposition III.2, we may use another diagonal procedure to find a subsequence $u_{n'}$ so that, for every $k = 1, 2, \cdots$, one has on S^p the weak convergences of Radon measures

$$\lim_{n\to\infty} |\nabla (u_{n'})_{c_k,r_k}|^p \, \mathcal{H}^p \llcorner S^p \quad = \quad \mu_k$$

and

$$\lim_{n\to\infty}(u_{n'})^K_{c_k,r_k} \quad = \quad u^K_{c_k,r_k} \; + \; \sum_{i=1}^{I(c_k,r_k)} m_i(c_k,r_k)\,\delta_{a_i(c_k,r_k)} \quad , \tag{IV.2}$$

where μ_k is a positive Radon measure on S^p, $I(C_k, r_k) \in \{0, 1, \cdots\}$ and $m_i(c_k, r_k) \in \mathbb{R}_+$ and $a_i(c_k, r_k) \in S^p$ for $i = 1, \cdots, I(c_k, r_k)$. Consider now an arbitrary point $c \in \mathbb{R}^{p+1}$. Since $u_{n'}$ converges strongly to u in L^1_{loc}, the exceptional set

$$X_c = \left\{ r \in \mathbb{R} \;:\; \mathcal{H}^p(\{x \in \partial B_r(c) \;:\; \lim_{n'\to\infty} u_{n'}(x) \neq u(x)\}) \;>\; 0 \right\} \tag{IV.3}$$

has, by Fubini's theorem, measure zero. Using now Fatou's lemma, we have that

$$\int_{r\in\mathbb{R}} \liminf_{n'\to 0} \int_{\partial B_r(c)} (|\nabla u| + |\nabla u_{n'}|^p) d\mathcal{H}^p \;\leq\; 2 \sup_{n'} \int_{\mathbb{R}^{p+1}} |\nabla u_{n'}|^p dx \;<\; \infty \;,$$

and the set

$$Y_c = \left\{ r \in \mathbb{R} \;:\; \int_{\partial B_r(c)} |\nabla u|^p d\mathcal{H}^p \;+\; \liminf_{n'\to 0} \int_{\partial B_r(c)} |\nabla u_{n'}|^p d\mathcal{H}^p \;=\; \infty \right\} \tag{IV.4}$$

also has measure zero. Finally,

$$Z_c = \{ r \in \mathbb{R} \;:\; \mu_k(\partial B_r(c)) \;>\; 0 \text{ for some } k = 1, 2, \cdots \} \tag{IV.5}$$

is countable. Removing these three sets and taking $r \in \mathbb{R} \setminus (X_c \cup Y_c \cup Z_c)$, we can find a subsequence, that we keep denoting $u_{n'}$, such that

$$u_{n'} \;\rightharpoonup\; u \quad \text{ in } W^{1,p}(\partial B_r(c), N) \quad .$$

By Proposition III.2, any subsequence of $u_{n'}$ contains a subsequence $u_{n''}$ giving the weak convergence of measures in the form

$$(u_{n''})^K_{c,r} \quad \rightharpoonup \quad u^K_{c,r} \; + \; \sum_{i=1}^{\bar{I}} \bar{m}_i \;\delta_{\bar{a}_i} \quad ,$$

where $\bar{I} \in \{0, 1, \cdots\}$ and $\bar{m}_i \in \mathbb{R} \setminus \{0\}$ and $\bar{a}_i \in S^p$ for $i = 1, \cdots, \bar{I}$. To prove the proposition, we need to establish the uniqueness of this limit. That is,

$$\sum_{j=1}^{J} n_j \;\delta_{b_j} \quad = \quad \sum_{i=1}^{\bar{I}} \bar{m}_i \;\delta_{\bar{a}_i} \tag{IV.6}$$

whenever $u_{n'''}$ is another subsequence of $u_{n'}$ giving a similar weakly convergent limit,

$$(u_{n'''})^K_{c,r} \quad \rightharpoonup \quad u^K_{c,r} \; + \; \sum_{j=1}^{J} n_j \; \delta_{b_j} \quad . \tag{IV.7}$$

To verify (IV.6), take any one point $\bar{a}_i \in \{\bar{a}_1, \cdots, \bar{a}_{\bar{I}}\}$ and fix a positive number

$$\rho \; < \; \min\{\, |\bar{a}_i \, - \, b| \; : \; \bar{a}_i \neq b \in \{\bar{a}_1, \cdots, \bar{a}_{\bar{I}}, b_1, \cdots, b_J\,\} \quad .$$

We may choose an element (c_k, r_k) of our countable family above so that $r_k < \frac{1}{2}\rho$ and $\bar{a}_i \in B_{r_k}(c_k)$. Thus $B_{r_k}(c_k)$ either does not intersect $\{b_1, \cdots, b_J\}$ or contains exactly one b_j, which then coincides with $\bar{a}_i$.

Consider now the Lipschitz sphere

$$\Sigma \;\; = \;\; \partial\,(B_{r_k}(c_k) \cap B_r(c)) \;\; = \overline{\Sigma_1} \cup \Sigma_2 \quad ,$$

where

$$\Sigma_1 \;\; = \;\; \Sigma \cap B_r(c) \;\; = \;\; \partial B_{r_k}(c_k) \cap B_r(c) \quad ,$$

$$\Sigma_2 \;\; = \;\; \Sigma \cap B_{r_k}(c_k) \;\; = \;\; \partial B_r(c) \cap B_{r_k}(c_k) \quad .$$

Letting $\Psi(x) \; = \; r_k^{-1}(x - c_k)$ for $x \in \overline{\Sigma_1}$, we see that Ψ extends to a bilipschitz homeomorphism of $\Psi : \Sigma \to S^p$ whose restriction to Σ_2 is a smooth diffeomorphism.

Concerning the region $\overline{\Sigma_1}$, we have the convergence, as Radon measures on S^p, of the original subsequence $(u_{n'})^K_{c_k,r_k}$. This convergence then restricts, by (III.19), to the open subset $\Psi(\Sigma_1)$ and, by (IV.5), to the closure $\Psi(\overline{\Sigma_1})$. Using (III.14), (III.17), and (III.14), we deduce that,

$$\begin{aligned} (u_{n'} \circ \Psi^{-1})^K \llcorner \Psi(\overline{\Sigma_1}) \;\; &= \;\; (u_{n'})^K_{c_k,r_k} \llcorner \Psi(\overline{\Sigma_1}) \\ \rightharpoonup \;\; u^K_{c_k,r_k} \llcorner \Psi(\overline{\Sigma_1}) \; + \; \mathcal{R} \;\; &= \;\; (u \circ \Psi^{-1})^K \llcorner \Psi(\overline{\Sigma_1}) \; + \; \mathcal{R} \; , \end{aligned} \tag{IV.8}$$

where

$$\mathcal{R} \;\; = \;\; \sum_{a_i(c_k,r_k) \in \Sigma_1} m_i(c_k, r_k) \delta_{a_i(c_k,r_k)} \; .$$

Concerning the complementary region Σ_2, we may restrict the convergences of the two subsequences $(u_{n''})^K_{c,r}$ and $(u_{n'''})^K_{c,r}$ to the open subset $\Psi(\Sigma_2)$ of S^p. By our choice of ρ, (III.17), and (III.19),

$$(u_{n''} \circ \Psi^{-1})^K \llcorner \Psi(\Sigma_2) \;\; \rightharpoonup \;\; (u \circ \Psi^{-1})^K \llcorner \Psi(\Sigma_2) \;\; + \bar{m}_i \delta_{\Psi(c + r\bar{a}_i)} \tag{IV.9}$$

and similarly combining (IV.2), (IV.7), (III.17) and (III.19), we get that

$$(u_{n'''} \circ \Psi^{-1})^K \llcorner \Psi(\Sigma_2) \ \rightharpoonup \ (u \circ \Psi^{-1})^K \llcorner \Psi(\Sigma_2) \ + \ m\delta_{\Psi(c+r\bar{a}_i)} \ , \qquad \text{(IV.10)}$$

where

$$\text{either} \quad m = 0 \quad \text{in case} \ \ B_{r_k}(c_k) \cap \{b_1, \cdots, b_J\} = \emptyset \ ,$$

$$\text{or} \qquad m = n_j \quad \text{and} \ \ b_j = \bar{a}_i \ .$$

Finally we note that

$$\int_{S^p} (u_n \circ \Psi^{-1})^K \ = \ 0 \quad . \qquad \text{(IV.11)}$$

for all n because $u_n \circ \Psi$ is, by the continuity of u_n on $\overline{B_r(c) \cap B_{r_k}(c_k)}$, homotopic to zero in $[S^p, N]$.

Adding (IV.8) (with n' replaced by n'') and (IV.9), integrating over S^p, using (IV.11), and taking $\lim_{n''\to\infty}$, we deduce that

$$0 \ = \ \lim_{n''\to\infty} \int_{S^p} (u_{n''} \circ \Psi^{-1})^K \ = \ \left(\int_{S^p} (u \circ \Psi^{-1})^K \right) \ + \ \mathcal{R}(1) \ + \ \bar{m}_i \ .$$

Similarly using (IV.8) (with n' replaced by n''') , (IV.10) and (IV.11), and taking $\lim_{n'''\to\infty}$, we see that

$$0 \ = \ \lim_{n'''\to\infty} \int_{S^p} (u_{n'''} \circ \Psi^{-1})^K \ = \ \left(\int_{S^p} (u \circ \Psi^{-1})^K \right) \ + \ \mathcal{R}(1) \ + \ m \ .$$

Combining the last two equations we see that $m = \bar{m}_i \neq 0$ so that $n_j = m = \bar{m}_i$ and $b_j = \bar{a}_i$.

By repeating this argument we deduce that the two sets $\{b_1, \cdots, b_J\}$ and $\{\bar{a}_1, \cdots, \bar{a}_{\bar{I}}\}$ coincide and that the associated multiplicities are equal, which completes the proof of (IV.6).

Finally the measurability of the limiting scan

$$(c, r) \quad \longmapsto \quad u^K_{c,r} \ + \ \sum_{i=1}^{I(c,r)} m_i(c,r)\, \delta_{a_i(c,r)} \quad ,$$

and hence of $I(c,r)$, $m_i(c,r)$ and $a_i(c,r)$, follows from the measurability of pointwise limits of sequences of measurable functions and the separability of $C^0(S^p)$. ■

IV.3 Connecting $\pi_p(N) \otimes \mathbb{R}$ Singularities.

In this section we prove our main result : Theorem I.1. This is an immediate consequence of Proposition III.2(i)(ii) and the following structure theorem on the rectifiability of the "bubbled scan" in Proposition IV.1 :

Theorem IV.1 *Suppose $z \in Hom(\pi_p(N), \mathbb{R})$, $K = K_z$ is a corresponding linear combination of tree-graphs of forms,*

$$u_n \in C^\infty(\mathbb{R}^{p+1}, N) \quad \rightharpoonup \quad u \in W^{1,p}(S^p, N) \quad \textit{weakly in} \quad W^{1,p} \ ,$$

and $u_{n'}$ is the subsequence with bubbled scan

$$\mathcal{S}(c,r) \quad = \quad u^K_{c,r} \ + \sum_{i=1}^{I(c,r)} m_i(c,r)\, \delta_{a_i(c,r)} \quad ,$$

as in Proposition IV.1. Then there exist a countable union Γ of C^1 curves with measurable orientation $\vec{\Gamma}$ and a nonnegative $\mathcal{H}^1$ measurable function θ from Γ into $z(\pi_p(N))$ such that

$$\int_\Gamma \theta^{\frac{p}{p+n_z}} \ d\mathcal{H}^1 \ \le \ C_z \liminf_{n\to\infty} \int_{R^{p+1}} |\nabla u_n|^p \, dx \quad , \qquad \text{(IV.12)}$$

(with C_z depending only on z and n_z as in Definition II.1) and, for almost all $(c,r) \in \mathbb{R}^{p+1} \times \mathbb{R}_+$, $\partial B_r(c)$ is transverse to Γ, and

$$\mathcal{S}(c,r) \ = \ u^K_{c,r} \ + \sum_{a \in \Gamma \cap \partial B_r(c)} sgn\,[\vec{\Gamma}(a) \cdot (a-c)]\, \theta(a)\, \delta_{\frac{a-c}{r}} \ . \qquad \text{(IV.13)}$$

Thus,

$$\{a_1(c,r), \cdots, a_{I(c,r)}(c,r)\} \ = \ \left\{ \ \frac{a-c}{r} \ : \ a \in \Gamma \cap \partial B_r(c), \, \theta(a) \neq 0 \right\} \ ,$$

and

$$m_i(c,r) \ = \ sgn\,[\vec{\Gamma}(c + r a_i(c,r)) \cdot a_i(c,r)]\, \theta(c + r a_i(c,r)) \ .$$

We will need the following elementary lemma, a proof of which can be found in [HR1] (Lemma 7.1).

Lemma IV.1 *For $\varepsilon > 0$ and for any sequence $u_n \in W^{1,p}(\mathbb{R}^{p+1}, N)$ with $L = sup_n \int_{\mathbb{R}^{p+1}} |\nabla u_n|^p dx$ being finite, the ε energy concentration set*

$$E_\varepsilon = \left\{ c \in \mathbb{R}^{p+1} \ : \ \limsup_{r\to 0} \liminf_{n\to\infty} \frac{1}{r} \int_{B_r(c)} |\nabla u_n|^p \, dx \ \ge \ \varepsilon \right\}$$

has $\mathcal{H}^1(E_\varepsilon) \ \le \ C_p \, \varepsilon^{-1} L$ where C_p is a constant depending only on p.

Proof of Theorem IV.1. Suppose $(c, r) \in \mathbb{R}^{p+1} \times \mathbb{R}_+$, $a \in \partial B_r(c)$ and $0 < \rho < r$. Then the boundary

$$\Sigma \ = \ \partial\,(B_\rho(a) \cap B_r(c))$$

is uniformly bilipschitz homeomorphic to $\partial B_\rho(a)$. In fact, one may define $\Psi : \Sigma \to \partial B_\rho(a)$ to be the identity on $(\partial(B_\rho(a)) \cap B_r(c)$ and, on $(B_\rho(a)) \cap \partial B_r(c)$, to be the radial projection away from the point $c + (\frac{r-\rho}{r})(a - c)$ onto $\partial B_\rho(a) \setminus B_r(c)$. One checks that $\operatorname{Lip}\Psi \le 4$ and $\operatorname{Lip}\Psi^{-1} \le 1$. Thus

$$4^{-p}\int_\Sigma |\nabla u|^p d\mathcal{H}^p \ \le \ \int_{\partial B_\rho(a)} |\nabla(u \circ \Psi^{-1})|^p d\mathcal{H}^p \ \le \ 4^p \int_\Sigma |\nabla u|^p d\mathcal{H}^p \ .$$

For the corresponding map $(u \circ \Psi^{-1})_{a,\rho} : S^p \to N$, we have the conformal invariance

$$\int_{S^p} |\nabla\,(\,(u \circ \Psi^{-1})_{a,\rho})\,|^p\, d\mathcal{H}^p \ = \ \int_{\partial B_\rho(a)} |\nabla(u \circ \Psi^{-1})|^p\, d\mathcal{H}^p \ .$$

With $\varepsilon_1 = 10^{-1}4^{-p}\varepsilon_{p,N}$ where $\varepsilon_{p,N}$ is the constant introduced in (III.20), it follows that

$$\int_\Sigma |\nabla u|^p \ \le \ 10\varepsilon_1 \quad \Longrightarrow \quad [u \,|\, \Sigma] \ \sim \ [(u\circ\Psi^{-1})_{a,\rho}] \ = \ 0 \text{ in } \pi_p(N) \quad . \tag{IV.14}$$

Suppose $c \in \mathbb{R}^{p+1}$ and $r \in \mathbb{R}_+ \setminus (X_c \cup Y_c \cup Z_c)$ are as before (IV.3), (IV.4), (IV.5), and consider the "bubbling points" in $\partial B_r(c)$,

$$A(c, r) \quad = \quad \{\, c + r\, a_1(c, r),\ c + r\, a_2(c, r),\ \ldots\ ,\ c + r\, a_{I(c,r)}(c, r)\,\}\ .$$

Also recall the classical fact (see e.g. [Gi] Th.2.2) that the set of energy density points of the $W^{1,p}$ map u

$$S_u \ = \ \left\{ c \in \mathbb{R}^{p+1} \ : \ \limsup_{r\to 0} \frac{1}{r}\int_{B_r(c)} |\nabla u|^p\, dx \quad > \quad 0 \right\} \tag{IV.15}$$

has $\mathcal{H}^1(S_u) = 0$.

We will complete the proof in three steps:

Step 1. $A(c, r) \setminus W \subset E_{\varepsilon_1}$.

Suppose, for contradiction, that there is a point

$$a \ = \ c \,+\, r\, a_i(c, r) \ \in \ A(c, r) \setminus (W \cup E_{\varepsilon_1}) \quad .$$

By IV.4 and the fact that $a \notin (W \cup E_{\varepsilon_1})$, we can also choose a positive σ small enough so that $B_\sigma(a) \cap A(c,r) = \{a\}$,

$$\int_{\partial B_r(c) \cap B_\sigma(a)} |\nabla u|^p \, d\mathcal{H}^p \ le \ \varepsilon_1 \tag{IV.16}$$

and

$$\sigma^{-1} \int_{B_\sigma(a)} |\nabla u|^p \, dx \ + \ \liminf_{n' \to \infty} \sigma^{-1} \int_{B_\sigma(a)} |\nabla u_{n'}|^p \, dx \ \leq \ \varepsilon_1 \ .$$

Using now Fubini's theorem and Proposition IV.1, we can find a subsequence $u_{n''}$ of u'_n and a radius $\rho \in [\sigma/2, \sigma]$ such that

$$(u_{n''})^K_{a,\rho} \ \rightharpoonup \ \mathcal{S}(a,\rho) \quad , \tag{IV.17}$$

and

$$\lim_{n'' \to \infty} \int_{\partial B_\rho(a)} |\nabla u_{n''}|^p \, d\mathcal{H}^p \ \leq \ 6\varepsilon_1 \quad . \tag{IV.18}$$

Combining this facts with Proposition III.2 and (IV.14), we deduce that $(u_{n''})^K_{a,\rho}$ cannot concentrate to produce any $m_i(a,\rho)\delta_{a_i(a,\rho)}$ so that

$$\mathcal{S}(a,\rho) \ = \ u^K_{a,\rho} \tag{IV.19}$$

Let $\Sigma = \ \partial\left(B_\rho(a) \cap B_r(c)\right)$ and $\Psi : \Sigma \to \partial B_\rho(a)$ be as above. Using (IV.2), (IV.17), (IV.19), (III.17) and (III.19) as in the proof of IV.10, we deduce that

$$(u_{n''} \circ \Psi^{-1})^K_{a,\rho} \ \rightharpoonup \ (u \circ \Psi^{-1})^K_{a,\rho} \ + \ m_i(c,r) \, \delta_{\rho^{-1}(\Psi(a)-a)} \tag{IV.20}$$

Combining now (IV.16) and (IV.18) and (IV.14), we also have that the p-energy of $(u \circ \Psi^{-1})_{a,\rho}$ on Σ is below the required energy for having a nonzero $\pi_p(N)$ homotopy class. Thus

$$\int_{S^p} (u \circ \Psi^{-1})^K_{a,\rho} \ = \ 0 \quad . \tag{IV.21}$$

Since the restriction of $u_{n''}$ to Σ is also null homotopic (because it extends as a smooth map on $B_r(c) \cap B_\rho(a)$), we also have

$$\int_{S^p} (u_{n''} \circ \Psi^{-1})^K_{a,\rho} = 0 \quad . \tag{IV.22}$$

Combining (IV.20),(IV.21) and (IV.22), we obtain that $m_i(r,c) = 0$, a contradiction which completes the proof of Step 1.

Step 2. *Choice of* Γ.

Since, by Lemma IV.1, $\mathcal{H}^1(E_{\varepsilon_1}) < \infty$, we see from the Besicovitch structure theorem [Fe] 3.3.13 that

$$\mathcal{H}^1 \left(E_{\varepsilon_1} \setminus (\Gamma \cup \Lambda)\right) = 0 , \qquad \text{(IV.23)}$$

for some some countable union Γ of C^1 curves and some *totally unrectifiable set* Λ. The latter set does not intersect almost all p-spheres so that, by Fubini's theorem, the set

$$\mathcal{B}_\Lambda = \left\{c \in \mathbb{R}^{p+1} \; : \; \mathcal{H}^1 \left(\{r \in \mathbb{R}_+ \; : \; \partial B_r(c) \cap \Lambda \neq \emptyset\}\right) > 0\right\} \qquad \text{(IV.24)}$$

has $(p+1)$ dimensional Lebesgue measure zero.

By Lemma V.1 below, the set

$$\mathcal{A}_\Gamma = \left\{c \in \mathbb{R}^{p+1} \; : \; \mathcal{H}^1(T_{\Gamma, c}) > 0\right\} \qquad \text{(IV.25)}$$

where

$$T_{\Gamma, c} = \left\{x \in \Gamma \; : \; \partial B_{|x-c|}(c) \text{ is not transverse to } \Gamma \text{ at } x\right\}$$

also has $(p+1)$-dimensional Lebesgue measure zero.

For a point $c \in \mathbb{R}^{p+1} \setminus (\mathcal{A}_\Gamma \cup \mathcal{B}_\Lambda)$, we deduce from (IV.15), (IV.25),)IV.23), and (IV.24) that the set

$$W_c = \{r \in \mathbb{R}_+ \; : \; \partial B_r(c) \cap \left(S_u \cup T_{\Gamma, c} \cup \Lambda \cup [\, E_{\varepsilon_1} \setminus (\Gamma \cup \Lambda)\,]\right) \neq \emptyset\} \qquad \text{(IV.26)}$$

has $\mathcal{H}^1(W_c) = 0$.

Step 3. *Choice of multiplicity* θ *and orientation* $\vec{\Gamma}$.

We fix one point $c_0 \in \mathbb{R}^{p+1} \setminus (\mathcal{A}_\Gamma \cup \mathcal{B}_\Lambda)$, and, recalling (IV.3), (IV.4), (IV.5), and (IV.26), let

$$R_0 = \bigcup \{A(c_0, r) \; : \; r \in \mathbb{R}_+ \setminus (W_{c_0} \cup X_{c_0} \cup Y_{c_0} \cup Z_{c_0} \,)\} \; .$$

By Step 1 and (IV.26),

$$R_0 \subset \Gamma \quad .$$

On the subset $\Gamma \setminus R_0$ we simply define $\theta \equiv 0$, and take any choice of measurable orientation $\vec{\Gamma}$ there.

For any point $a \in R_0$, $a \in A(c_0, |a - c_0|\,)$ and

$$a = c_0 + |a - c_0| \, a_i(c_0, |a - c_0|\,)$$

for some $i \in \{1, \cdots, I(c_0, r)\}$, and we may define

$$\theta(a) \;=\; |m_i(c_0, |a - c_0|\,)\,| \quad ,$$

and choose the unique unit tangent vector $\vec{\Gamma}(a)$ so that

$$\operatorname{sgn}[\vec{\Gamma}(a) \cdot (a - c_0)] \;=\; \operatorname{sgn} m_i(c_0, |a - c_0|\,)$$

(because Γ is transverse to $\partial B_r(c)$ at a). This implies the $\mathcal{H}^1$ measurability of θ and $\vec{\Gamma}$, and inequality (IV.12) follows from (III.14) and (III.16). The functions θ and $\vec{\Gamma}$ were chosen so that the formula (IV.13) holds in case the center $c = c_0$ and the radius $r \in \mathbb{R}_+ \setminus (W_{c_0} \cup X_{c_0} \cup Y_{c_0} \cup Z_{c_0})$.

It only remains to verify that, with this choice of θ and $\vec{\Gamma}$, the formula (IV.13) is still true for an arbitrary c in $\mathbb{R}^{p+1} \setminus (\mathcal{A}_\Gamma \cup \mathcal{B}_\Lambda)$ and almost every $r > 0$. Since $c_0 \notin \mathcal{A}_\Gamma$, transversality a.e. implies that the set

$$\Upsilon c_0 \;=\; \{x \in \Gamma \;:\; |x - c_0| \in W_{c_0} \cup X_{c_0} \cup Y_{c_0} \cup Z_{c_0}\}$$

has measure $\mathcal{H}^1(\Upsilon_{c_0}) = 0$. It follows that

$$V_c \;=\; \{r \in \mathbb{R}_+ \;:\; \partial B_r(c) \cap \Upsilon_{c_0} \neq 0\}$$

also has $\mathcal{H}^1(V_c) = 0$. Now, if choose any radius

$$r \;\in\; \mathbb{R}_+ \setminus (V_c \cup W_c \cup X_c \cup Y_c \cup Z_c) \quad ,$$

then for any point $a \in \Gamma \cap \partial B_r(c)$, the distance $r_0 = |a - c_0|$ is also a good radius for c_0, that is, $r_0 \notin (W_{c_0} \cup X_{c_0} \cup Y_{c_0} \cup Z_{c_0})$. In particular, we have the transversality of Γ at a with respect to both spheres $\partial B_r(c)$ and $\partial B_{r_0}(c_0)$.

For the orientations, there are four possibilities,

$$\operatorname{sgn}[\vec{\Gamma}(a) \cdot (a - c)] \;=\; \pm 1 \;, \quad \operatorname{sgn}[\vec{\Gamma}(a) \cdot (a - c_0)] \;=\; \pm 1 \quad .$$

To simplify the presentation, we assume that

$$\operatorname{sgn}[\vec{\Gamma}(a) \cdot (a - c)] \;=\; \operatorname{sgn}[\vec{\Gamma}(a) \cdot (a - c_0)] \;=\; +1 \tag{IV.27}$$

(the other cases can be treated in a similar way). As above, we have, with $r = |a - c|$ and $r_0 = |a - c_0|$, that

$$a \;=\; c_0 \;+\; r_0\, a_i(c_0, r_0) \;=\; c \;+\; r\, a_j(c, r)$$

for some $i \in \{1, \cdots, I(c_0, r_0)\}$ and $j \in \{1, \cdots, I(c, r)\}$. To establish (IV.13), it now suffices to show that

$$m_j(c, r) \;=\; m_i(c_0, r_0) \quad .$$

First note that for positive ρ less than

$$\delta \ = \ \min\{\mathrm{dist}(a, A(c,r)\setminus\{a\})\ ,\ \mathrm{dist}(a, A(c_0,r_0)\setminus\{a\})\}\ ,$$

one has

$$B_\rho(a)\cap A(c,r) \ = \ \{a\} \ = \ B_\rho(a)\cap A(c_0,r_0)\ . \tag{IV.28}$$

Let $\Omega \ = \ [B_r(c)\setminus B_{r_0}(c_0)]\cup[B_{r_0}(c)\setminus B_r(c)]$. By (IV.27), the unit tangent vectors $\pm\vec{\Gamma}(a)$ lie outside the (closed) tangent cone $\mathrm{Tan}(\Omega,a)$. It follows that

$$\lim_{\sigma\to 0}\sigma^{-1}\mathcal{H}^1\left(\left\{\rho\in[0,\sigma]\ :\ \Gamma\cap\partial B_\rho(a)\cap\tilde{\Omega}\neq\emptyset\right\}\right) \ = \ 0 \tag{IV.29}$$

for some (slightly larger) open cone $\tilde{\Omega}$ about a containing $a+\mathrm{Tan}(\Omega,a)$.

Also, since $x\notin S_u$,

$$\lim_{\sigma\to 0}\sigma^{-1}\int_\sigma^{2\sigma}\int_{\partial B_\rho(x)}|\nabla u|^p\,d\mathcal{H}^p\,d\rho \ \le\ \lim_{\sigma\to 0}\sigma^{-1}\int_{B_\sigma(a)}|\nabla u|^p dx \ = \ 0\ , \tag{IV.30}$$

and clearly

$$\int_{\partial B_r(c)\cap B_\rho(a)}|\nabla u|^p\,d\mathcal{H}^p \ + \ \int_{\partial B_{r_0}(c_0)\cap B_\rho(a)}|\nabla u|^p\,d\mathcal{H}^p \ \to \ 0 \tag{IV.31}$$

as $\rho\to 0$.

We will now argue as in Step 1 using the two Lipschitz spheres

$$\Sigma \ = \ \partial\left(B_\rho(a)\cap B_r(c)\right) \quad \text{and} \quad \Sigma_0 \ = \ \partial\left(B_\rho(a)\cap B_{r_0}(c_0)\right) \quad .$$

¿From (IV.29), (IV.30), (IV.31), and (IV.14), we see that we may choose a sufficiently small positive $\rho<\delta$ so that

$$\Gamma\ \cap\ \partial B_\rho(a)\ \cap\ (\Omega\cup\tilde{\Omega}) \ = \ \emptyset\ , \tag{IV.32}$$

so that the restrictions

$$u\,|\,\Sigma \quad \text{and} \quad u\,|\,\Sigma_0$$

have sufficiently small p energies so that their $\Pi_p(N)$ homotopy classes both vanish, and so that there is, as in Proposition IV.1, weak convergence of subsequences of the p forms $(u_{n'})^K_{a,\rho}$ to $\mathcal{S}(a,\rho)$.

Let

$$\Psi:\Sigma\to\partial B_\rho(a) \quad \text{and} \quad \Psi_0:\Sigma_0\to\partial B_\rho(a)$$

be bilipschitz homeomorphisms as in Step 1. Now, after translation and rescaling, we are considering the five sequences of maps from S^p to N,

$$(u_n)_{c,r}\ ,\quad (u_n)_{c_0,r_0}\ ,\quad (u_n)_{a,\rho}\ ,\quad (u_n\circ\Psi^{-1})_{a,\rho}\ ,\quad (u_n\circ\Psi_0^{-1})_{a,\rho}\quad .$$

Since $u_n\,|\,\Sigma$ and $u_n\,|\,\Sigma_0$ have smooth extensions to $\mathbb{R}^{p+1}$, they are null-homotopic and the integrals of the corresponding tree-graph forms vanish,

$$\int_{S^p} (u_n \circ \Psi^{-1})^K_{a,\rho} \;=\; 0 \;=\; \int_{S^p} (u_n \circ \Psi_0^{-1})^K_{a,\rho} \;.$$

Similarly,

$$\int_{S^p} (u \circ \Psi^{-1})^K_{a,\rho} \;=\; 0 \;=\; \int_{S^p} (u \circ \Psi_0^{-1})^K_{a,\rho} \;.$$

because, as we saw above, the p homotopy classes of $u\,|\,\Sigma$ and $u\,|\,\Sigma_0$ also vanish.

Our Proposition IV.1 allows us to pass to consecutive subsequences. Concerning the bubbling, we may, as before, consider the partition

$$\Sigma \;=\; (\Sigma \cap B_r(c)) \cup \left(\Sigma \cap \overline{B_\rho(a)}\right) \;,$$

and similarly for Σ_0. On $\Sigma \cap \overline{B_\rho(a)}$ (respectively, $\Sigma_0 \cap \overline{B_\rho(a)}$), we have, by (IV.1) and (IV.28), the single term

$$m_j(c,r)\delta_{a_j(c,r)} \quad \text{(respectively,} \quad m_i(c_0,r_0)\delta_{a_i(c_0,r_0)}\;)$$

while, on $\Sigma \cap B_r(c)$ or $\Sigma_0 \cap B_{r_0}(c_0)$ all the bubbling occurs, by (IV.32), on the intersection

$$\Sigma \cap B_r(c) \,\cap\, \Sigma_0 \cap B_{r_0}(c) \;=\; (\partial B_\rho(a)) \,\cap\, B_r(c) \cap B_{r_0}(c_0) \;.$$

Putting this all together, we have for some subsequence $u_{n''}$

$$\begin{aligned} 0 \;&=\; \lim_{n\to\infty} \int_{S^p} (u_{n''} \circ \Psi^{-1})^K_{a,\rho} \\ &=\; \int_{S^p} (u_\circ \Psi^{-1})^K_{a,\rho} \;+\; m_j(c,r) \;+\; \sum_{a+\rho a_k(a,\rho)\in B_r(c)} m_k(a,\rho) \\ &=\; 0 \;+\; m_j(c,r) \;+\; \sum_{a+\rho a_k(a,\rho)\in B_r(c)\cap B_{r_0}(c_0)} m_k(a,\rho) \quad . \end{aligned}$$

Similarly,

$$0 \;=\; 0 \;+\; m_i(c_0,r_0) \;+\; \sum_{a+\rho a_k(a,\rho)\in B_{r_0}(c_0)\cap B_r(c)} m_k(a,\rho) \quad .$$

The last two equations now give the desired equality $m_j(c,r) = m_i(c_0,r_0)$. ∎

V Appendix

Suppose $\gamma : [0,L] \to \mathbb{R}^n$ is a C^1 curve, $t \in [0,1]$, and $a \in \mathbb{R}^n$. Then γ is, at $\gamma(t)$, *transverse* to the intersecting sphere $\partial B_{|\gamma(t)-a|}(a)$ if and only if

$$\dot\gamma(t) \cdot (\gamma(t) - a) \neq 0 \quad .$$

For most centers a, one has such transversality at most points of the curve. More precisely,

Lemma V.1 *For any C^1 embedded curve $\gamma : [0,L] \to \mathbb{R}^n$, the set*

$$\mathcal{A}_\gamma = \left\{ a \in \mathbb{R}^n : \mathcal{H}^1\left(\{t \in [0,L] : \dot\gamma(t) \cdot (\gamma(t) - a) = 0\}\right) > 0 \right\} \quad ,$$

is countably $(n-2)$ rectifiable, and thus has n dimensional Lebesgue measure zero.

Proof of Lemma V.1 : For $i = 0, \cdots, n$, let G_i denote the Grassmann of i dimensional vector subspaces of $\mathbb{R}^n$, and consider the projection

$$\pi_i : \mathbb{R}^n \times G_i \longrightarrow \mathbb{R}^n \quad , \qquad \pi_i(a,P) = a$$

for $(a,P) \in \mathbb{R}^n \times G_i$. For each such (a,P), also let

$$T_{(a,P)} = \{t \in [0,L] \quad : \quad \dot\gamma(t) \cdot (\gamma(t) - a) = 0\,,\ \dot\gamma(t) \subset P\} \quad .$$

Letting

$$S_i = \{(a,P) \in \mathbb{R}^n \times G_i \quad : \quad \mathcal{H}^1(T_{(a,P)}) > 0\ \} \quad ,$$

we readily infer that $S_0 = \emptyset$. We also claim that $S_1 = \emptyset$. In fact otherwise, there is a convergent sequence $t_i \to t_0$ in some $T_{(a,P)}$ with $\dim P = 1$. Assuming γ is parameterized by arc-length, $\dot\gamma(t_i)$ is, for $i = 0$ and i sufficiently large, the same unit vector v whose span is the line P. But then

$$1 = v \cdot \dot\gamma(t_0) = \lim_{i\to\infty} v \cdot \frac{\gamma(t_i) - \gamma(t_0)}{t_i - t_0}$$

$$= \lim_{i\to\infty} \frac{\dot\gamma(t_i)\cdot\gamma(t_i) - \dot\gamma(t_0)\cdot\gamma(t_0)}{t_i - t_0} = \lim_{i\to\infty} \frac{\dot\gamma(t_i)\cdot a - \dot\gamma(t_0)\cdot a}{t_i - t_0} = 0\ ,$$

a contradiction. Thus, $S_1 = \emptyset$.

Next we note that

$$\mathcal{A}_\gamma = \pi_n(S_n) \supset \pi_{n-1}(S_{n-1}) \supset \cdots \supset \pi_1(S_1) = \emptyset \quad ,$$

so that it suffices to show that each set

$$\mathcal{A}_i \quad = \quad \pi_i(S_i) \setminus \pi_{i-1}(S_{i-1})$$

is countably $n-2$ rectifiable for $i = 2, 3, \cdots, n$.

Consider two points (a, P), (a', P') in S_i with $P \neq P'$. Then

$$\mathcal{H}^1(T_{(a,P)} \cap T_{(a',P')}) \;=\; 0 \quad , \tag{V.1}$$

because otherwise the inclusion

$$T_{(a,P)} \cap T_{(a',P')} \subset T_{(a,P\cap P')}$$

would imply that $\mathcal{H}^1(T_{(a,P\cap P')}) \;>\; 0$ and $a \in \pi_j(S_j)$ with $j = \dim(P\cap P') < i$, contradicting $a \notin \pi_{i-1}(S_{i-1})$. It now follows from (V.1) that distinct P give $\mathcal{H}^1$ essentially disjoint positive measure subsets of $[0, L]$. Thus

$$\mathcal{P}_i \quad = \quad \{P \;:\; (a, P) \in S_i \text{ for some } a \in \mathbb{R}^n\}$$

is countable.

Fixing $P \in \mathcal{P}_i$, we now claim that for any two distinct points a and a' satisfying

$$\mathcal{H}^1(T_{(a,P)} \cap T_{(a',P)}) \;>0 \quad ,$$

the vector $a - a'$ is orthogonal to P. In fact, otherwise the $n-1$ dimensional vectorspace H orthogonal to $a - a'$ would not contain P, $\dim(P \cap H) = i-1$, and, as before, the inclusion

$$T_{(a,P)} \cap T_{(a',P)} \subset T_{(a,P\cap H)}$$

would imply that $\mathcal{H}^1(T_{(a,P\cap H)}) > 0$ and $a \in \pi_{i-1}(S_{i-1})$, contradicting $a \in \mathcal{A}_i$. It follows that the unique affine $(n-i)$ plane Q_a which is orthogonal to P and passes through a must coincide with the corresponding $(n-i)$ plane $Q_{a'}$. In other words, if $Q_a \neq Q_{a''}$ for two points a and a'' for which both $\mathcal{H}^1(T_{(a,P)})$ and $\mathcal{H}^1(T_{(a'',P)})$ are positive, then necessarily

$$\mathcal{H}^1(T_{(a,P)} \cap T_{(a'',P)}) = 0 \quad .$$

It follows, as before, that the family $\mathcal{Q}_P$ of all such orthogonal affine $(n-i)$ planes Q_a with $\mathcal{H}^1(T_{(a,P)}) > 0$ is countable because distinct ones give $\mathcal{H}^1$ essentially disjoint positive measure subsets of $[0, L]$.

We now conclude that

$$\mathcal{A}_\gamma \quad \subset \quad \bigcup_{i=2}^{n} \bigcup_{P\in\mathcal{P}_i} \bigcup_{Q\in\mathcal{Q}_P} Q$$

is countably $(n-2)$ rectifiable. ∎

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