Math 499 Exercises (01/25/05)

Use of computers, friends, and myself are encouraged.

1. Computers often have trouble graphing near singularities. Knowing the tangent cone can help you 'fill in the gaps' where the computer went awry. For each of the following curves find ALL singular points, the multiplicity of the singular points, and the tangent cones. Draw a picture of the curves, with tangent cones. Make the computer do as much of the work as possible.

(a) $2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$
(b) $2y^2(x^2 + y^2) - 3y^2 - x^2 + 1 = 0$
(c) $2y^2(x^2 + y^2) - 2y^2(x + y) - 2y^2 - x^2 + 2x + 2y = 0$

**Definition 0.1.** Let $f = a_n x^n + \cdots + a_1 x + a_0$ ($a_n \neq 0$) be a degree $n$ polynomial. The **discriminant** of $f$ is defined to be $\frac{1}{a_n} \text{res}(f, df/dx)$ and vanishes precisely when $f$ has multiple roots.

2. Let $f$ be a degree 3 polynomial in $x$. By a suitable affine transformation $x \rightarrow ax + b$ we can write $f$ in the standard form $x^3 + px + q$.

(a) Compute the discriminant of the standard form.
(b) Convert $g = x^3 + a \cdot x^2 + 1$ into standard form. When does $g$ have a double root?

3. There is some subtlety in dealing with resultants in multiple variables. We have to be careful to compute the determinant in the correct degree. For example, $\text{res}(ax^2 + bx + c, 2ax + b)$ will be $a(b^2 - 4ac)$ which gives us a condition for multiple roots as long as $a \neq 0$. If $a = 0$ we CAN'T have a multiple root even though it might appear our resultant is zero. In fact, since the degree has changed, the resultant is calculated in a different manner. As a similar example, let $f = x^2y + x - 1$ and $g = x^2y + x + y^2 - 4$. Compute $h(y) = \text{Res}(f, g, x)$ (i.e. the resultant in terms of the variable $x$). Show that $h(0)$ and $\text{Res}(f(0), g(0))$ have different roots.

4. Prove that $\text{res}(f_1, f_2, g) = 0$ iff $\text{res}(f_1, g) \ast \text{res}(f_2, g) = 0$. Compute $\text{res}(f_1, f_2, g)$ and $\text{res}(f_1, g) \ast \text{res}(f_2, g)$ explicitly in the case $f_1 = ax^2 + b, f_2 = cx + d, g = px^2 + q$. 

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