

# Important Facts about Projective Space

## 1 Terminology

Henceforth, a 'standard' curve living in  $\mathbb{C} \times \mathbb{C}$  will be called an *affine curve*.

A *projective curve* is defined to be the zero set of a homogeneous polynomial in  $\mathbb{P}^2$ , i.e it is of the form  $X(f) = \{f(x, y, z) = 0\} \subset \mathbb{P}^2$ .

Let  $X(f)$  be an affine curve and let  $\tilde{f}$  be it's homogenization. The projective curve  $X(\tilde{f})$  is called the *projectivization* of  $X(f)$ . The points of  $X(\tilde{f})$  with  $z \neq 0$  correspond precisely to our original affine curve. The points with  $z = 0$  are called the *points at infinity* of  $X(f)$  and correspond to limiting behavior of the graph as it goes towards infinity.

**Definition 1.1.** A *projective change of coordinates* has the form 
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where  $A$  is an invertible  $3 \times 3$  matrix. Two curves which differ by a coordinate change are called *projectively equivalent*.

## 2 Philosophy, etc.

We gain much by extending curves to projective space. As I've said before, projectivizing a curve keeps track of what our affine curve does as it tends towards infinity. Besides this, projective curves are compact (an extremely important property throughout mathematics) and many classical geometric problems can be naturally solved in projective space.

An example of a classical geometric question is "how many points in the plane does it take to determine a unique conic going through them?" This kind of question only has simple answers if you replace 'plane' with 'projective plane'.

A final reason to study projective space is that it often makes things much simpler.

*Example 2.1.* Let  $y = x^2$  be a parabola. It has a homogenization  $zy = x^2$ . By doing the coordinate changes  $z = z' + y'$ ,  $y = z' - y'$  we get  $z'^2 = x^2 + y'^2$ .

Now let's look at the circle  $x^2 + y^2 = 1$ . It has homogenization  $x^2 + y^2 = z^2$  which is the same equation we just saw!!

Thus projectivizations of the parabola and the circle are projectively equivalent. Therefore, for all intents and purposes, they are the same!

**We lose no information by projectivizing;** we can always set  $z = 1$  and regain our original curve. Coordinate changes can also be inverted. Moreover, all properties that we care about such as singularities will be preserved under coordinate changes. Thus, to study parabolas and circles, it's really enough to just study circles!! This may seem counterintuitive, but it is simply an example of how the projective plane simplifies things without losing information. In general, we can study large groups of curves all at once by studying a nice curve which is projectively equivalent to the whole group.

# Math 499 Exercises (02/08/06)

**Theorem 2.2 (Euler).**

Let  $F = \sum_{i+j+k=n} a_{ij} x^i y^j z^k$  be an arbitrary homogeneous polynomial of degree  $n$ .

Then: 
$$nF = x \frac{dF}{dx} + y \frac{dF}{dy} + z \frac{dF}{dz}.$$

**Definition 2.3.** Let  $f(x, y, z)$  be homogeneous. The projective curve  $X(f)$  is *singular* at points such that  $\frac{df}{dx} = \frac{df}{dy} = \frac{df}{dz} = 0$ . If it has no singularities, it is *smooth*. Euler's theorem proves that we get  $f = 0$  for free.

1. Prove Euler's theorem.
2. Let  $\tilde{f}(x, y, z)$  be the homogenization of  $f(x, y)$ . Suppose that  $X(f)$  is singular at the point  $(a, b) \in \mathbb{C}^2$ . Prove that  $X(\tilde{f})$  is singular at  $(a, b, 1) \in \mathbb{P}^2$ .
3. Homogenize the following equations and find which points at infinity are added to projectivize the associated curve. Draw the graph and "draw-in" as best you can the points at infinity.
  - (a)  $y = 2x - 3$
  - (b)  $2y^3 + 4x + 1 = 0$
  - (c)  $x^3 = y^2$
  - (d)  $x^2y + y^3 = x^3$
  - (e)  $x^2 + 2xy = 1$
4. Find an example of a smooth affine curve which has a singular projectivization.
5. Determine whether any of the following curves have singularities at any points of infinity. If so, determine their multiplicity and (if degree 2) whether they are nodes or cusps.
  - (a)  $(x^2 + y^2)^2 = x^2 - y^2$
  - (b)  $y^2 = x^2 + x^3$
  - (c)  $y = x^2y + x^3$

**CHALLENGE PROBLEM:** (optional)

- (a) Prove that any projective conic is projectively equivalent to  $x^2 = 0$  (a line),  $x^2 + y^2 = 0$  (2 lines), or  $x^2 + y^2 + z^2 = 0$  (a nondegenerate conic).
- (a) Let  $p_1, \dots, p_5$  be points in the projective plane, no three on a line. Prove that there is a unique conic that goes through these points.