

# Math 499 Exercises (02/01/06)

Use of computers, friends, and myself are encouraged.

1. Rational parametrization is yet another way which can, in certain circumstances, simplify finding intersection points of curves. It is a technique used often in computer graphics. Find the intersection points of  $x^2 + y^2 = 1$  and  $y^2 = x^2 + x^3$  by parameterizing the 2nd. equation and solving.

2. Find a rational parametrization for the following curves. Which points (if any) have we lost?

- (a)  $X(y^2 - x^3)$ , the curves defined by  $y^2 - x^3 = 0$ .
- (b)  $X(x^2 - y^2 - (x - 2y)(x^2 + y^2))$
- (c)  $X(x^5 - xy^2 + y^3)$
- (d)  $X(x^2 + 50yx^7 + x^{52} + 1)$

3. The cardioid has the polar equation  $r = 1 - \cos(\theta)$ . Find an implicit polynomial equation for this in terms of  $x$  and  $y$ .

It should be clear that affine transformations preserve rationality. For example, if  $x = f(t)$ ,  $y = g(t)$  then  $x' = ax + by + c = af(t) + bg(t) + c$ , etc. The important consequence is that rationality, like all non-trivial properties we study in this class, can be proved for a whole set of curves at once by finding a 'nice form' for the set.

4. Using the last comment, we can prove that all irreducible conics are rational. The following method to do this generalizes.

(a) Let  $f = (ax + by) + (cx^2 + dxy + ey^2) = f_1 + f_2$  be irreducible, where  $f_i$  is the purely degree  $i$  part of the polynomial. Prove that  $f$  is rational.

(b) In the notation of part a, let  $f = f_{n-1} + f_n$  be a degree  $n$  polynomial with pure degree  $n$  and  $n - 1$  terms. Assuming that  $f$  is irreducible, prove that the associated curve is *rational*.