Note: Implicit in our definition of intersection multiplicity is that it is preserved under coordinate change. This is fairly easy to prove using the properties from the handout. In particular we define $I_p(f, g)$ to be the intersection multiplicity $I_0(\tilde{f}, \tilde{g})$ where $\tilde{f}, \tilde{g}$ are simply $f$ and $g$ ‘shifted’ by the coordinate change sending $p$ to the origin. Also note that our definition works equally well for both projective and affine curves. If you have a pair of projective curves meeting at the origin you may set $z = 1$ (i.e de-homogenize it) to calculate $I_0$.

**Theorem 0.1 (Bezout).** Let $f(x, y, z), g(x, y, z)$ be homogeneous of degrees $n, m$. Then $$\sum_{p \in \mathbb{P}^2} I_p(f, g) = nm.$$ 

0. Read the handout, which is, by the way, from the book ‘conics and cubics’, by Robert Bix.

1. Use the properties and theorems from the handout to compute the intersection multiplicity of the following pairs of curves at the origin:

   (a) $y = x^3, y^4 + 6x^3y + x^8 = 0$
   (b) $y = x^2 - 2x, y^2 + 5y = 4x^3$
   (c) $y^2 + x^2y - x^3 = 0, y^2 + x^3y + 2x = 0$
   (d) $y^5 = x^7, y^2 = x^3$

2. Verify Bezout’s theorem with the curves $x^2 = y^3 + y^2$ and $x = y^2$ by homogenizing if necessary and computing all intersection multiplicities. Draw these curves and all of their intersections. Try to explain the intersection numbers graphically.

3. Let $f(x, y) = 0$ be nonsingular. Let $L$ be a line which meets $f$ at the origin. Show that $I_0(f, L) > 1$ if and only if $L$ is tangent to $f$ at the origin. **Hint:** you may assume $L$ is $\{x = 0\}$. Explain why you may assume this.

4. Similarly, Let $C$ and $C'$ be two circles meeting at a point $p$. Show that $I_p(C, C') = 2$ if and only if the circles have the same tangent line at $p$.

5. Let $f = y^2 - x^3$. Show that $I_0(f, ax + by) > 1$ for all $a, b$. When is it equal to 2? When is it equal to 3? Can you think of a statement about the intersection number of singular curves at singularities which generalizes this result?