Bubbling Phenomena and Weak Convergence for Maps in $W^{2,2}(\mathbf{B}^5,\mathbf{S}^3)$

(version 1)

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§0. Introduction.

§1. Preliminaries.

We will let c denote an absolute constant whose value may change from statement to statement and which is usually easily estimable.

Let $\mathbf{H}:\mathbf{S}^3\to\mathbf{S}^2$ be the standard Hopf map [HR], and $\mathbf{SH}:\mathbf{S}^4\to\mathbf{S}^3$ be its suspension:

$$\mathbf{SH}(x_0,\ldots,x_4) \;=\; \big(x_0\,,\,\sqrt{1-x_0^2}\,\cdot\,\Pi\big(\frac{x_1}{\sqrt{x_1^2+\ldots+x_4^2}},\ldots,\frac{x_4}{\sqrt{x_1^2+\ldots+x_4^2}}\big)\big)$$

for $(x_0,\ldots,x_4)\in\mathbf{S}^5$. The latter map generates the nonzero element of $\pi_4(\mathbf{S}^3)\simeq\mathbf{Z}_2$. Also, its homogeneous degree 0 extension

$$\mathbf{SH}\big(\frac{x}{|x|}\big)\;\in\;W^{2,2}(\mathbf{B}^5,\mathbf{S}^3)\;.$$

Let $\mathcal{R}_\infty(\mathbf{B}^5,\mathbf{S}^3)$ denote the class of maps that are smooth except for finitely many suspension Hopf singularities. That is,

$$u\in\mathcal{R}_\infty(\mathbf{B}^5,\mathbf{S}^3)\quad\Longleftrightarrow$$

$$u\in\mathcal{C}^\infty(\mathbf{B}^5\setminus\{a_1,\ldots,a_m\},\mathbf{S}^3)\ \text{ and }\ u(x)\;=\;\mathbf{SH}\big(\frac{x-a_i}{|x-a_i|}\big)\ \text{on}\ \mathbf{B}_{\delta_0}(a_i)\setminus\{a_i\}$$

for some finite subset $\{a_1,\ldots,a_m\}$ of $\mathbf{B}^5$ and some $\delta_0>0$.

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Lemma 1.1. $\mathcal{R}_\infty(\mathbf{B}^5,\mathbf{S}^3)$ *is strongly dense in* $W^{2,2}(\mathbf{B}^5,\mathbf{S}^3)$.

(Proof to be written): For covers of the bad set, homogeneous extension is to be replaced by the following:

Lemma 1.2. Given $f \in W^{2,2}(\partial\mathbf{B}^5,\mathbf{S}^3)$ and $g \in W^{1,2}(\partial\mathbf{B}^5,\mathbf{R}^4)$ with $g\cdot f = 0$ a.e. on $\partial\mathbf{B}^5$, there exists $v \in W^{2,2}(\mathbf{B}^5,\mathbf{S}^3)$ such that

$$v \;=\; f \;\text{ and }\; \frac{\partial v}{\partial \nu} \;=\; g \;\text{ on }\; \partial\mathbf{B}^5 \;,$$

$$\|\, v\,\|_{W^{2,2}} \;\leq\; c\big(\;\|\, f\,\|_{W^{2,2}} \;+\; \|\, g\,\|_{W^{1,2}}\;\big).$$

(Proof to be written): Replace $\mathbf{B}^5$ by the upper half-space in space-time $\mathbf{R}^4\times\mathbf{R}$. Let $v = \frac{w}{|w|}$ where

$$w(x,t) \;=\; f(x) \;-\; t\,h(x,t)$$

with h being a solution of the heat equation $h_t = \Delta h$ with $h(x,0) = g(x)$.

§2. Connecting Singularities with Controlled Length.

Suppose $u \in \mathcal{R}^\infty(\mathbf{B}^5,\mathbf{S}^3)$ with $\operatorname{Sing} u = \{a_1,a_2,\ldots,a_m\}$ as above. Our goal in this section is to connect the singular points a_i in pairs by some union of curves whose total length is bounded by a constant multiple of the *Hessian energy*

$$\int_{\mathbf{B}^5} |\nabla^2 u|^2\,dx\;.$$

These curves allow one to "topologically cancel" the singularities of u. Specifically one may then slightly modify u in consecutively smaller tubular neighborhoods of these curves to obtain a sequence of completely smooth maps which weakly approach u in $W^{2,2}$. The extra Hessian energy required for this construction will be proportional to the total length of the curves and hence to the Hessian energy of u. This construction, along with Lemma 1.1, will establish the weak density of $\mathcal{C}^\infty(\mathbf{B}^5,\mathbf{S}^3)$ in $W^{2,2}(\mathbf{B}^5,\mathbf{S}^3)$.

By the surjectivity of the Hopf map (and its suspension) each *regular* value $p \in \mathbf{S}^3\setminus\{(-1,0,0,0),\,(1,0,0,0)\}$ of u gives a level surface

$$\Sigma = u^{-1}\{p\}$$

which necessarily contains all the singular points a_i of u. Note that $\Sigma = u^{-1}\{p\}$ is smoothly embedded away from the a_i with standard orientation $\omega_\Sigma \equiv *u^\#\omega_{\mathbf{S}^3}/|u^\#\omega_{\mathbf{S}^3}|$, induced from u. Concerning the behavior near a_i, the neighborhood

$$\Sigma\cap\mathbf{B}_{\delta_0}(a_i)\setminus\{a_i\}$$

is simply a truncated cone whose boundary

$$\Gamma_i \;=\; \Sigma \cap \partial \mathbf{B}_{\delta_0}(a_i)$$

is a planar circle in the 3-sphere $\partial \mathbf{B}_{\delta_0}(a_i) \cap \big(\{p_0\} \times \mathbf{R}^4\big)$ where $p = (p_0, p_1, p_2, p_3)$.

We will eventually choose the desired "topologically-cancelling" curves all to lie on such a level surface Σ.

2.1 *Estimates for Choosing the Surface* $\Sigma = u^{-1}\{p\}$.

We first recall the 3 Jacobian $J_3 u = \| \wedge_3 Du\|$ and apply the coarea formula to

$$\frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u}\ ,$$

we obtain the relation

$$\int_{\mathbf{S}^3} \int_{u^{-1}\{p\}} \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u}\, d\mathcal{H}^2\, d\mathcal{H}^3 p \;=\; \int_{\mathbf{B}^5} |\nabla u|^4 + |\nabla^2 u|^2\ . \tag{2.1}$$

Moreover, since $\|u\|_{L^\infty} = 1$, we also have (see [MR]) the integral inequality

$$\int_{\mathbf{B}^5} |\nabla u|^4 \le\; c \int_{\mathbf{B}^5} |\nabla^2 u|^2\ . \tag{2.2}$$

In case u is constant on $\partial \mathbf{B}^5$, we verify this by computing

$$\begin{aligned}
\int_{\mathbf{B}^5} |\nabla u|^4 \;&=\; \int_{\mathbf{B}^5} \big(\nabla u \cdot \nabla u\big) |\nabla u|^2 \\
&=\; \int_{\mathbf{B}^5} \operatorname{div}\big(u \nabla u |\nabla u|^2\big) \;-\; u \cdot \Delta u |\nabla u|^2 \;-\; u \nabla u \cdot \nabla\big(|\nabla u|^2\big) \\
&\le\; 0 \;+\; 5 \int_{\mathbf{B}^5} |\nabla^2 u| |\nabla u|^2 \;+\; 2 \int_{\mathbf{B}^5} |\nabla^2 u| |\nabla u|^2 \\
&\le\; \frac{1}{2} \int_{\mathbf{B}^5} |\nabla u|^4 \;+\; \frac{49}{2} \int_{\mathbf{B}^5} |\nabla^2 u|^2\ .
\end{aligned}$$

In the general case we write $u = \sum_{i=1}^{\infty} \lambda_i u$ where $\{\lambda_i\}$ is a partition of unity adapted to a family of Whitney cubes for $\mathbf{B}^5$. See [MR]. (The above inequality is true even with the constraint $\|u\|_{BMO} \le 1$ in place of $\|u\|_{L^\infty} \le 1$ [MR].)

By (2.1) and (2.2) we may now choose a regular value $p \in \mathbf{S}^3 \backslash \{(-1,0,0,0),\ (1,0,0,0)\}$ of u so that

$$\int_{u^{-1}\{p\}} \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u}\, d\mathcal{H}^2 \;\le\; c \int_{\mathbf{B}^5} |\nabla^2 u|^2\ . \tag{2.3}$$

2.2 *Pull-back Normal Framing for* $\Sigma = u^{-1}\{p\}$.

Suppose again that $p = (p_0, p_1, p_2, p_3) \in \mathbf{S}^3 \setminus \{(-1,0,0,0),\ (1,0,0,0)\}$ is a regular value of u. Then

$$\eta_1 \;=\; \Big(-\sqrt{1-p_0^2}\,,\,\frac{p_0p_1}{\sqrt{1-p_0^2}}\,,\,\frac{p_0p_2}{\sqrt{1-p_0^2}}\,,\,\frac{p_0p_3}{\sqrt{1-p_0^2}}\Big)$$

is the unit vector tangent at p to the geodesic that runs from $(1,0,0,0)$ through p to $(-1,0,0,0)$. We may choose two other vectors

$$\eta_2\,,\ \eta_3 \;\in\; \mathrm{Tan}\big(\{p_0\}\times\sqrt{1-p_0^2}\,\mathbf{S}^2, p\big) \;\subset\; \mathrm{Tan}\,(\mathbf{S}^3, p)$$

so that η_1, η_2, η_3 becomes an orthonormal basis for $\mathrm{Tan}\,(\mathbf{S}^3, p)$. Since p is a regular value for u, these three vectors lift to three unique smooth linearly independent normal vectorfields τ_1, τ_2, τ_3 along $\Sigma = u^{-1}\{p\}$. That is, at each point $x \in \Sigma$,

$$\tau_j(x)\;\perp\;\Sigma \text{ at } x \quad\text{and}\quad Du(x)\big[\tau_j(x)\big] \;=\; \eta_j$$

for $j = 1, 2, 3$.

Near each singularity a_i the lifted vectorfields τ_1, τ_2, τ_3 are also orthonormal. In fact, for $x \in \Sigma \cap \mathbf{B}_{\delta_0}(a_i)$, $\frac{x_0 - a_{i0}}{|x-a_i|} = p_0$, and

$$\tau_1(x) \;=\; \Big(-\sqrt{1-p_0^2}\,,\,\frac{p_0}{\sqrt{1-p_0^2}}\frac{x_1-a_{i1}}{|x-a_i|}\,,\,\frac{p_0}{\sqrt{1-p_0^2}}\frac{x_2-a_{i2}}{|x-a_i|}\,,\,\frac{p_0}{\sqrt{1-p_0^2}}\frac{x_3-a_{i3}}{|x-a_i|}\Big)\ . \tag{2.4}$$

Also $\tau_1(x), \tau_2(x), \tau_3(x)$ are orthonormal for such x because the Hopf map is horizontally orthogonal and the lifts $\tau_2(x), \tau_3(x)$ are tangent to the 3 sphere $\{p_0\}\times\sqrt{1-p_0^2}\,\mathbf{S}^3$.

On the remainder of the surface $\Sigma \setminus \cup_{i=1}^m \mathbf{B}_{\delta_0}(a_i)$, the linearly independent vectorfields τ_1, τ_2, τ_3 are not necessarily orthonormal, and we use their Gram-Schmidt orthonormalizations

$$\begin{aligned}
\tilde\tau_1 &= \frac{\tau_1}{|\tau_1|}\ ,\\
\tilde\tau_2 &= \frac{\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1|} \;=\; \frac{\tau_2 - (\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tilde\tau_1\wedge\tau_2|}\ ,\\
\tilde\tau_3 &= \frac{\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2}{|\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2|} \;=\; \frac{\tau_3 - (\tilde\tau_1\cdot\tau_3)\tilde\tau_1 - (\tilde\tau_2\cdot\tau_3)\tilde\tau_2}{|\tilde\tau_1\wedge\tilde\tau_2\wedge\tau_3|}
\end{aligned}$$

which provide an *orthonormal framing* for the unit normal bundle of Σ.

We need to estimate the total variation of these orthonormalizations. Noting that

$|\nabla\big(\frac{a}{|a|}\big)| \le 2\frac{|\nabla a|}{|a|}$ for any differentiable a, we see that

$$
\begin{aligned}
|\nabla\tilde\tau_1| &\le 2\frac{|\nabla\tau_1|}{|\tau_1|} \le 2\frac{|\nabla\tau_1||\tau_1||\tau_2||\tau_3|}{|\tau_1||\tau_1\wedge\tau_2\wedge\tau_3|} = 2\frac{|\tau_2||\tau_3||\nabla\tau_1|}{|\tau_1\wedge\tau_2\wedge\tau_3|} ,\\
|\nabla\tilde\tau_2| &= 2\Big[\frac{\tau_2-(\tilde\tau_1\cdot\tau_2)\tilde\tau_1}{|\tilde\tau_1\wedge\tau_2|}\Big] \le 2\Big[\frac{2|\nabla\tau_2|+2|\tau_2||\nabla\tilde\tau_1|}{|\tau_1\wedge\tau_2||\tau_1|^{-1}}\Big]\\
&\le 8\Big[\frac{|\tau_1\|\nabla\tau_2|+|\tau_2||\nabla\tau_1|}{|\tau_1\wedge\tau_2|}\cdot\frac{|\tau_1\wedge\tau_2||\tau_3|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\Big]\\
&= 8\Big[\frac{|\tau_2||\tau_3||\nabla\tau_1|+|\tau_1||\tau_3||\nabla\tau_2|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\Big] ,\\
|\nabla\tilde\tau_3| &\le 2\Big[\frac{3|\nabla\tau_3|+2|\tau_3||\nabla\tilde\tau_1|+2|\tau_3||\nabla\tilde\tau_2|}{|\tilde\tau_1\wedge\tilde\tau_2\wedge\tau_3|}\Big]\\
&\le 32\Big[\frac{|\nabla\tau_3|+|\tau_3||\tau_1|^{-1}|\nabla\tau_1|+|\tau_3|\big(\frac{|\tau_1\|\nabla\tau_2|+|\tau_2||\nabla\tau_1|}{|\tau_1\wedge\tau_2|}\big)}{|\frac{\tau_1}{|\tau_1|}\wedge\big(\frac{\tau_2}{||\tau_1|^{-1}\tau_1\wedge\tau_2|}\big)\wedge\tau_3|}\Big]\\
&\le 32\Big[\frac{|\tau_1||\tau_2||\nabla\tau_3|+|\tau_2||\tau_3||\nabla\tau_1|+|\tau_1||\tau_3||\nabla\tau_2|}{|\tau_1\wedge\tau_2\wedge\tau_3|}\Big] .
\end{aligned}
$$

Inasmuch as

$$|\tau_j| \le |\nabla u| , \quad |\nabla\tau_j| \le |\nabla^2 u| , \quad |\tau_1\wedge\tau_2\wedge\tau_3| = J_3 u ,$$

we deduce the general pointwise estimate

$$|\nabla\tilde\tau_j| \le c\frac{|\nabla u|^2|\nabla^2 u|}{|J_3 u|} \le c\frac{|\nabla u|^4+|\nabla^2 u|^2}{|J_3 u|} ,$$

which we may integrate using (2.3) to obtain the variation estimate along $\Sigma = u^{-1}\{p\}$,

$$\int_\Sigma |\nabla\tilde\tau_j|\, d\mathcal{H}^2 \le c\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx . \tag{2.5}$$

2.3 *Twisting of the Normal Frame* $\tilde\tau_1,\tilde\tau_2,\tilde\tau_3$ *About Each Singularity* a_i.

First we recall from [M] that the Grassmannian

$$\tilde G_2(\mathbf{R}^5)$$

of *oriented* 2 planes through the origin in $\mathbf{R}^5$ is a compact smooth manifold of dimension 6. It may be identified with the set of simple unit 2 vectors in $\mathbf{R}^5$,

$$\{v\wedge w \in \wedge_2\mathbf{R}^5 \ :\ v\in\mathbf{S}^4,\ w\in\mathbf{S}^4,\ v\cdot w = 0\} .$$

We will use the distance $|P-Q|$ on $\tilde G_2(\mathbf{R}^5)$ given by this embedding into $\wedge_2\mathbf{R}^5 \approx \mathbf{R}^{10}$.

For a fixed plane $P \in \tilde{G}_2(\mathbf{R}^5)$, the set of *nontransverse* 2 planes

$$\mathcal{Q}_P \;=\; \{Q \in \tilde{G}_2(\mathbf{R}^5) \;:\; P \cap Q \neq \{0\}\}$$

is a (Schubert) subvariety of dimension $1+3=4$ because every $Q \in \mathcal{Q}_P \setminus \{P\}$ equals $v \wedge w$ for some $w \in \mathbf{S}^4 \cap P$ and some $v \in \mathbf{S}^4 \cap w^\perp$. These subvarieties are all orthogonally isomorphic and, in particular, have the same finite 4 dimensional Hausdorff measure. Also

$$Y_P \;=\; \{Q \in \mathcal{Q}_P \;:\; P^\perp \cap Q \neq \{0\}\}$$

is a closed subvariety of dimension 3, and $\mathcal{Q}_P \setminus Y_P$ is a smooth submanifold.

For each point $x \in \Sigma \setminus \{a_1, \ldots, a_m\}$, we abbreviate

$$\mathcal{Q}_x \;\equiv\; \mathcal{Q}_{\mathrm{Tan}\,(\Sigma,x)} \;.$$

Then, near each singularity a_i, the set of 2 planes nontransverse to the cone $\Sigma \cap \mathbf{B}_{\delta_0}(a_i) \setminus \{a_i\}$,

$$W \;=\; \cup_{x \in \Sigma \cap \mathbf{B}_{\delta_0}(a_i)\setminus\{a_i\}} \mathcal{Q}_x \;=\; \cup_{x \in \Gamma_i} \mathcal{Q}_x \;,$$

has dimension only $1+4=5<6=\dim \tilde{G}_2(\mathbf{R}^5)$. It also does not depend on i because u has identical behavior near each a_i.

We now describe explicitly how the framing $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$ *twists once* as x goes around each circle Γ_i. The problem is that the vectors $\tilde{\tau}_j(x)$ lie in the normal space $\mathrm{Nor}\,(\Sigma, x)$ which also varies with x. To measure the rotation of the frame $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$, as x traverses the circle Γ_i, it is necessary to use some *reference frame* for $\mathrm{Nor}\,(\Sigma, x)$.

We can induce such a frame from some fixed unit vectors in $\mathbf{R}^5$ as follows:. Consider a fixed $Q \in \tilde{G}_2(\mathbf{R}^5) \setminus W$, and suppose $Q = v \wedge w$ with v, w being an orthonormal basis for Q. For each $x \in \Gamma_i$, the orthogonal projections of v, w onto $\mathrm{Nor}\,(\Sigma, x)$ are linearly independent; let $\sigma_1(x)$, $\sigma_2(x)$ be their Gram-Schmidt orthonormalizations. We then get $\sigma_3(x)$ by using the map u to pull-back the orientation of $\mathbf{S}^3$ to $\mathrm{Nor}\,(\Sigma, x)$ so that the resulting orienting 3 vector is $\sigma_1(x) \wedge \sigma_2(x) \wedge \sigma_3(x)$ for a unique unit vector $\sigma_3(x) \in \mathrm{Nor}\,(\Sigma, x)$ orthogonal to $\sigma_1(x)$, $\sigma_2(x)$. We view

$$\sigma_1(x)\,,\;\sigma_2(x)\,,\;\sigma_3(x)$$

as the *reference frame* determined by the fixed vectors v, w. For each $x \in \Gamma_i$, there is then a unique rotation $\gamma(x) \in \mathbf{SO}(3)$ so that

$$\gamma(x)\big[\sigma_j(x)\big] \;=\; \tilde{\tau}_j(x) \quad \text{for } j = 1, 2, 3 \;.$$

In the next paragraph we will check that $\gamma : \Gamma_i \to \mathbf{SO}(3)$ is a single geodesic circle in $\mathbf{SO}(3)$. The twisting of the frame $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ around the circle Γ_i is reflected in the fact that such a circle induces the nonzero element in $\pi_1(\mathbf{SO}(3)) \simeq \mathbf{Z}_2$.

In the special case $v = (1,0,0,0)$, the normalized orthogonal projection of v onto $\mathrm{Nor}\,(\Sigma, x)$ is, by (2.4), simply

$$\sigma_1(x) \;=\; \tilde{\tau}_1(x) \;.$$

So in this case, each orthogonal matrix $\gamma(x)$ is a rotation about the first axis, and one checks that, as x traverses the circle Γ_i once, these rotations complete a single geodesic circle in $\mathbf{SO}(3)$. For another choice of v, the geodesic circle $\gamma : \Gamma_i \to \mathbf{SO}(3)$ involves a circle of rotations about a different axis combined with a single orthogonal change of coordinates.

2.4 *Reference Normal Framing for* $\Sigma = u^{-1}\{p\}$.

The above calculations near the a_i suggest comparing on the *whole* surface $u^{-1}\{p\}$ the pull-back normal framing $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$ with some reference normal framing $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ induced by two fixed vectors v, w. Unfortunately, there may not exist fixed vectors v, w so that the corresponding reference framing σ_1, σ_2, σ_3 is defined *everywhere* on Σ. In this section we show that any orthonormal basis v, w of almost every oriented 2 plane $Q \in \tilde{G}_2(\mathbf{R}^5)$ gives a reference framing on Σ which is well-defined and smooth except at finitely many *discontinuities*

$$b_1, b_2, \ldots, b_n \;.$$

We will then need to connect the original singularities a_i to the b_j and, in §2.6, choose other curves to connect the b_j to each other, with all curves having total length bounded by a multiple of $\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx$.

To find a suitable $Q = v \wedge w$ we will first rule out the exceptional planes that contain some nonzero vector normal to Σ at some point $x \in \Sigma$. The really exceptional 2 planes that lie completely in some normal space

$$X \;=\; \cup_{x\in\Sigma} X_x \;,\quad X_x \;=\; \{Q \in \tilde{G}_2(\mathbf{R}^5) \;:\; Q \subset \mathrm{Nor}\,(\Sigma, x)\}$$

has dimension at most $2+2 = 4 < 6 = \dim \tilde{G}_2(\mathbf{R}^5)$ because $\dim \Sigma = 2$ and $\dim \tilde{G}_2(\mathbf{R}^3) = 2$. The remaining exceptional planes

$$Y \;=\; \cup_{x\in\Sigma} Y_x \;,\quad Y_x \;=\; \{Q \in \tilde{G}_2(\mathbf{R}^5) \;:\; \dim\big(Q \cap \mathrm{Nor}\,(\Sigma, x)\big) = 1\}$$

has dimension at most $2+2+1 = 5 < 6 = \dim \tilde{G}_2(\mathbf{R}^5)$ because

$$Y_x = \;\{e \wedge w \;:\; e \in \mathbf{S}^4 \cap \mathrm{Nor}\,(\Sigma, x) \text{ and } w \in \mathbf{S}^4 \cap \mathrm{Tan}\,(\Sigma, x)\} \;.$$

In terms of our previous notation, $Y_{\mathrm{Tan}\,(\Sigma,x)} = X_x \cup Y_x$.

Any unit vector $e \notin \mathrm{Nor}\,(\Sigma, x)$ has a nonzero orthogonal projection

$$e_T(x)$$

onto $\mathrm{Tan}\,(\Sigma, x)$.

Normalizing

$$\tilde{e_T}(x) \;=\; \frac{e_T(x)}{|e_T(x)|} \;,$$

we find a unique unit vector $e_\Sigma(x) \in \mathrm{Tan}\,(\Sigma, x)$ orthogonal to $e_T(x)$ so that $\tilde{e_T}(x) \wedge e_\Sigma(x)$ is the standard orientation of $\mathrm{Tan}\,(\Sigma, x)$. Then

$$e \cdot e_\Sigma(x) \;=\; \big(e - e_T(x)\big) \cdot e_\Sigma(x) \;+\; e_T(x) \cdot e_\Sigma(x) \;=\; 0 \;+\; 0$$

because $e - e_T(x) \in \mathrm{Nor}\,(\Sigma, x)$. Thus,

$$\tilde{e_T}(x),\; e_\Sigma(x), \tilde{\tau}_1(x),\; \tilde{\tau}_2(x),\; \tilde{\tau}_3(x),$$

is an orthonormal basis for $\mathbf{R}^5$.

Away from the 4 dimensional unit normal bundle

$$\mathcal{N}_\Sigma \;=\; \{(x, e) \;:\; x \in \Sigma,\; e \in \mathbf{S}^4 \cap \mathrm{Nor}\,(\Sigma, x)\} \;,$$

we now define the basic map

$$\Phi \;:\; (\Sigma \times \mathbf{S}^4\big) \setminus \mathcal{N}_\Sigma \;\to\; G_2(\mathbf{R}^5) \;,\quad \Phi(x, e) = e \wedge e_\Sigma(x) \;,$$

to parameterize the planes *nontransverse to* Σ in $\tilde{G}_2(\mathbf{R}^5)\setminus Y$. Incidentally, these do include the 2 dimensional family of tangent planes

$$Z \;=\; \{Q \in \tilde{G}_2(\mathbf{R}^5) \;:\; Q = \mathrm{Tan}\,(\Sigma, x) \text{ for some } x \in \Sigma\} \;.$$

In terms of the notation at the beginning of this section, for any 2 plane $Q \notin Y$,

$$Q \in \mathcal{Q}_x = \mathcal{Q}_{\mathrm{Tan}\,(\Sigma,x)} \iff Q \;=\; \Phi(x, e) \text{ for some } e \in \mathbf{S}^4 \setminus \mathrm{Nor}\,(\Sigma, x) \;.$$

Note that $\Phi(x, -e) = \Phi(x, e)$, and, in fact,

$$\Phi(x, e') = \Phi(x, e) \in \tilde{G}_2(\mathbf{R}^5) \setminus Y \quad\iff\quad e' = \pm\, e \;.$$

It is also easy to describe the behavior of Φ at the singular set $\mathcal{N}_\Sigma$. A 2 plane Q belongs to Y, that is, $Q = v \wedge w$ for some $v \in \mathrm{Nor}\,(\Sigma, x) \cap \mathbf{S}^4$ and $w \in \mathrm{Tan}\,(\Sigma, x) \cap \mathbf{S}^4$, if and only if $Q = \lim_{n\to\infty} \Phi(x_n, v_n)$ for some sequence $(x_n, v_n) \in (\Sigma \times \mathbf{S}^4\big)\setminus \mathcal{N}_\Sigma$ approaching (x, v). The map Φ essentially "blows-up" the 4 dimensional $\mathcal{N}_\Sigma$ to the 5 dimensional Y, and, in particular, *any smooth curve in* $\tilde{G}_2(\mathbf{R}^5)$ *transverse to* Y *lifts by* Φ *to a pair of antipodal curves in* $\Sigma \times \mathbf{S}^4$ *extending continuously transversally across* $\mathcal{N}_\Sigma$.

We now choose and fix $Q \in G^2(\mathbf{R}^5)$ so that *neither Q nor $-Q$ belong to the* 5 *dimensional exceptional set* $W \cup X \cup Y \cup Z$ *and both are regular values of* Φ.

In particular, since

$$\dim(\Sigma \times \mathbf{S}^4) = 6 = \dim \tilde{G}_2(\mathbf{R}^5) ,$$

$\Phi^{-1}\{Q, -Q\}$ is a finite set, say

$$\Phi^{-1}\{Q,-Q\} \;=\; \{(b_1,e_1),(b_1,-e_1),(b_2,e_2),(b_2,-e_2),\ldots,(b_n,e_n),(b_n,-e_n)\} \ .$$

We now see that the reference framing $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ of $\operatorname{Nor}(\Sigma, x)$ corresponding to any fixed orthonormal basis v, w of Q fails to exist precisely at the points $b_1, b_2, \ldots, b_n$. As before, we now have the smooth mapping

$$\gamma : \Sigma \setminus \{a_1,\ldots,a_m,b_1,\ldots,b_n\} \;\to\; \mathbf{SO}(3) \ ,$$

which is defined by the condition $\gamma(x)\big[\sigma_j(x)\big] \;=\; \tilde{\tau}_j(x)$ for $j = 1,2,3$ or, in column-vector notation,

$$\gamma \;=\; \big[\sigma_1\sigma_2\sigma_3\big]^{-1}\big[\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3\big] \ .$$

2.5 *Asymptotic Behavior of* γ *Near the Singularities* a_i *and* b_j.

As discussed in §3.1, the map u, the surface $\Sigma = u^{-1}\{p\}$, the frames $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ and $\sigma_1, \sigma_2, \sigma_3$, and the rotation field γ are all precisely known near a singularity a_i in the cone neighborhood $\Sigma \cap \mathbf{B}_{\delta_0}(a_i) \setminus \{a_i\}$. In particular, γ is homogeneous of degree 0 on $\Sigma \cap \mathbf{B}_{\delta_0}(a_i) \setminus \{a_i\}$; on its boundary $\gamma | \Gamma_i$ is a constant-speed geodesic circle.

At each b_j, the frame $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ is smooth, but the frame $\sigma_1, \sigma_2, \sigma_3$, and hence the rotation γ, has an essential discontinuity. Nevertheless, we may deduce some of the asymptotic behavior at b_j because $\pm Q$ were chosen to be regular values of Φ. In fact, we'll verify:

The tangent map γ_j *of* γ *at* b_j,

$$\gamma_j : \operatorname{Tan}(\Sigma, b_j) \cap \mathbf{B}_1(0) \to \mathbf{SO}(3) \ , \quad \gamma_j(x) = \lim_{r\to 0} \gamma\big[\exp^{\Sigma}_{b_j}(rx)\big] \ ,$$

exists and is the homogeneous degree 0 *extension of some reparameterization of a geodesic circle in* $\mathbf{SO}(3)$.

In particular, for small positive δ, $\gamma \,|\, \big(\Sigma \cap \partial \mathbf{B}_\delta(b_j)\big)$ is an embedded circle inducing the nonzero element of $\pi_1(\mathbf{SO}(3)) \simeq \mathbf{Z}_2$.

To check this, we use, as above, the more convenient orthonormal basis $\{e_j, e_{j\Sigma}\}$ for Q; that is,

$$e_{j\Sigma} = e_{j\Sigma}(b_j) \in \operatorname{Tan}(\Sigma, b_j) \quad \text{and} \quad Q = e_j \wedge e_{j\Sigma} = \Phi(b_j, \pm e_j) \ .$$

Then, for $x \in \Sigma$, let

$$e_j^N(x) \ , \quad e_{j\Sigma}^N(x)$$

denote the orthogonal projections of the fixed vectors e_j, $e_{j\Sigma}$ onto $\mathrm{Nor}\,(\Sigma, x)$, and

$$\hat{e}_j^N(x)$$

denote the cross-product of $e_{j\Sigma}^N(x)$ and $e_j^N(x)$ in $\mathrm{Nor}\,(\Sigma, x)$. These three vectorfields are smooth near b_j with

$$e_j^N(b_j) \ \neq \ 0 \ , \quad e_{j\Sigma}^N(b_j) \ = \ 0 \ , \quad \hat{e}_j^N(b_j) \ = \ 0 \ .$$

Here our insistence that $\pm Q \notin Z$ guarantees that Q is not tangent to Σ at b_j. Let g_j denote the orthogonal projection of $\mathbf{R}^5$ onto the 2 plane

$$P_j \ = \ \mathrm{Nor}\,(\Sigma, b_j) \cap \left[e_j^N(b_j)\right]^\perp \ .$$

Then $G_j(x) \ = \ g_j \circ e_j^N(x)$ defines a smooth map from a Σ neighborhood of b_j to P_j, which has, by the regularity of Φ at $(b_j, \tilde{e}_j)$, a simple, nondegenerate zero at b_j (of degree ± 1). It follows that as x circulates $\Sigma \cap \partial \mathbf{B}_\delta(b_j)$ once, for δ small, $G_j(x)$ and similarly $g_j \circ \hat{e}_j^N(x)$, circulate 0 once in P_j. Returning to the original basis v, w of Q, we now check that, as x circulates $\Sigma \cap \partial \mathbf{B}_\delta(b_j)$ once, the frame $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ approximately, and asymptotically as $\delta \to 0$, rotates once about the vector $e_j^N(b_j)$. Since the frame $\tilde{\tau}_1(x), \tilde{\tau}_2(x), \tilde{\tau}_3(x)$ is smooth at b_j, we see that the map γ has, at b_j, a tangent map γ_j as described above.

2.6 *Connecting the Singularities a_i to the b_j.*

Here we will find curves reaching all the a_i and b_j. Concerning the a_i, we recall from [B,§III,10] that $\mathbf{SO}(3)$ is isometric to $\mathbf{R}P^3 \simeq \mathbf{S}^3/\{x \sim -x\}$. Any geodesic circle Γ in $\mathbf{SO}(3)$ generates $\pi_1(\mathbf{SO}(3)) \simeq \mathbf{Z}_2$ and lifts to a great circle $\tilde{\Gamma}$ in $\mathbf{S}^3$. The rotations at maximal distance from Γ form another geodesic circle $\Gamma^\perp$ and the nearest point retraction

$$\rho_\Gamma : \mathbf{SO}(3) \setminus \Gamma \ \to \ \Gamma^\perp$$

is induced by the standard nearest point retraction

$$\rho_{\tilde{\Gamma}} : \mathbf{S}^3 \setminus \tilde{\Gamma} \ \to \ \tilde{\Gamma}^\perp \ .$$

In particular,

$$|\nabla \rho_\Gamma(\zeta)| \ \leq \ \frac{c}{\mathrm{dist}\,(\zeta, \Gamma)} \quad \text{for } \zeta \in \mathbf{SO}(3) \ . \tag{2.6}$$

Any geodesic circle Γ' in $\mathbf{SO}(3)$ that does *not* intersect Γ is mapped diffeomorphically by ρ_Γ onto the circle $\Gamma^\perp$. We deduce that if Γ is chosen to miss the asymptotic circles

$$\gamma(\Gamma_i) \quad \text{and} \quad \gamma_j\big(\mathrm{Tan}\,(\Sigma, b_j) \cap \mathbf{S}^4\big)$$

associated with the singularities a_i and b_j, then, on Σ, the composition $\rho_\Gamma \circ \gamma$ maps every sufficiently small circle

$$\Sigma \cap \partial \mathbf{B}_\delta(a_i) \ \text{ and } \ \Sigma \cap \partial \mathbf{B}_\delta(b_j)$$

diffeomorphically onto the circle $\Gamma^\perp$.

Under the identification of $\mathbf{SO}(3)$ with $\mathbf{R}P^3$, $\mathbf{O}(4)$ acts transitively by isometry on

$$\mathcal{G} \ = \ \{\text{geodesic circles } \Gamma \subset \mathbf{SO}(3)\ \}\ .$$

Then $\mathcal{G}$ is compact and admits a positive invariant measure $\mu_\mathcal{G}$. For $\mu_\mathcal{G}$ almost every circle Γ,

$$\Gamma \cap \gamma(\Gamma_i) = \emptyset \ \text{ for } \ i = 1, \ldots, m\ , \qquad \Gamma \cap \gamma_j\big(\mathrm{Tan}\,(\Sigma, b_j) \cap \mathbf{S}^4\big) = \emptyset \ \text{ for } \ j = 1, \ldots, n\ ,$$

and Γ is transverse to the map γ. In particular, $\gamma^{-1}(\Gamma)$ is a finite subset

$$\{c_1, c_2, \ldots, c_\ell\}$$

of Σ. For such a circle Γ and any regular value $z \in \Gamma^\perp$ of

$$\rho_\Gamma \circ \gamma \ : \ \Sigma \setminus \{a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_\ell\} \ \to \ \Gamma^\perp\ ,$$

the fiber

$$A \ = \ (\rho_\Gamma \circ \gamma)^{-1}\{z\}$$

is a smooth embedded 1 dimensional submanifold with

$$\overline{A} \setminus A \ \subset \ \{a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_\ell\}\ .$$

We also can deduce the local behavior of A near each of the points a_i, b_j, c_k. From the above description of the asymptotic behavior of γ near a_i and b_j, we see that

$$\overline{A} \cap \mathbf{B}_{\delta_0}(a_i)$$

is simply a *single line segment with one* <u>*endpoint*</u> a_i while

$$\overline{A} \cap \mathbf{B}_\delta(b_j)$$

is, for δ sufficiently small, a *single smooth segment with one* <u>*endpoint*</u> b_j. On the other hand,

$$\overline{A} \cap \mathbf{B}_\delta(c_k)$$

is, for δ sufficiently small, a *single smooth segment with an* <u>*interior*</u> *point* c_k. To see this, observe that, for the lifted map $\rho_{\tilde{\Gamma}} : \mathbf{S}^3 \setminus \tilde{\Gamma} \to \tilde{\Gamma}^\perp$ and any point $\tilde{z} \in \tilde{\Gamma}^\perp$, the fiber $\rho_{\tilde{S}}^{-1}\{z\}$ is an open great hemisphere, centered at z, with boundary $\tilde{\Gamma}$. It follows for the downstairs map ρ_Γ that $E_z = \operatorname{Clos}\left(\rho_\Gamma^{-1}\{z\}\right\}$ is a full geodesic 2 sphere containing z and the circle Γ. Since the surface $\gamma(\Sigma)$ intersects the circle Γ transversely at a finite set, this sphere E_z is also transverse to $\gamma(\Sigma)$ near this set. Thus, for δ sufficiently small, $\overline{A} \cap \mathbf{B}_\delta(c_k)$, being mapped diffeomorphically by γ onto the intersection $E_z \cap \gamma\big(\Sigma \cap \mathbf{B}_\delta(c_k)\big)$, is an open smooth segment containing c_k in its interior.

Combining this boundary behavior with the interior smoothness of the 1 manifold A, we now conclude that

$\overline{A}$ globally consists of finitely many disjoint smooth segments joining pairs of points in $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$, and each such point is joined by a unique segment to another such point.

2.7 *Estimating the Length of the Connecting Set A.*

The definition of the A depends on many choices:

(1) the point $p \in \mathbf{S}^3$, which determines the surface $\Sigma = u^{-1}\{p\}$,

(2) the vectors $\eta_2, \eta_2, \eta_3 \in \operatorname{Tan}(\mathbf{S}^3, p)$, which determine the pull-back normal framing $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$,

(3) the vectors $v, w \in \mathbf{S}^4$, which determine the reference normal framing $\sigma_1, \sigma_2, \sigma_3$ and the rotation field $\gamma = \big[\sigma_1\sigma_2\sigma_3\big]^{-1}\big[\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3\big] \ : \ \Sigma \setminus \{b_1, \ldots, b_m\} \to \mathbf{SO}(3)$,

(4) the circle $\Gamma \subset \mathbf{SO}(3)$, which determines the retraction $\rho_\Gamma : \mathbf{SO}(3) \setminus \Gamma \to \Gamma^\perp$, and

(5) the point $z \in \Gamma^\perp$, which finally gives $A = (\rho_\Gamma \circ \gamma)^{-1}\{z\}$.

We need to make suitable choices of these to get the desired length estimate for A. In §2.1, we already used one coarea formula to choose $p \in \mathbf{S}^3$ to give the basic estimate (2.3)

$$\int_\Sigma \frac{|\nabla u|^4 + |\nabla^2 u|^2}{J_3 u}\, d\mathcal{H}^2 \ \le \ c \int_{\mathbf{B}^5} |\nabla^2 u|^2 \ ,$$

and the pull-back frame estimate (2.5)

$$\int_\Sigma |\nabla \tilde{\tau}_j|\, dx \ \le \ c \int_{\mathbf{B}^5} |\nabla^2 u|^2 \ ,$$

independent of the choice of η_1, η_2, η_3, then followed.

For the choice of $z \in S^\perp$, we want to use another coarea formula,

$$\int_{\Gamma^\perp} \mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\}\,dz \;=\; \int_\Sigma |\nabla(\rho_\Gamma \circ \gamma)|\,d\mathcal{H}^2\ . \tag{2.7}$$

To bound the righthand integral, we first use the chain rule and (2.6) for the pointwise estimate

$$|\nabla(\rho_\Gamma \circ \gamma)(x)| \;=\; |\nabla(\rho_\Gamma)\big(\gamma(x)\big)||\nabla\gamma(x)| \;\le\; \frac{c|\nabla\gamma(x)|}{\operatorname{dist}\big(\gamma(x),\Gamma\big)}\ . \tag{2.8}$$

Next we observe the finiteness of the integral

$$C \;=\; \int_{\mathcal{G}} \frac{1}{\operatorname{dist}(\zeta,\Gamma)}\,d\mu_{\mathcal{G}}\Gamma \;<\; \infty\ ,$$

independent of the point $\zeta \in \mathbf{SO}(3)$. To verify this, we note that $\mu_{\mathcal{G}}(\mathcal{G}) < \infty$ and choose a smooth coordinate chart for $\mathbf{SO}(3)$ near ζ that maps ζ to $0 \in \mathbf{R}^3$ and that transforms circles into affine lines in $\mathbf{R}^3$. Distances are comparable, and an affine line in $\mathbf{R}^3 \setminus \{0\}$ is described by its nearest point a to the origin and a direction in the plane $a^\perp$. Since

$$\mu_{\mathcal{G}}\{\Gamma \in \mathcal{G}\ :\ \zeta \in \Gamma\} \;=\; 0\ ,$$

the finiteness of C now follows from the finiteness of the 3 dimensional integral

$$\int_{\mathbf{R}^3 \cap \mathbf{B}_1} |y|^{-1}\,dy\ .$$

We deduce from Fubini's Theorem, (2.7), and (2.8) that

$$\begin{aligned}\int_{\mathcal{G}}\int_{\Gamma^\perp} \mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\}\,dz\,d\mu_{\mathcal{G}}\Gamma \;&\le\; c\int_\Sigma |\nabla\gamma(x)|\int_{\mathcal{G}} \frac{1}{\operatorname{dist}\big(\gamma(x),\Gamma\big)}\,d\mu_{\mathcal{G}}\Gamma\,d\mathcal{H}^2x\\ &\le\; c\,C\int_\Sigma |\nabla\gamma(x)|\,d\mathcal{H}^2x\ .\end{aligned}$$

Thus there exists a $\Gamma \in \mathcal{G}$ and $z \in \Gamma^\perp$ so that

$$\mathcal{H}^1(\rho_\Gamma \circ \gamma)^{-1}\{z\} \;\le\; c\int_\Sigma |\nabla\gamma(x)|\,d\mathcal{H}^2x\ . \tag{2.9}$$

To estimate the righthand side, recall the matrix formula

$$\gamma \;=\; \big[\sigma_1\sigma_2\sigma_3\big]^{-1}\big[\tilde\tau_1\tilde\tau_2\tilde\tau_3\big]\ .$$

and use Cramer's rule and the product and quotient rules to deduce the pointwise bound

$$|\nabla\gamma(x)| \;\le\; c\sum_{j=1}^3 \big(\,|\nabla\sigma_j(x)| \;+\; |\nabla\tilde\tau_j(x)|\,\big)\ . \tag{2.10}$$

In light of (2.5), it remains to bound each term $\int_\Sigma |\nabla \sigma_j(x)|\, d\mathcal{H}^2 x$ for $j = 1, 2, 3$.

For the first one, note that

$$|\nabla \sigma_1| \;=\; |\nabla \big(\frac{v^N}{|v^N|}\big)| \;\leq\; 2\frac{|\nabla v^N|}{|v^N|} \tag{2.11}$$

where $v^N(x)$ is the orthogonal projection of v onto the normal space $\mathrm{Nor}\,(\Sigma, x)$ for each $x \in \Sigma$. The formula

$$v^N \;=\; \sum_{j=1}^{3} (v \cdot \tilde{\tau}_j)\tilde{\tau}_j$$

and the product rule give the pointwise estimate for the numerator,

$$|\nabla v^N| \;\leq\; c \sum_{j=1}^{3} |\nabla \tilde{\tau}_j| \;, \tag{2.12}$$

independent of the choice of $v \in \mathbf{S}^4$.

To estimate the denominator, we let v^L denote the orthogonal projection of v to any *fixed* 3 dimensional subspace L of $\mathbf{R}^5$, and observe the finiteness

$$C_1 \;=\; \int_{\mathbf{S}^4} \frac{1}{|v^L|}\, d\mathcal{H}^4 v \;<\; \infty \;,$$

independent of L. To verify this, we note that the projection of $\mathbf{S}^4$ to L vanishes along a great circle, and, near any point of this circle, the projection is bilipschitz equivalent to an orthogonal projection of $\mathbf{R}^4$ to $\mathbf{R}^3$. So the finiteness of C_1 again follows from the finiteness of the 3 dimensional integral $\int_{\mathbf{R}^3 \cap \mathbf{B}_1} |y|^{-1}\, dy$.

By Fubini's Theorem, (2.11), (2.12), and (2.5),

$$\begin{aligned}
\int_{\mathbf{S}^4} \int_\Sigma |\nabla \sigma_1(x)|\, d\mathcal{H}^2 x\, d\mathcal{H}^4 v \;&\leq\; 2 \int_\Sigma |\nabla v^N(x)| \int_{\mathbf{S}^4} \frac{1}{|v^N(x)|}\, d\mathcal{H}^4 v\, d\mathcal{H}^2 x \\
&\leq\; 2C_1 \int_\Sigma |\nabla v^N(x)|\, d\mathcal{H}^2 x \\
&\leq\; c \sum_{j=1}^{3} \int_\Sigma |\nabla \tilde{\tau}_j(x)|\, d\mathcal{H}^2 x \\
&\leq\; c \int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx \;.
\end{aligned}$$

So there exists a $v \in \mathbf{S}^4$ giving the σ_1 estimate

$$\int_\Sigma |\nabla \sigma_1(x)|\, d\mathcal{H}^2 x \;\leq\; c \int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx \;. \tag{2.13}$$

Next we observe that $\sigma_2 = \frac{w_2}{|w_2|}$ where $w_2(x)$ is the orthogonal projection onto the 2 dimensional subspace $\mathrm{Nor}\,(\Sigma, x) \cap \sigma_1^\perp$. We again find

$$|\nabla \sigma_2| \;=\; |\nabla (\frac{w_2}{|w_2|})| \;\le\; 2\frac{|\nabla w_2|}{|w_2|} \;. \tag{2.14}$$

Now the formula

$$w_2 \;=\; \big[\sum_{j=1}^{3} (w\cdot\tilde\tau_j)\tilde\tau_j\big] \;-\; (w\cdot\sigma_1)\sigma_1 \;,$$

and the product rule give the pointwise estimate for the numerator,

$$|\nabla w_2| \;\le\; c\big(|\nabla\sigma_1| \;+\; \sum_{j=1}^{3} |\nabla\tilde\tau_j|\big) \;, \tag{2.15}$$

independent of the choice $w \in \mathbf{S}^4$.

To estimate the denominator, we let w^M denote the orthogonal projection of w to any *fixed* 2 dimensional subspace M of the hyperplane $v^\perp = \sigma_1^\perp$, and observe the finiteness of the integral

$$C_2 \;=\; \int_{\mathbf{S}^4\cap v^\perp} \frac{1}{|w^M|}\, d\mathcal{H}^3 w \;<\; \infty \;,$$

independent of the choices of v or M. To verify this, we note that the projection of the 3 sphere $\mathbf{S}^4 \cap v^\perp$ to M vanishes along a great circle, where it is now bilipschitz equivalent to an orthogonal projection of $\mathbf{R}^3$ to $\mathbf{R}^2$. So the finiteness of C_2 this time follows from the finiteness of the 2 dimensional integral $\int_{\mathbf{R}^2\cap \mathbf{B}_1} |y|^{-1}\, dy$.

By Fubini's Theorem, (2.5), (2.12), (2.13), (2.14) and (2.15),

$$\begin{aligned}
\int_{\mathbf{S}^4\cap v^\perp}\int_\Sigma |\nabla\sigma_2(x)|\, d\mathcal{H}^2 x\, d\mathcal{H}^3 w \;&\le\; 2\int_\Sigma |\nabla w_2(x)| \int_{\mathbf{S}^4\cap v^\perp} \frac{1}{|w_2(x)|}\, d\mathcal{H}^3 w\, d\mathcal{H}^2 x\\
&\le\; 2C_2 \int_\Sigma |\nabla w_2(x)|\, d\mathcal{H}^2 x\\
&\le\; c\int_\Sigma \big(|\nabla\sigma_1(x)| \;+\; \sum_{j=1}^{3} |\nabla\tilde\tau_j(x)|\big)\, d\mathcal{H}^2 x\\
&\le\; c\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx \;.
\end{aligned}$$

So there exists a $w \in \mathbf{S}^4 \cap v^\perp$ giving the σ_2 estimate

$$\int_\Sigma |\nabla\sigma_2(x)|\, d\mathcal{H}^2 x \;\le\; c\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx \;. \tag{2.16}$$

Finally we may use the product rule and the formula

$$\begin{aligned}\sigma_3 \;=\; & \big[(\sigma_1\cdot\tilde\tau_2)(\sigma_2\cdot\tilde\tau_3)-(\sigma_1\cdot\tilde\tau_3)(\sigma_2\cdot\tilde\tau_2)\big]\tilde\tau_1\\ &+\;\big[(\sigma_1\cdot\tilde\tau_3)(\sigma_2\cdot\tilde\tau_1)-(\sigma_1\cdot\tilde\tau_1)(\sigma_2\cdot\tilde\tau_3)\big]\tilde\tau_2\\ &+\;\big[(\sigma_1\cdot\tilde\tau_1)(\sigma_2\cdot\tilde\tau_2)-(\sigma_1\cdot\tilde\tau_2)(\sigma_2\cdot\tilde\tau_1)\big]\tilde\tau_3\end{aligned}$$

along with (2.5), (2.13), and (2.16) to obtain the σ_3 estimate

$$\int_\Sigma |\nabla\sigma_3(x)|\,d\mathcal{H}^2x \;\le\; c\int_{\mathbf{B}^5}|\nabla^2u|^2\,dx\;. \tag{2.17}$$

Now we may combine (2.9), (2.10), (2.5), (2.13), (2.16), and (2.17) to obtain the desired length estimate

$$\mathcal{H}^1(A) \;=\; \mathcal{H}^1(\rho_\Gamma\circ\gamma)^{-1}\{z\} \;\le\; c\int_{\mathbf{B}^5}|\nabla^2u|^2\,dx\;. \tag{2.18}$$

2.8 *Connecting the Singularities* b_j *to* $b_{j'}$.

Although we now have a good description and length estimate for A, we are not done. The problem is that the set $\overline{A}$ does not necessarily connect each of the original singularities a_i to another $a_{i'}$. The path in $\overline{A}$ starting at a_i may end at some b_j. To complete the connections between pairs of a_i, it will be sufficient to find a *different* union B of curves which connect each frame singularity b_j to a another unique frame singularity $b_{j'}$. Then adding to $\overline{A}$ some components of B will give the desired curves connecting every a_i to a distinct $a_{i'}$. In this section we will use the map Φ from §2.4 to construct this additional connecting set B, and we will, in the next section §2.9, obtain the required estimate on the length of B.

First we recall the description in [M] of $\tilde G_2(\mathbf{R}^5)$ as a 2 sheeted cover of the Grassmannian of *unoriented* 2 planes in R^5. With $Q\in\tilde G_2(\mathbf{R}^5)$ chosen as before in §3.2, consider the 5 dimensional Schubert cycle

$$\mathcal{S}_Q \;=\; \{P\in\tilde G_2(\mathbf{R}^5)\;:\;\dim\big(P\cap Q^\perp\big)\ge 1\}$$

and the 4 dimensional subcycle

$$\mathcal{T}_Q \;=\; \{P\in\tilde G_2(\mathbf{R}^5)\;:\;\dim\big(P\cap Q^\perp\big)\ge 2\} \;=\; \{P\in\tilde G_2(\mathbf{R}^5)\;:\;P\subset Q^\perp\}\;.$$

As in [M], we see that $\tilde G_2(\mathbf{R}^5)\setminus\mathcal{S}_Q$ has two 6 dimensional antipodal cells, D_+ centered at Q and D_- centered at $-Q$.

Next we will carefully define a (nearest-point) retraction map

$$\Pi_{\mathbf{Q}}\;:\;\tilde G_2(\mathbf{R}^5)\setminus\{Q,-Q\}\;\to\mathcal{S}_Q\;.$$

For $P \in D_+ \setminus \{Q\}$, there is a unique vector $v \in P \cap \mathbf{S}^4$ which is at maximal distance in $P \cap \mathbf{S}^4$ from $Q \cap \mathbf{S}^4$ and a unique vector w in $Q \cap \mathbf{S}^4$ that is closest to v; in particular, $0 < w \cdot v < 1$. Choose $A_P \in \mathrm{so}\,(5)$ so that the corresponding rotation $\exp A_P \in SO(5)$ maps w to v and maps $\tilde{w}$ to $\tilde{v}$ where $P = v \wedge \tilde{v}$ and $Q = w \wedge \tilde{w}$. Thus $\exp A_P$ maps Q to P, preserving orientation. Here $(\exp tA_P)(w)$ defines a geodesic circle in $\mathbf{S}^4$, and

$$t_P \ \equiv \ \inf\{t > 0 \ : \ w \cdot (\exp tA_P)(w) = 0 \ \} \ > \ 1 \ .$$

Then $(\exp 2t_pA_P)(w) = -w$ and $\exp 4t_pA_P = \mathrm{id}$. It follows that, in $\tilde{G}_2(\mathbf{R}^5)$, as t increases,

$$(\exp tA_P)(Q) \in D_+ \text{ for } 0 \leq t < t_P \quad \text{and} \quad (\exp tA_P)(Q) \in D_- \text{ for } t_P < t \leq 2t_P \ ,$$

$$(\exp 0A_P)(Q) = Q, \quad (\exp A_P)(Q) = P, \quad (\exp t_pA_P)(Q) \in \mathcal{S}_Q, \quad (\exp 2t_pA_P)(Q) = -Q \ ,$$

and we let $\Pi_Q(P) = (\exp t_pA_P)(Q)$.

As P approaches $\partial D_+ = \mathcal{S}_Q$, $t_P \downarrow 1$ and $|\Pi_Q(P) - P| \to 0$. Thus, let

$$\Pi_Q(P) \ = \ P \ \text{ for } \ P \in \mathcal{S}_Q \ .$$

Also, let

$$\Pi_Q(P) \ = \ -\Pi_Q(-P) \ \text{ for } \ P \in D_- \setminus \{-Q\} \ .$$

For $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$, the intersection $P \cap Q^\perp \cap \mathbf{S}^4$ consists of 2 antipodal points in $P \cap \mathbf{S}^4$ that are uniquely of maximal distance from $Q \cap \mathbf{S}^4$, and one sees that

$$\mathrm{Clos}\, \Pi_Q^{-1}\{P\}$$

is a single semi-circular geodesic arc joining Q and $-Q$. For almost all $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$, this semi-circle meets transversely both Y, and, near $\pm Q$, each small surface

$$\Phi\big(\,[\Sigma \cap \mathbf{B}_\delta(b_j)] \times \{e_j\}\big) \ .$$

We will choose $P \in \mathcal{S}_Q \setminus \mathcal{T}_Q$ also to be a regular value of $\Pi_Q \circ \Phi$. It follows (see §2.4) that the set

$$\big(\Pi_Q \circ \Phi\big)^{-1}\{P\} \ = \ \Phi^{-1}\big(\Pi_Q^{-1}\{P\}\big)$$

is an embedded 1 dimensional submanifold, with endpoints in $\{(b_1, \pm e_1), \ldots, (b_m, \pm e_m)\}$. In small neighborhoods of any two points (b_j, e_j), $(b_j, -e_j)$ the set $\mathrm{Clos}\,\big(\Pi_Q \circ \Phi\big)^{-1}\{P\}$ consists of two smooth segments (antipodoal in the $\mathbf{S}^4$ factor) which both project, under the projection

$$p_\Sigma \ : \ \Sigma \times \mathbf{S}^4 \ \to \ \Sigma \ ,$$

onto a *single* smooth segment in Σ with endpoint b_j. These two segments upstairs continue in $\big(\Pi_Q \circ \Phi)^{-1}\{P\}$ to form two embedded antipodal paths whose final endpoints are $(b_{j'}, e_{j'})$, $(b_{j'}, -e_{j'})$ for some j' *distinct from* j. Here

$$e_{j'} \wedge e_{j'\Sigma} \ = \ \Phi\big(b_{j'}, \pm e_{j'}\big) \ = \ -\Phi\big(b_j, \pm e_j\big) \ = \ -e_j \wedge e_{j\Sigma} \ .$$

Composing either path with the projection p_Σ gives the same path connecting b_j and $b_{j'}$ Thus the the whole set

$$B \ = \ p_\Sigma\big[\big(\Pi_Q \circ \Phi)^{-1}\{P\}\big]$$

provides the desired connection in Σ.

Also note that these two paths upstairs have similar orientations induced as fibers of the map $\Pi_Q \circ \Phi$. That is, in the notation of slicing currents [F,4.3],

$$p_{\Sigma\#}\big\langle\, [[\Sigma \times \mathbf{S}^4]]\, , \Pi_Q \circ \Phi\, , Q\ \big\rangle \ = \ 2(\mathcal{H}^2 \llcorner B) \wedge \vec{B}\ , \tag{2.19}$$

where $\vec{B}$ is a unit tangent vectorfield along B (in the direction running from b_j to $b_{j'}$).

2.9 *Estimating the Length of the Connecting Set* B.

The definition of B depends on the choices of:

(1) the point $p \in \mathbf{S}^3$ which gives the surface $\Sigma = u^{-1}\{p\}$ and the map

$$\Phi \ : \ (\Sigma \times \mathbf{S}^4\big) \setminus \mathcal{N}_\Sigma \ \to \ \tilde{G}_2(\mathbf{R}^5)\ , \quad \Phi(x, e) = e \wedge e_\Sigma(x)\ ,$$

(2) the 2 plane $Q \in \tilde{G}_2(\mathbf{R}^5)$ which determines the retraction $\Pi_{\mathbf{Q}}$ of $\tilde{G}_2(\mathbf{R}^5) \setminus \{Q, -Q\}$ onto the 5 dimensional Schubert cycle $\mathcal{S}_Q$, and

(3) the 2 plane $P \in \mathcal{S}_Q$ which gives $B = p_\Sigma\big[\big(\Pi_Q \circ \Phi)^{-1}\{P\}\big]$.

Having chosen $p \in \mathbf{S}^3$ as before to obtain estimate (2.5), we need to chose Q and P to get the desired length estimate for B.

Concerning Q, we first readily verify that the retraction Π_Q is locally Lipschitz in $\tilde{G}_2(\mathbf{R}^5) \setminus \{Q, -Q\}$ and deduce the estimate

$$|\nabla \Pi_Q(S)| \ \le \ \frac{c}{|S - Q||S + Q|} \quad \text{for} \quad S \in \tilde{G}_2(\mathbf{R}^5) \setminus \{Q, -Q\}\ . \tag{2.21}$$

Using (2.19) and [F,4.3.1], we may integrate the slices to find that

$$\begin{aligned}\int_{\mathcal{S}_Q} p_{\Sigma\#}\big\langle\, [[\Sigma \times \mathbf{S}^4]], \Pi_Q \circ \Phi, P\,\big\rangle\, d\mathcal{H}^5 P \ &= \ p_{\Sigma\#} \int_{\mathcal{S}_Q} \big\langle\, [[\Sigma \times \mathbf{S}^4]], \Pi_Q \circ \Phi, P\,\big\rangle\, d\mathcal{H}^5 P \\ &= \ p_{\Sigma\#}\big([[\Sigma \times \mathbf{S}^4]] \llcorner (\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q}\big)\ ,\end{aligned}$$

where $\omega_{\mathcal{S}_Q}$ is the volume element of $\mathcal{S}_Q$. By (2.19) and Fatou's Lemma,

$$\int_{\mathcal{S}_Q} 2\mathcal{H}^1\big(p_\Sigma[(\Pi_Q \circ \Phi)^{-1}\{P\}]\big)\, d\mathcal{H}^5 P \;=\; \int_{\mathcal{S}_Q} \mathbf{M}[p_{\Sigma\#}\big\langle\, [[\Sigma \times \mathbf{S}^4]], \Pi_Q \circ \Phi, P\,\big\rangle] d\mathcal{H}^5 P$$
$$\leq\; \mathbf{M}\big[p_{\Sigma\#}\big(\, [[\Sigma \times \mathbf{S}^4]] \llcorner\; (\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q}\big)\big]$$
$$=\; \sup_{\alpha \in \mathcal{D}^1(\Sigma), |\alpha| \leq 1} \int_\Sigma \int_{\mathbf{S}^4} (\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q} \wedge p_\Sigma^{\#}\alpha\ . \qquad (2.22)$$

To estimate this last double integral, we recall from §2.3 that, for each fixed $x \in \Sigma \setminus \{a_1, \ldots, a_m\}$,

$$\Phi(x, \cdot)\ :\ \mathbf{S}^4 \setminus \mathrm{Nor}\,(\Sigma, x)\ \to\ \mathcal{Q}_x\ \equiv\ \mathcal{Q}_{\mathrm{Tan}\,(\Sigma,x)} \setminus Y_x$$

is a the smooth, orientation-preserving, 2-sheeted cover map. Each map $\Phi(x, \cdot)$ depends only on $\mathrm{Tan}\,(\Sigma, x)$, and any two such maps are orthogonally conjugate. We will derive the formula

$$\big[(\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q} \wedge p_\Sigma^{\#}\alpha\big](x, \cdot)\ =\ \beta(x, \cdot) p_\Sigma^{\#}\omega_\Sigma(x) \wedge \Phi(x, \cdot)^{\#}\omega_{\mathcal{Q}_x} \qquad (2.23)$$

where ω_Σ and $\omega_{\mathcal{Q}_x}$ denote the volume elements of Σ and $\mathcal{Q}_x$ and $\beta(x, \cdot)$ is a smooth function on $\mathbf{S}^4 \setminus \mathrm{Nor}\,(\Sigma, x)$ satisfying

$$|\beta(x, e)|\ \leq\ \frac{c}{|\Phi(x, e) - Q|^5 |\Phi(x, e) + Q|^5} \sum_{j=1}^{3} |\nabla \tilde{\tau}_j(x)| \quad \text{for } e \in \mathbf{S}^4\ . \qquad (2.24)$$

Before proving (2.23), note that the decomposition on the righthand side is not necessarily smooth in x since the different $\mathcal{Q}_x$ may overlap for x near a critical point of $\Phi(\cdot, e)$ for some $e \in \mathbf{S}^4$. Nevertheless, the formula does imply the measurability of $\beta(x, e)$ in x, and so may be integrated over Σ.

To derive (2.23), we first note that, with the factorization $\Sigma \times \mathbf{S}^4$, there are only two terms in the (p, q) decomposition of the 5 form,

$$(\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q}\ =\ \Omega_{2,3}\ +\ \Omega_{1,4}\ .$$

Thus,

$$(\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q} \wedge p_\Sigma^{\#}\alpha\ =\ 0\ +\ \Omega_{1,4} \wedge p_\Sigma^{\#}\alpha \qquad (2.25)$$

because the term $\Omega_{2,3} \wedge p_\Sigma^{\#}\alpha$, being of type $(2+1, 3)$, must vanish.

For each $S = \Phi(x, \pm e) \in \mathcal{Q}_x \setminus Y_x$, we also have the factorization

$$\mathrm{Tan}\,(\tilde{G}_2(\mathbf{R}^5), S)\ =\ \mathrm{Nor}\,(\mathcal{Q}_x, S)\ \times\ \mathrm{Tan}\,(\mathcal{Q}_x, S)\ .$$

Let $\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2$ be an orthonormal basis of $\wedge^1 \mathrm{Tan}\,(\tilde{G}_2(\mathbf{R}^5), S)$ so that

$$\mu_1, \mu_2, \mu_3, \mu_4 \in \wedge^1 \mathrm{Tan}\,(\mathcal{Q}_x, S)\ ,\ \ \nu_1, \nu_2 \in \wedge^1 \mathrm{Nor}\,(\mathcal{Q}_x, S)\ ,\ \ \mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \mu_4 = \omega_{\mathcal{Q}_x}(S)\ ;$$

thus, $0=\nu_1(v)=\nu_2(v)=\mu_1(w)=\mu_2(w)=\mu_3(w)=\mu_4(w)$ whenever $v\in \mathrm{Tan}\,(\mathcal{Q}_x,S)$ and $w\in \mathrm{Nor}\,(\mathcal{Q}_x,S)$. We may expand the 5 covector

$$\begin{aligned}
&\Pi_Q^{\#}(\omega_{\mathcal{S}_Q})(S) \;=\\
&\lambda_1\,\nu_2\wedge\mu_1\wedge\mu_2\wedge\mu_3\wedge\mu_4+\lambda_2\,\nu_1\wedge\mu_1\wedge\mu_2\wedge\mu_3\wedge\mu_4+\lambda_3\,\nu_1\wedge\nu_2\wedge\mu_2\wedge\mu_3\wedge\mu_4\\
&+\lambda_4\,\nu_1\wedge\nu_2\wedge\mu_1\wedge\mu_3\wedge\mu_4+\lambda_5\,\nu_1\wedge\nu_2\wedge\mu_1\wedge\mu_2\wedge\mu_4+\lambda_6\,\nu_1\wedge\nu_2\wedge\mu_1\wedge\mu_2\wedge\mu_3
\end{aligned}$$

where

$$|\lambda_i|\le\frac{c}{|S-Q|^5|S+Q|^5}\ , \tag{2.26}$$

by (2.21). Applying $\Phi^{\#}$ (that is, $\wedge^1 D\Phi(x,e)$) to all covectors and taking the $(1,4)$ component, we find that only the first two terms survive so that

$$\begin{aligned}
\Omega_{1,4}(x,e) &= \big[\lambda_1\Phi^{\#}\nu_2+\lambda_2\Phi^{\#}\nu_1\big]_{(1,0)}\wedge\Phi^{\#}\mu_1\wedge\Phi^{\#}\mu_2\wedge\Phi^{\#}\mu_3\wedge\Phi^{\#}\mu_4\\
&= \big[\lambda_1\Phi^{\#}\nu_2+\lambda_2\Phi^{\#}\nu_1\big]_{(1,0)}\wedge\Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x}(S)\ .
\end{aligned}\tag{2.27}$$

Being of type $(2,0)$, the 2 covector

$$\big(\big[\lambda_1\Phi^{\#}\nu_2+\lambda_2\Phi^{\#}\nu_1\big]_{1,0}\wedge p_\Sigma^{\#}\alpha\big)(x,e) \;=\; \beta(x,e)p_\Sigma^{\#}\omega_\Sigma(x) \tag{2.28}$$

for some scalar $\beta(x,e)$, and (2.25) and (2.27) now give the desired formula (2.23). This formula readily implies the smoothness of $\beta(x,\cdot)$ on $\mathbf{S}^4\setminus\mathrm{Nor}\,(\Sigma,x)$.

To verify the bound (2.24), observe that

$$|\ [\Phi^{\#}\nu_i]_{1,0}| \;=\; \sup_{v\in\mathbf{S}^4\cap\mathrm{Tan}\,(\Sigma,x)}\nu_i[\nabla_v\Phi(x,e)]\ , \tag{2.29}$$

where $\nabla_v\Phi(x,e)=D\Phi_{(x,e)}(v,0)\in\mathrm{Tan}\,(\tilde G_2(\mathbf{R}^5),S)$. For any unit vector $v\in\mathrm{Tan}\,(\Sigma,x)$ and any $w\in\mathbf{R}^5$,

$$v\wedge w\in\mathrm{Tan}\,(\mathcal{Q}_x,S)$$

because we may assume $w\notin\mathrm{Tan}\,(\Sigma,x)$ and then choose a curve $y(t)$ in $\mathbf{S}^4\cap v^\perp\setminus\mathrm{Nor}\,(\Sigma,x)$ with $y'(0)=w-(w\cdot v)v$, hence,

$$v\wedge w\;=v\wedge y'(0)\;=\;\frac{d}{dt}_{t=0}\big(v\wedge y(t)\big)\;=\;-\frac{d}{dt}_{t=0}\Phi\big(x,y(t)\big)\ .$$

Thus, for any 2 vector $\xi\in\mathrm{Nor}\,(\mathcal{Q}_x,S)$, $|\xi|\;=\;|\xi\wedge v|$; in particular, $|\xi|=|\xi\wedge\tilde{e_T}(x)|$, $|\xi|=|\xi\wedge e_\Sigma(x)|$, and hence,

$$|\xi|\;=|\,\xi\wedge\big(\tilde{e_T}(x)\wedge e_\Sigma(x)\big)\,|\ .$$

Since $\nu_i \in \wedge^1 \mathrm{Nor}\,(\mathcal{Q}_x, S)$ and $|\nu_i| = 1$, we now find that

$$\nu_i[\nabla_v \Phi(x,e)] \ = \nu_i\big[\big(\nabla_v \Phi(x,e)\big)_{\mathrm{Nor}\,(\Sigma,x)}\big] \ \leq |\nabla_v \Phi(x,e) \wedge \big(\tilde{e_T}(x) \wedge e_\Sigma(x)\big)|\ . \tag{2.30}$$

Moreover,

$$\begin{aligned} |\,\nabla_v \Phi \wedge (\tilde{e_T} \wedge e_\Sigma)| \ &\leq \ |\,\big(\nabla_v (e \wedge e_\Sigma)\big) \wedge (\tilde{e_T} \wedge e_\Sigma)| \ \leq \ |\,(\nabla_v e_\Sigma) \wedge (\tilde{e_T} \wedge e_\Sigma)| \\ &= \ |\,e_\Sigma \wedge \nabla_v(\tilde{e_T} \wedge e_\Sigma)| \ \leq \ |\nabla_v(\tilde{e_T} \wedge e_\Sigma)| \\ &= \ |\,\nabla_v\big(*\,(\,\tilde\tau_1 \wedge \tilde\tau_2 \wedge \tilde\tau_3)\big)\,| \ \leq \ c\sum_{j=1}^3 |\nabla \tilde\tau_j|\ , \end{aligned} \tag{2.31}$$

where $*$ is the Hodge $* : \wedge_3 \mathbf{R}^5 \to \wedge_2 t\mathbf{R}^5 \approx \mathbf{R}^5$ [F,1.7.8]) The desired pointwise bound (2.24) now follows by combining (2.26), (2.28), (2.29), (2.30) and (2.31).

For each $x \in \Sigma$, the pull-back $\Phi(x,\cdot)^\# \omega_{\mathcal{Q}_x}$ is point-wise a positive multiple of the volume form of $\mathbf{S}^4$. So we may first integrate over $\mathbf{S}^4$ and use (2.24) to see that

$$\begin{aligned} \int_{\mathbf{S}^4} \beta(x,\cdot)\Phi(x,\cdot)^\# \omega_{\mathcal{Q}_x} \ &\leq \ \int_{\mathbf{S}^4} |\beta(x,\cdot)|\Phi(x,\cdot)^\# \omega_{\mathcal{Q}_x} \\ &\leq \ c\big(\sum_{j=1}^3 |\nabla \tilde\tau_j(x)|\big) \int_{\mathbf{S}^4} \frac{\Phi(x,\cdot)^\# \omega_{\mathcal{Q}_x}}{|\Phi(x,\cdot) - Q|^5 |\Phi(x,\cdot) + Q|^5} \\ &= \ c\big(\sum_{j=1}^3 |\nabla \tilde\tau_j(x)|\big) \int_{\mathcal{Q}_x} \frac{\omega_{\mathcal{Q}_x}(S)}{|S-Q|^5|S+Q|^5} \\ &\leq \ c\big(\sum_{j=1}^3 |\nabla \tilde\tau_j(x)|\big) \int_{\mathcal{Q}_x} \frac{d\mathcal{H}^4 S}{|S-Q|^5|S+Q|^5}\ . \end{aligned} \tag{2.32}$$

To handle the denominator, we note that the Grassmannian $\tilde G_2(\mathbf{R}^5)$ is a 6 dimensional homogeneous space, and we readily use local coordinates to verify that

$$C_3 \ = \ \int_{\tilde G_2(\mathbf{R}^5)} \frac{1}{|S-Q|^5|S+Q|^5}\, d\mathcal{H}^6 Q \ < \ \infty\ , \tag{2.33}$$

independent of S.

Now we recall (2.22) and fix a sequence of 1 forms $\alpha_i \in \mathcal{D}^1(\Sigma)$ with $|\alpha_i| \leq 1$ so that

$$\mathbf{M}\big[p_{\Sigma\#}\big([[\Sigma \times \mathbf{S}^4]] \llcorner (\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q}\big)\big] \ = \ \lim_{i\to\infty} \int_\Sigma \int_{\mathbf{S}^4} (\Pi_Q \circ \Phi)^\# \omega_{\mathcal{S}_Q} \wedge p_\Sigma^\# \alpha_i\ ,$$

let β_i be the corresponding function from the formula (2.23), and use (2.22), Fatou's Lemma, (2.23), (2.32), Fubini's Theorem, (2.33), and (2.5) to obtain our final integral estimate

$$\begin{aligned}
&\int_{\tilde{G}_2(\mathbf{R}^5)} \int_{\mathcal{S}_Q} 2\mathcal{H}^1\big(p_\Sigma[(\Pi_Q \circ \Phi)^{-1}\{P\}]\big)\, d\mathcal{H}^5 P\, d\mathcal{H}^6 Q \\
&\le \int_{\tilde{G}_2(\mathbf{R}^5)} \lim_{i\to\infty} \int_\Sigma \int_{\mathbf{S}^4} (\Pi_Q \circ \Phi)^{\#}\omega_{\mathcal{S}_Q} \wedge p_\Sigma^{\#}\alpha_i \\
&\le \liminf_{i\to\infty} \int_{\tilde{G}_2(\mathbf{R}^5)} \int_\Sigma \int_{\mathbf{S}^4} \beta_i(x,\cdot)\omega_\Sigma(x) \wedge \Phi(x,\cdot)^{\#}\omega_{\mathcal{Q}_x} \\
&\le c\int_\Sigma \big(\sum_{j=1}^{3} |\nabla\tilde{\tau}_j(x)|\big) \int_{\mathcal{Q}_x} \int_{\tilde{G}_2(\mathbf{R}^5)} \frac{1}{|S-Q|^5|S+Q|^5}\, d\mathcal{H}^6 Q\ d\mathcal{H}^4 S\ d\mathcal{H}^2 x \\
&\le c\, C_3 \sum_{j=1}^{3} \int_\Sigma |\nabla\tilde{\tau}_j(x)|\, d\mathcal{H}^2 x \\
&\le c\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx\ .
\end{aligned}$$

The Schubert cycles $\mathcal{S}_Q$ are all orthogonally equivalent with the same positive 5 dimensional Hausdorff measure. So we can use the final integral inequality to choose first a 2 plane $Q \in \tilde{G}_2(\mathbf{R}^5)$ and then a 2 plane $P \in \mathcal{S}_Q$ so that the corresponding connecting set

$$B\ = p_\Sigma[(\Pi_Q \circ \Phi)^{-1}\{P\}]$$

satisfies the desired length estimate

$$\mathcal{H}^1(B)\ \le\ c\int_{\mathbf{B}^5} |\nabla^2 u|^2\, dx\ .$$