

Cartier isomorphism and Hodge Theory in the non-commutative case

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ABSTRACT. These lectures attempt to give an elementary introduction to my recent paper “Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie”.

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1. Introduction

One of the standard ways to compute the cohomology groups of a smooth complex manifold X is by means of the de Rham theory: the de Rham cohomology groups

$$(1.1) \quad H_{DR}^\bullet(X) = \mathbb{H}^\bullet(X, \Omega_{DR}^\bullet)$$

are by definition the hypercohomology groups of X with coefficients in the (holomorphic) de Rham complex Ω_{DR}^\bullet , and since, by the Poincaré Lemma, Ω_{DR}^\bullet is a resolution of the constant sheaf \mathbb{C} , we have $H_{DR}^\bullet(X) \cong H^\bullet(X, \mathbb{C})$. If X is in fact algebraic, then Ω_{DR}^\bullet can also be defined algebraically, so that the right-hand side in (1.1) can be understood in two ways: either as the hypercohomology of an analytic space, or as the hypercohomology of a scheme equipped with the Zariski topology. One can show that the resulting groups $H_{DR}^\bullet(X)$ are the same (for compact X , this is just the GAGA principle; in the non-compact case this is a difficult but true fact established by Grothendieck [Gro66]).

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Of course, an algebraic version of the Poincaré Lemma is false, since the Zariski topology is not fine enough—no matter how small a Zariski neighborhood of a point one takes, it usually has non-trivial higher de Rham cohomology. However, the Lemma survives on the formal level: the completion $\widehat{\Omega_{DR}^\bullet}$ of the de Rham complex near a closed point $x \in X$ is quasi-isomorphic to \mathbb{C} placed in degree 0.

Assume now that our X is a smooth algebraic variety over a perfect field k of characteristic $p > 0$. Does the de Rham cohomology still make sense?

The de Rham complex Ω_{DR}^\bullet itself is well-defined: Ω^1 is just the sheaf of Kähler differentials, which makes sense in any characteristic and comes equipped with the universal derivation $d : \mathcal{O}_X \rightarrow \Omega^1$, and Ω_{DR}^\bullet is its exterior algebra, which is also well-defined in characteristic p . However, the Poincaré Lemma breaks down completely—the homology of the de Rham complex remains large even after taking completion at a closed point.

In degree 0, this is actually very easy to see: for any local function f on X , we have $df^p = pf^{p-1}df = 0$, so that all the p -th powers of functions are closed with respect to the de Rham differential. Since we are in characteristic p , these powers form a subsheaf of algebras in \mathcal{O}_X which we denote by $\mathcal{O}_X^p \subset \mathcal{O}_X$. This is a large subsheaf. In fact, if we denote by $X^{(1)}$ the scheme X with \mathcal{O}_X^p as the structure sheaf, then $X \cong X^{(1)}$ as abstract schemes, with the isomorphism given by the Frobenius map $f \mapsto f^p$. Fifty years ago P. Cartier proved that in fact *all* the functions in \mathcal{O}_X closed with respect to the de Rham differential are contained in \mathcal{O}_X^p , and moreover, one has a similar description in higher degrees: there exist natural isomorphisms

$$(1.2) \quad C : \mathcal{H}_{DR}^\bullet \cong \Omega_{X^{(1)}}^\bullet,$$

where on the left we have the homology sheaves of the de Rham complex, and on the right we have the sheaves of differential forms on the scheme $X^{(1)}$. These isomorphisms are known as *Cartier isomorphisms*.

The Cartier isomorphism has many applications, but one of the most unexpected was discovered in 1987 by P. Deligne and L. Illusie: one can use the Cartier isomorphism to give a purely algebraic proof of the following purely algebraic statement, which is normally proved by the highly transcendental Hodge Theory.

THEOREM 1.1 ([DI87]). *Assume given a smooth proper variety X over a field K of characteristic 0. Then the Hodge-to-de Rham spectral sequence*

$$H^\bullet(X, \Omega^\bullet) \Rightarrow H_{DR}^\bullet(X)$$

associated to the stupid filtration on the de Rham complex Ω^\bullet degenerates at the first term.

The proof of Deligne and Illusie was very strange, because it worked by reduction to positive characteristic, where the statement is not true for a general X . What they proved is that if one imposes two additional conditions on X , then the Cartier isomorphisms can be combined together into a quasi-isomorphism

$$(1.3) \quad \Omega_{DR}^\bullet \cong \bigoplus_i \mathcal{H}_{DR}^i[-i] \cong \bigoplus_i \Omega_{X^{(1)}}^i[-i]$$

in the derived category of coherent sheaves on $X^{(1)}$. The degeneration follows from this immediately for dimension reasons. The additional conditions are:

- (i) X can be lifted to a smooth scheme over $W_2(k)$, the ring of second Witt vectors of the perfect field k (e.g. if $k = \mathbb{Z}/p\mathbb{Z}$, X has to be liftable to $\mathbb{Z}/p^2\mathbb{Z}$), and
- (ii) we have $p > \dim X$.

To deduce Theorem 1.1, one finds by the standard argument a proper smooth model X_R of X defined over a finitely generated subring $R \subset K$, one localizes R so that it is unramified over \mathbb{Z} and all its residue fields have characteristic greater than $\dim X$, and one deduces that all the special fibers of X_R/R satisfy the assumptions above; hence the differentials in the Hodge-to-de Rham spectral sequence vanish at all closed points of $\text{Spec } R$, which means they are identically 0 by Nakayama.

The goal of these lectures is to present in a down-to-earth way the results of two recent papers [Kal05], [Kal08], where the story summarized above has been largely transferred to the setting of *non-commutative geometry*.

To explain what I mean by this, let us first recall that a non-commutative version of differential forms has been known for quite some time now. Namely, assume given an associative unital algebra A over a field k , and an A -bimodule M . Then its *Hochschild homology* $HH_*(A, M)$ of A with coefficients in M is defined as

$$(1.4) \quad HH_*(A) = \text{Tor}_{A^{opp} \otimes A}^*(A, M),$$

where $A^{opp} \otimes A$ is the tensor product of A and the opposite algebra A^{opp} , and the A -bimodule M is treated as a left module over $A^{opp} \otimes A$. Hochschild homology $HH_*(A)$ is the Hochschild homology of A with coefficients in itself.

Assume for a moment that A is in fact commutative, and $\text{Spec } A$ is a smooth algebraic variety over K . Then it has been proved back in 1962 in the paper [HKR62] that we have canonical isomorphisms $HH_i(A) \cong \Omega^i(A/k)$ for any $i \geq 0$. Thus for a general A , one can treat Hochschild homology classes as a replacement for differential forms.

Moreover, in the early 1980s it was discovered by A. Connes [Con83], J.-L. Loday and D. Quillen [LQ83], and B. Feigin and B. Tsygan [FT83], that the de Rham differential also makes sense in the general non-commutative setting. Namely, these authors introduced a new invariant of an associative algebra A called *cyclic homology*; cyclic homology, denoted $HC_*(A)$, is related to the Hochschild homology $HH_*(A)$ by a spectral sequence

$$(1.5) \quad HH_*(A)[u^{-1}] \Rightarrow HC_*(A),$$

which in the smooth commutative case reduces to the Hodge-to-de Rham spectral sequence (here u is a formal parameter of cohomological degree 2, and $HH_*(A)[u^{-1}]$ is shorthand for “polynomials in u^{-1} with coefficients in $HH_*(A)$ ”).

It has been conjectured for some time now that the spectral sequence (1.5), or a version of it, degenerates under appropriate assumptions on A (which imitate the assumptions of Theorem 1.1). Following [Kal08], we will attack this conjecture by the method of Deligne and Illusie. To do this, we will introduce a certain non-commutative version of the Cartier isomorphism, or rather, of the “globalized” isomorphism (1.3) (in the process of doing it, we will need to introduce some conditions on A which precisely generalize the conditions (i), (ii) above). Then we prove a version of the degeneration conjecture as stated by M. Kontsevich and Ya.

Soibelman in [KS06] (we will have to impose an additional technical assumption which, fortunately, is not very drastic).

The paper is organized as follows. In Section 2 we recall the definition of the cyclic homology and some versions of it needed for the Cartier isomorphism (most of this material is quite standard; the reader can find good expositions in [Lod98] or [FT87]). One technical result needed in the main part of the paper has been separated into Section 3. In Section 4, we construct the Cartier isomorphism for an algebra A equipped with some additional piece of data which we call the *quasi-Frobenius map*. It exists only for special classes of algebras—e.g. for free algebras, or for the group algebra $k[G]$ of a finite group G —but the construction illustrates nicely the general idea. In Section 5, we show what to do in the general case. Here the conditions (i), (ii) emerge, and in a somewhat surprising way—as it turns out, they essentially come from algebraic topology, and the whole theory has a distinctly topological flavor. Finally, in Section 6 we show how to apply our generalized Cartier isomorphism to the Hodge-to-de Rham degeneration. The exposition in Sections 2-4 is largely self-contained. In the rest of the paper, we switch to a more descriptive style, with no proofs, and not many precise statements; this part of the paper should be treated as a companion to [Kal08].

Acknowledgments. This paper is a write-up (actually quite an enlarged write-up) of two lectures given in Göttingen in August 2006, at a summer school organized by Yu. Tschinkel and funded by the Clay Institute. I am very grateful to all concerned for making it happen, and for giving me an opportunity to present my results. In addition, I would like to mention that a large part of the present paper is written in overview style; many, if not most, of the things overviewed are certainly *not* my results. This especially concerns Section 2, on the one hand, and Section 6, on the other hand. Given the chosen style, it is difficult to provide exact attributions; however, I should at least mention that I've learned much of this material from A. Beilinson, A. Bondal, M. Kontsevich, B. Toën and B. Tsygan.

2. Cyclic homology package

2.1. Basic definitions. The fastest and most down-to-earth way to define cyclic homology is by means of an explicit complex. Namely, assume given an associative unital algebra A over a field k . To compute its Hochschild homology with coefficients in some bimodule M , one has to find a flat resolution of M . One such is the *bar resolution*—it is rather inconvenient in practical computations, but it is completely canonical, and it exists without any assumptions on A and M . The terms of this resolution are of the form $A^{\otimes n} \otimes M$, $n \geq 0$, and the differential $b' : A^{\otimes n+1} \otimes M \rightarrow A^{\otimes n} \otimes M$ is given by

$$(2.1) \quad b' = \sum_{0 \leq i \leq n} (-1)^i \text{id}^{\otimes i} \otimes m \otimes \text{id}^{\otimes n-i},$$

where $m : A \otimes A \rightarrow A$, $m : A \otimes M \rightarrow M$ are the multiplication maps. Substituting this resolution into (1.4) gives a complex which computes $HH_*(A, M)$; its terms are also $A^{\otimes i} \otimes M$, but the differential is given by

$$(2.2) \quad b = b' + (-1)^{n+1} t,$$

with the correction term t being equal to

$$t(a_0 \otimes \cdots \otimes a_{n+1} \otimes m) = a_1 \otimes \cdots \otimes a_{n+1} \otimes ma_0$$

for any $a_0, \dots, a_{n+1} \in A$, $m \in M$. Geometrically, one can think of the components a_0, \dots, a_{n-1}, m of some tensor in $A^{\otimes n} \otimes M$ as having been placed at $n+1$ points on the unit interval $[0, 1]$, including the endpoints $0, 1 \in [0, 1]$; then each of the terms in the differential b' corresponds to contracting an interval between two neighboring points and multiplying the components sitting at its endpoints. To visualize the differential b in a similar way, one has to take $n+1$ points placed on the unit circle S^1 instead of the unit interval, including the point $1 \in S^1$, where we put the component m .

In the case $M = A$, the terms in the bar complex are just $A^{\otimes n+1}$, $n \geq 0$, and they acquire an additional symmetry: we let $\tau : A^{\otimes n+1} \rightarrow A^{\otimes n+1}$ be the cyclic permutation multiplied by $(-1)^n$. Note that in spite of the sign change, we have $\tau^{n+1} = \text{id}$, so that it generates an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on every $A^{\otimes n+1}$. The fundamental fact here is the following.

LEMMA 2.1 ([FT87],[Lod98]). *For any n , we have*

$$\begin{aligned} (\text{id} - \tau) \circ b' &= -b \circ (\text{id} - \tau), \\ (\text{id} + \tau + \dots + \tau^{n-1}) \circ b &= -b' \circ (\text{id} + \tau + \dots + \tau^n) \end{aligned}$$

as maps from $A^{\otimes n+1}$ to $A^{\otimes n}$.

Proof. Denote $m_i = \text{id}^i \otimes m \otimes \text{id}^{n-i} : A^{\otimes n+1} \rightarrow A^{\otimes n}$, $0 \leq i \leq n-1$, so that $b' = m_0 - m_1 + \dots + (-1)^{n-1} m_{n-1}$, and let $m_n = t = (-1)^n (b - b')$. Then we obviously have

$$m_{i+1} \circ \tau = \tau \circ m_i$$

for $0 \leq i \leq n-1$, and $m_0 \circ \tau = (-1)^n m_n$. Formally applying these identities, we conclude that

$$\begin{aligned} \sum_{0 \leq i \leq n} (-1)^i m_i \circ (\text{id} - \tau) &= \sum_{0 \leq i \leq n} (-1)^i m_i - m_0 - \sum_{1 \leq i \leq n} (-1)^i \tau \circ m_{i-1} \\ (2.3) \qquad \qquad \qquad &= -(\text{id} - \tau) \circ \sum_{0 \leq i \leq n-1} (-1)^i m_i, \end{aligned}$$

$$\begin{aligned} (2.4) \quad b' \circ (\text{id} + \tau + \dots + \tau^n) &= \sum_{0 \leq i \leq n-1} \sum_{0 \leq j \leq n} (-1)^i m_i \circ \tau^j \\ &= \sum_{0 \leq j \leq i \leq n-1} (-1)^i \tau^j \circ m_{i-j} + \sum_{1 \leq i \leq j \leq n} (-1)^{i+n} \tau^{j-1} \circ m_{n+i-j} \\ &= -(\text{id} + \tau + \dots + \tau^{n-1}) \circ b, \end{aligned}$$

which proves the claim. □

As a corollary, the following diagram is in fact a bicomplex.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & A & \xrightarrow{\text{id}} & A & \xrightarrow{0} & A \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
 \cdots & \longrightarrow & A \otimes A & \xrightarrow{\text{id} + \tau} & A \otimes A & \xrightarrow{\text{id} - \tau} & A \otimes A \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
 (2.5) & & \vdots & & \vdots & & \vdots \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
 \cdots & \longrightarrow & A^{\otimes n} & \xrightarrow{\text{id} + \tau + \cdots + \tau^{n-1}} & A^{\otimes n} & \xrightarrow{\text{id} - \tau} & A^{\otimes n} \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Here it is understood that the whole thing extends indefinitely to the left, all the even-numbered columns are the same, all odd-numbered columns are the same, and the bicomplex is invariant with respect to the horizontal shift by 2 columns. The total homology of this bicomplex is called the *cyclic homology* of the algebra A , and denoted by $HC_*(A)$.

We see right away that the first, the third, and so on column when counting from the right is the bar complex which computes $HH_*(A)$, and the second, the fourth, and so on column is acyclic (the top term is A , and the rest is the bar resolution for A). Thus the spectral sequence for this bicomplex has the form given in (1.5) (modulo obvious renumbering). On the other hand, the rows of the bicomplex are just the standard 2-periodic complexes which compute the cyclic group homology $H_*(\mathbb{Z}/n\mathbb{Z}, A^{\otimes n})$ (with respect to the $\mathbb{Z}/n\mathbb{Z}$ -action on $A^{\otimes n}$ given by τ).

Shifting (2.5) to the right by 2 columns gives the *periodicity map*

$$u : HC_{\bullet+2}(A) \rightarrow HC_{\bullet}(A),$$

which fits into an exact triangle

$$(2.6) \quad HH_{\bullet+2} \longrightarrow HC_{\bullet+2}(A) \longrightarrow HC_{\bullet}(A) \longrightarrow ,$$

known as the *Connes exact sequence*. One can also invert the periodicity map—in other words, extend the bicomplex (2.5) not only to the left, but also to the right. This gives the *periodic cyclic homology* $HP_*(A)$. Since the bicomplex for $HP_*(A)$ is infinite in both directions, there is a choice involved in taking the total complex: we can take either the product, or the sum of the terms. We take the product. In characteristic 0, the sum is actually acyclic (because so is every row).

If A is commutative, $X = \text{Spec}(A)$ is smooth, and $\text{char } k$ is either 0 or greater than $\dim X$, then the only non-trivial differential in the Hodge-to-de Rham spectral sequence (1.5) is the first one, and it is the de Rham differential. Consequently, we have $HP_{\bullet}(A) = H_{DR}^{\bullet}(X)((u))$ (where as before, u is a formal variable of cohomological degree 2).

2.2. The p -cyclic complex. All of the above is completely standard; however, we will also need to use another way to compute $HC_{\bullet}(A)$, which is less

standard. Namely, fix an integer $p \geq 2$, and consider the algebra $A^{\otimes p}$. Let $\sigma : A^{\otimes p} \rightarrow A^{\otimes p}$ be the cyclic permutation, and let $A_{\sigma}^{\otimes p}$ be the diagonal $A^{\otimes p}$ -bimodule with the bimodule structure twisted by σ —namely, we let

$$a \cdot b \cdot c = ab\sigma(c)$$

for any $a, b, c \in A^{\otimes p}$.

LEMMA 2.2. *We have $HH_*(A^{\otimes p}, A_{\sigma}^{\otimes p}) \cong HH_*(A)$.*

Proof. Induction on p . We may compute the tensor product in (1.4) over each of the factors A in $A^{\otimes p}$ in turn; this shows that

$$HH_*(A^{\otimes p}, A_{\sigma}^{\otimes p}) \cong \text{Tor}_{(A^{\otimes(p-1)})^{\text{opp}} \otimes A^{\otimes(p-1)}}^{\bullet} \left(A^{\otimes(p-1)}, \text{Tor}_{A^{\text{opp}} \otimes A}^{\bullet}(A, A_{\sigma}^{\otimes p}) \right),$$

and one checks easily that as long as $p \geq 2$, so that $A_{\sigma}^{\otimes p}$ is flat over $A^{\text{opp}} \otimes A$, $\text{Tor}_{A^{\text{opp}} \otimes A}^i(A, A_{\sigma}^{\otimes p})$ is naturally isomorphic to $A_{\sigma}^{\otimes(p-1)}$ if $i = 0$, and trivial if $i \geq 1$. \square

By virtue of this Lemma, we can use the bar complex for the algebra $A^{\otimes p}$ to compute $HH_*(A)$. The resulting complex has terms $A^{\otimes pn}$, $n \geq 0$. The differential $b'_p : A^{\otimes p(n+2)} \rightarrow A^{\otimes p(n+1)}$ is given by essentially the same formula as (2.1):

$$b'_p = \sum_{0 \leq i \leq n} (-1)^i m_i^p = \sum_{0 \leq i \leq n} (-1)^i \text{id}^{\otimes pi} \otimes m^{\otimes p} \otimes \text{id}^{\otimes p(n-i)},$$

where we decompose $A^{\otimes p(n+1)} = (A^{\otimes p})^{\otimes(n+1)}$. The correction term $t_p = m_{n+1}^p$ in (2.2) is given by $m_0 \circ \tau$ (where, as before, $\tau : A^{\otimes p(n+2)}$ is the cyclic permutation of order $p(n+2)$ twisted by a sign). Geometrically, the component m_i^p of the Hochschild differential b_p corresponds to contracting simultaneously the i -th, $(i+p)$ -th, $(i+2p)$ -th, and so on, intervals in the unit circle, divided into $p(n+2)$ intervals by $p(n+2)$ points. On the level of bar complexes, the comparison isomorphism $HH_*(A^{\otimes p}, A_{\sigma}^{\otimes p}) \cong HH_*(A)$ of Lemma 2.2 is represented by the map

$$(2.7) \quad M = m \circ (\text{id} \otimes m) \circ (\text{id}^{\otimes 2} \otimes m) \circ \dots \circ (\text{id}^{\otimes pn-2} \otimes m) : A^{\otimes pn} \rightarrow A^{\otimes n};$$

explicitly, we have

$$\begin{aligned} M(a_{1,1} \otimes a_{2,1} \otimes \dots \otimes a_{n,1} \otimes a_{1,2} \otimes a_{2,2} \otimes \dots \otimes a_{n,2} \otimes \dots \otimes a_{1,p} \otimes a_{2,p} \otimes \dots \otimes a_{n,p}) \\ = a_{1,1} \otimes a_{2,1} \otimes \dots \otimes a_{n-1,1} \otimes \left(a_{n,1} \cdot \prod_{2 \leq j \leq p} \prod_{1 \leq i \leq n} a_{i,j} \right) \end{aligned}$$

for any $a_{1,1} \otimes a_{2,1} \otimes \dots \otimes a_{n,1} \otimes a_{1,2} \otimes a_{2,2} \otimes \dots \otimes a_{n,2} \otimes \dots \otimes a_{1,p} \otimes a_{2,p} \otimes \dots \otimes a_{n,p} \in A^{\otimes pn}$ —in other words, $M : A^{\otimes pn} \rightarrow A^{\otimes n}$ leaves the first $n-1$ terms in the tensor product intact and multiplies the remaining $pn-n+1$ terms. We leave it to the interested reader to check explicitly that $M \circ b_p = b \circ M$.

LEMMA 2.3. *For any n , we have*

$$\begin{aligned} (\text{id} - \tau) \circ b'_p &= -b_p \circ (\text{id} - \tau), \\ (\text{id} + \tau + \dots + \tau^{pn-1}) \circ b_p &= -b'_p \circ (\text{id} + \tau + \dots + \tau^{p(n+1)-1}) \end{aligned}$$

as maps from $A^{\otimes p(n+1)}$ to $A^{\otimes pn}$.

Proof. One immediately checks that, as in the proof of Lemma 2.1, we have

$$m_{i+1}^p \circ \tau = -\tau \circ m_i^p$$

for $0 \leq i \leq n$, and we also have $m_0^p \circ \tau = m_{n+1}^p$. Then the first equality follows from (2.3), and (2.4) gives

$$(\text{id} + \tau + \dots + \tau^{n-1}) \circ b_p = -b'_p \circ (\text{id} + \tau + \dots + \tau^n)$$

(note that the proof of these two equalities does *not* use the fact that $\tau^{n+1} = \text{id}$ on $A^{\otimes(n+1)}$). To deduce the second equality of the Lemma, it suffices to notice that

$$\text{id} + \tau + \dots + \tau^{p(n+1)-1} = (\text{id} + \tau + \dots + \tau^n) \circ (\text{id} + \sigma + \dots + \sigma^{p-1}),$$

and σ commutes with all the maps m_i^p . □

Using Lemma 2.3, we can construct a version of the bicomplex (2.5) for $p > 1$:

$$(2.8) \quad \begin{array}{ccccccc} \dots & \longrightarrow & A^{\otimes p} & \xrightarrow{\text{id} + \tau + \dots + \tau^{p-1}} & A^{\otimes p} & \xrightarrow{\text{id} - \tau} & A^{\otimes p} \\ & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\ \dots & \longrightarrow & A^{\otimes 2p} & \xrightarrow{\text{id} + \dots + \tau^{2p-1}} & A^{\otimes 2p} & \xrightarrow{\text{id} - \tau} & A^{\otimes 2p} \\ & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\ \dots & \longrightarrow & A^{\otimes pn} & \xrightarrow{\text{id} + \tau + \dots + \tau^{pn-1}} & A^{\otimes pn} & \xrightarrow{\text{id} - \tau} & A^{\otimes pn} \\ & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\ & & \vdots & & \vdots & & \vdots \end{array}$$

By abuse of notation, we denote the homology of the total complex of this bicomplex by $HC_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$. (This is really abusive, since in general one *cannot* define cyclic homology with coefficients in a bimodule—unless the bimodule is equipped with additional structure, such as e.g. in [Kal07], which lies beyond the scope of this paper.) As for the usual cyclic complex, we have the periodicity map, the Connes exact sequence, and we can form the periodic cyclic homology $HP_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$.

2.3. Small categories. Unfortunately, this is as far as the down-to-earth approach takes us. While it is true that the isomorphism $HH_\bullet(A^{\otimes p}, A_\sigma^{\otimes p}) \cong HH_\bullet(A)$ given in Lemma 2.2 can be extended to an isomorphism

$$HC_\bullet(A^{\otimes p}, A_\sigma^{\otimes p}) \cong HC_\bullet(A),$$

it is not possible to realize this extended isomorphism by an explicit map of bicomplexes. Indeed, already in degree 0 the comparison map M of (2.7) which realized the isomorphism

$$HH_0(A^{\otimes p}, A_\sigma^{\otimes p}) \rightarrow HH_0(A)$$

on the level of bar complexes is given by the multiplication map $A^{\otimes p} \rightarrow A$, and to define this multiplication map, one has to break the cyclic symmetry of the product $A^{\otimes p}$. The best one can obtain is a map between total complexes computing $HC_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$ and $HC_\bullet(A)$ which preserves the filtration, but not the second

grading; when one tries to write the map down explicitly, the combinatorics quickly gets completely out of control.

For this reason, in [Kal05] and [Kal08] one follows [Con83] and uses a more advanced approach to cyclic homology which is based on the technique of *homology of small categories* (see e.g. [Lod98, Section 6]). Namely, for any small category Γ and any base field k , the category $\text{Fun}(\Gamma, k)$ of functors from Γ to k -vector spaces is an abelian category, and the direct limit functor \varinjlim_{Γ} is right exact. Its derived functors are called *homology functors* of the category Γ and denoted by $H_{\bullet}(\Gamma, E)$ for any $E \in \text{Fun}(\Gamma, k)$. For instance, if Γ is a groupoid with one object with automorphism group G , then $\text{Fun}(\Gamma, k)$ is the category of k -representations of the group G ; the homology $H_{\bullet}(\Gamma, -)$ is then tautologically the same as the group homology $H_{\bullet}(G, -)$. Another example is the category Δ^{opp} , the opposite to the category Δ of finite non-empty totally ordered sets. It is not difficult to check that for any simplicial k -vector $E \in \text{Fun}(\Delta^{opp}, k)$, the homology $H_{\bullet}(\Delta^{opp}, E)$ can be computed by the standard chain complex of E .

For applications to cyclic homology, one introduces special small categories Λ_{∞} and Λ_p , $p \geq 1$. The objects in the category Λ_{∞} are numbered by the positive integers and denoted $[n]$, $n \geq 1$. For any $[n], [m] \in \Lambda_{\infty}$, the set of maps $\Lambda_{\infty}([n], [m])$ is the set of all maps $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$(2.9) \quad f(a) \leq f(b) \quad \text{whenever } a \leq b, \quad f(a+n) = f(a) + m,$$

for any $a, b \in \mathbb{Z}$. For any $[n] \in \Lambda_{\infty}$, denote by $\sigma : [n] \rightarrow [n]$ the endomorphism given by $f(a) = a + n$. Then σ commutes with all maps in Λ_{∞} . The category Λ_p has the same objects as Λ_{∞} , and the set of maps is

$$\Lambda_p([n], [m]) = \Lambda_{\infty}([n], [m]) / \sigma^p$$

for any $[n], [m] \in \Lambda_p$. The category Λ_1 is denoted simply by Λ ; this is the original cyclic category introduced by A. Connes in [Con83]. By definition, we have projections $\Lambda_{\infty} \rightarrow \Lambda_p$ and $\pi : \Lambda_p \rightarrow \Lambda$.

If we only consider those maps in (2.9) which send $0 \in \mathbb{Z}$ to 0, then the resulting subcategory in Λ_{∞} is equivalent to Δ^{opp} . This gives a canonical embedding $j : \Delta^{opp} \rightarrow \Lambda_{\infty}$, and consequently, embeddings $j : \Delta^{opp} \rightarrow \Lambda_p$.

The category Λ_p conveniently encodes the maps m_i^p and τ between various tensor powers $A^{\otimes pn}$ used in the complex (2.8): m_i^p corresponds to the map $f \in \Lambda_p([n+1], [n])$ given by

$$f(a(n+1) + b) = \begin{cases} an + b, & b \leq i, \\ an + b - 1, & b > i, \end{cases}$$

where $0 \leq b \leq n$, and τ is the map $a \mapsto a + 1$, twisted by the sign (alternatively, one can say that m_i^p are obtained from face maps in Δ^{opp} under the embedding $\Delta^{opp} \subset \Lambda_p$). The relations between these maps which we used in the proof of Lemma 2.3 are encoded in the composition laws of the category Λ_p . Thus for any object $E \in \text{Fun}(\Lambda_p, k)$ —they are called *p-cyclic objects*—one can form the

bicomplex of the type (2.8) (or (2.5), for $p = 1$):

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & E([1]) & \xrightarrow{\text{id} + \tau + \cdots + \tau^{p-1}} & E([1]) & \xrightarrow{\text{id} - \tau} & E([1]) \\
 & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\
 \cdots & \longrightarrow & E([2]) & \xrightarrow{\text{id} + \cdots + \tau^{2p-1}} & E([2]) & \xrightarrow{\text{id} - \tau} & E([2]) \\
 & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\
 (2.10) & & \vdots & & \vdots & & \vdots \\
 & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\
 \cdots & \longrightarrow & E([n]) & \xrightarrow{\text{id} + \tau + \cdots + \tau^{pn-1}} & E([n]) & \xrightarrow{\text{id} - \tau} & E([n]) \\
 & & \uparrow b_p & & \uparrow b'_p & & \uparrow b_p \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Just as for the complex (2.8), we have periodicity, the periodic version of the complex, and the Connes exact sequence (2.6) (the role of Hochschild homology is played by the standard chain complex of the simplicial vector space $j^* E \in \text{Fun}(\Delta^{opp}, k)$).

LEMMA 2.4. *For any $E \in \text{Fun}(\Lambda_p, k)$, the homology $H_*(\Lambda_p, E)$ can be computed by the bicomplex (2.10).*

Proof. The homology of the total complex of (2.10) is obviously a homological functor from $\text{Fun}(\Lambda_p, k)$ to k (that is, short exact sequences in $\text{Fun}(\Lambda_p, k)$ give long exact sequences in homology). Therefore it suffices to prove the claim for a set of projective generators of the category $\text{Fun}(\Lambda_p, k)$. For instance, it suffices to consider all the representable functors E_n , $n \geq 1$ —that is, the functors given by

$$E_n([m]) = k[\Lambda_p([n], [m])],$$

where in the right-hand side we take the k -linear span. Then on one hand, for general tautological reasons—essentially by the Yoneda Lemma— $H_*(\Lambda_p, E_n)$ is k in degree 0 and 0 in higher degrees. On the other hand, the action of the cyclic group $\mathbb{Z}/pm\mathbb{Z}$ generated by $\tau \in \Lambda_p([m], [m])$ on $\Lambda_p([n], [m])$ is obviously free, and we have

$$\Lambda_p([n], [m])/\tau \cong \Delta^{opp}([n], [m])$$

—every $f : \mathbb{Z} \rightarrow \mathbb{Z}$ can be uniquely decomposed as $f = \tau^j \circ f_0$, where $0 \leq j < pm$, and f_0 sends 0 to 0. The rows of the complex (2.10) compute

$$H_*(\mathbb{Z}/pm\mathbb{Z}, E_n([m])) \cong k[\Delta^{opp}([n], [m])],$$

and the first term in the corresponding spectral sequence is the standard complex for the simplicial vector space $E_n^\Delta \in \text{Fun}(\Delta^{opp}, k)$ represented by $[n] \in \Delta^{opp}$. Therefore this complex computes $H_*(\Delta^{opp}, E_n^\Delta)$, which is again k . \square

The complex (2.5) is the special case of (2.10) for $p = 1$ and the following object $A_\# \in \text{Fun}(\Lambda, k)$: we set $A_\#([n]) = A^{\otimes n}$, where the factors are numbered by

elements in the set $V([n]) = \mathbb{Z}/n\mathbb{Z}$, and any $f \in \Lambda([n], [m])$ acts by

$$A_{\#}(f) \left(\bigotimes_{i \in V([n])} a_i \right) = \bigotimes_{j \in V([m])} \prod_{i \in f^{-1}(j)} a_i,$$

(if $f^{-1}(i)$ is empty for some $i \in V([m])$, then the right-hand side involves a product numbered by the empty set; this is defined to be the unity element $1 \in A$). To obtain the complex (2.8), we note that for any p , we have a functor $i : \Lambda_p \rightarrow \Lambda$ given by $[n] \mapsto [pn]$, $f \mapsto f$. Then (2.10) applied to $i^*A_{\#} \in \text{Fun}(\Lambda_p, k)$ gives (2.8). By Lemma 2.4 we have

$$\begin{aligned} HC_{\bullet}(A) &\cong H_{\bullet}(\Lambda, k), \\ HC_{\bullet}(A^{\otimes p}, A_{\sigma}^{\otimes p}) &\cong H_{\bullet}(\Lambda_p, k). \end{aligned}$$

LEMMA 2.5 ([Kal08, Lemma 1.12]). *For any $E \in \text{Fun}(\Lambda, k)$, we have a natural isomorphism*

$$H_{\bullet}(\Lambda_p, i^*E) \cong H_{\bullet}(\Lambda, E),$$

which is compatible with the periodicity map and with the Connes exact sequence (2.6). □

Thus $HC_{\bullet}(A^{\otimes p}, A_{\sigma}^{\otimes p}) \cong HC_{\bullet}(A)$. The proof of this Lemma is not difficult. First of all, a canonical comparison map $H_{\bullet}(\Lambda_p, i^*E) \rightarrow H_{\bullet}(\Lambda, E)$ exists for tautological adjunction reasons. Moreover, the periodicity homomorphism for $H_{\bullet}(\Lambda_p, -)$ is induced by the action of a canonical element $u_p \in H^2(\Lambda_p, k) = \text{Ext}^2(k, k)$, where k means the constant functor $[n] \mapsto k$ from Λ_p to k . One checks explicitly that $i^*u = u_p$, so that the comparison map is indeed compatible with periodicity, and then it suffices to prove that the comparison map

$$H_{\bullet}(\Delta^{opp}, i^*E) \rightarrow H_{\bullet}(\Delta^{opp}, E)$$

is an isomorphism. When E is of the form $A_{\#}$, this is Lemma 2.2; in general, one shows that $\text{Fun}(\Lambda, k)$ has a projective generator of the form $A_{\#}$. For details, we refer the reader to [Kal08].

3. One vanishing result

For our construction of the Cartier map, we will need one vanishing-type result on periodic cyclic homology in prime characteristic—we want to claim that the periodic cyclic homology $HP_{\bullet}(E)$ of a p -cyclic object E vanishes under some assumptions on E .

First, consider the cyclic group $\mathbb{Z}/np\mathbb{Z}$ for some $n, p \geq 1$, with the subgroup $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/pn\mathbb{Z}$ and the quotient $\mathbb{Z}/n\mathbb{Z} = (\mathbb{Z}/pn\mathbb{Z})/(\mathbb{Z}/p\mathbb{Z})$. It is well-known that for any representation V of the group $\mathbb{Z}/pn\mathbb{Z}$, we have the Hochschild-Serre spectral sequence

$$H_{\bullet}(\mathbb{Z}/n\mathbb{Z}, H_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V)) \Rightarrow H_{\bullet}(\mathbb{Z}/pn\mathbb{Z}, -).$$

To see it explicitly, one can compute the homology $H_{\bullet}(\mathbb{Z}/np\mathbb{Z}, V)$ by a complex which is slightly more complicated than the standard one. Namely, write down the

diagram

$$\begin{array}{ccccc}
 \xrightarrow{\text{id}-\sigma} & V & \xrightarrow{d_\sigma} & V & \xrightarrow{\text{id}-\sigma} & V \\
 & \uparrow \text{id}-\tau & & \uparrow \text{id}-\tau & & \uparrow \text{id}-\tau \\
 \xrightarrow{\text{id}-\sigma} & V & \xrightarrow{d_\sigma} & V & \xrightarrow{\text{id}-\sigma} & V \\
 (3.1) & \uparrow d_\tau & & \uparrow d_\tau & & \uparrow d_\tau \\
 \xrightarrow{\text{id}-\sigma} & V & \xrightarrow{d_\sigma} & V & \xrightarrow{\text{id}-\sigma} & V \\
 & \uparrow \text{id}-\tau & & \uparrow \text{id}-\tau & & \uparrow \text{id}-\tau
 \end{array}$$

where τ is the generator of $\mathbb{Z}/pn\mathbb{Z}$, $\sigma = \tau^n$ is the generator of $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/pn\mathbb{Z}$, and $d_\sigma = \text{id} + \sigma + \dots + \sigma^p$, $d_\tau = \text{id} + \tau + \dots + \tau^{n-1}$. This is not quite a bicomplex since the vertical differential squares to $\text{id} - \sigma$, not to 0; to correct this, we add to the total differential the term $\text{id} : V \rightarrow V$ of bidegree $(-1, 2)$ in every term in the columns with odd numbers (when counting from the right). The result is a filtered complex which computes $H_*(\mathbb{Z}/pn\mathbb{Z}, V)$, and the Hochschild-Serre spectral sequence appears as the spectral sequence of the filtered complex (3.1).

One feature which is apparent in the complex (3.1) is that it has two different periodicity endomorphisms: the endomorphism which shifts the diagram to the left by two columns (we will denote it by u), and the endomorphism which shifts the diagram downwards by two rows (we will denote it by u').

Assume now given a field k and a p -cyclic object $E \in \text{Fun}(\Lambda_p, k)$, and consider the complex (2.10). Its n -th row is the standard periodic complex which computes $H_*(\mathbb{Z}/pn\mathbb{Z}, E([n]))$, and we can replace all these complexes by the corresponding complex (3.1). By virtue of Lemma 2.3, the result is a certain filtered bicomplex of the form

$$\begin{array}{ccccc}
 \xrightarrow{\text{id}-\sigma} & C_\bullet(E) & \xrightarrow{\text{id}+\sigma+\dots+\sigma^{p-1}} & C_\bullet(E) & \xrightarrow{\text{id}-\sigma} & C_\bullet(E) \\
 & \uparrow B & & \uparrow B & & \uparrow B \\
 \xrightarrow{\text{id}-\sigma} & C'_\bullet(E) & \xrightarrow{\text{id}+\sigma+\dots+\sigma^{p-1}} & C'_\bullet(E) & \xrightarrow{\text{id}-\sigma} & C'_\bullet(E) \\
 (3.2) & \uparrow B & & \uparrow B & & \uparrow B \\
 \xrightarrow{\text{id}-\sigma} & C_\bullet(E) & \xrightarrow{\text{id}+\sigma+\dots+\sigma^{p-1}} & C_\bullet(E) & \xrightarrow{\text{id}-\sigma} & C_\bullet(E), \\
 & \uparrow B & & \uparrow B & & \uparrow B
 \end{array}$$

with id of degree $(-1, 2)$ added to the total differential, where $C_\bullet(E)$, resp. $C'_\bullet(E)$, is the complex with terms $E([n])$ and the differential b_p , resp. b'_p , and B is the horizontal differential in the complex (2.10) written down for $p = 1$. The complex $C_\bullet(E)$ computes the Hochschild homology $HH_\bullet(E)$, the complex $C'_\bullet(E)$ is acyclic, and the whole complex (3.2) computes the cyclic homology $HC_\bullet(E)$.

We see that the cyclic homology of the p -cyclic object E actually admits two periodicity endomorphisms: u and u' . The horizontal endomorphism u is the usual periodicity map; the vertical map u' is something new. However, we have the following.

LEMMA 3.1. *In the situation above, assume that $p = \text{char } k$. Then the vertical periodicity map $u' : HC_{\bullet}(E) \rightarrow HC_{\bullet-2}(E)$ is equal to 0.*

Sketch of a proof. It might be possible to write explicitly a contracting homotopy for the map u' , but this is very complicated; instead, we will sketch the “scientific” proof which uses small categories (for details, see [Kal08]). For any small category Γ , its cohomology $H^{\bullet}(\Gamma, k)$ is defined as

$$H^{\bullet}(\Gamma, k) = \text{Ext}_{\text{Fun}(\Gamma, k)}^{\bullet}(k, k),$$

where k in the right-hand side is the constant functor. This is an algebra which obviously acts on $H_{\bullet}(\Gamma, E)$ for any $E \in \text{Fun}(\Gamma, k)$.

The cohomology $H^{\bullet}(\Lambda, k)$ of the cyclic category Λ is the algebra of polynomials in one generator u of degree 2, $u \in H^2(\Lambda, k)$; the action of this u on the cyclic homology $HC_{\bullet}(-)$ is the periodicity map. The same is true for the m -cyclic categories Λ_m for all $m \geq 1$.

Now, recall that we have a natural functor $\pi : \Lambda_p \rightarrow \Lambda$, so that there are two natural elements in $H^2(\Lambda_p, k)$ —the generator u and the preimage $\pi^*(u)$ of the generator $u \in H^2(\Lambda, k)$. The action of u gives the horizontal periodicity endomorphism of the complex (3.2), and the action of $\pi^*(u)$ gives the vertical periodicity endomorphism u' . We have to prove that if $\text{char } k = p$, then $\pi^*(u) = 0$.

To do this, one uses a version of the Hochschild-Serre spectral sequence associated to π —namely, we have a spectral sequence

$$H^{\bullet}(\Lambda) \otimes H^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k) \Rightarrow H^{\bullet}(\Lambda_p, k).$$

If $\text{char } k = p$, then the group cohomology algebra $H^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k)$ is the polynomial algebra $k[u, \varepsilon]$ with two generators: an even generator $u \in H^2(\mathbb{Z}/p\mathbb{Z}, k)$ and an odd generator $\varepsilon \in H^1(\mathbb{Z}/p\mathbb{Z}, k)$. Since $H^{\bullet}(\Lambda_p, k) = k[u]$, the second differential d_2 in the spectral sequence must send ε to $\pi^*(u)$, so that indeed, $\pi^*(u) = 0$ in $H^2(\Lambda_p, k)$. \square

Consider now the version of the complex (3.2) which computes the periodic cyclic homology $HP_{\bullet}(E)$ —to obtain it, one has to extend the diagram to the right by periodicity. The rows of the extended diagram then become the standard complexes which compute the Tate homology $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, C_{\bullet}(E))$. We remind the reader that the Tate homology $\check{H}_{\bullet}(G, -)$ is a certain homological functor defined for any finite group G which combines together homology $H_{\bullet}(G, -)$ and cohomology $H^{\bullet}(G, -)$, and that for a cyclic group $\mathbb{Z}/m\mathbb{Z}$ with generator σ , the Tate homology $\check{H}_{\bullet}(\mathbb{Z}/m\mathbb{Z}, W)$ with coefficients in some representation W may be computed by the 2-periodic standard complex

$$(3.3) \quad \dots \xrightarrow{d_-} W \xrightarrow{d_+} W \xrightarrow{d_-} W \xrightarrow{d_+} \dots$$

with $d_+ = \text{id} + \sigma + \dots + \sigma^{m-1}$ and $d_- = \text{id} - \sigma$.

If W is a free module over the group algebra $k[G]$, then the Tate homology vanishes in all degrees, $\check{H}_{\bullet}(G, W) = 0$. When $G = \mathbb{Z}/m\mathbb{Z}$, this means that the standard complex is acyclic. If m is prime and equal to the characteristic of the base field k , the converse is also true— $\check{H}_{\bullet}(\mathbb{Z}/m\mathbb{Z}, W) = 0$ if and only if W is free over $k[\mathbb{Z}/m\mathbb{Z}]$. We would like to claim a similar vanishing for Tate homology $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, W_{\bullet})$ with coefficients in some complex W_{\bullet} of $k[\mathbb{Z}/p\mathbb{Z}]$ -modules; however, this is not possible unless we impose some finiteness conditions on W_{\bullet} .

DEFINITION 3.2. A complex W_\bullet of $k[\mathbb{Z}/p\mathbb{Z}]$ -modules is *effectively finite* if it is chain homotopic to a complex of finite length. A p -cyclic object $E \in \text{Fun}(\Lambda_p, k)$ is *small* if its standard complex $C_\bullet(E)$ is effectively finite.

Now we can finally state our vanishing result for periodic cyclic homology.

PROPOSITION 3.3. *Assume that $p = \text{char } k$. Assume that a p -cyclic object $E \in \text{Fun}(\Lambda_p, k)$ is small, and that $E([n])$ is a free $k[\mathbb{Z}/p\mathbb{Z}]$ -module for every object $[n] \in \Lambda_p$. Then $HP_\bullet(E) = 0$.*

Proof. To compute $HP_\bullet(E)$, let us use the periodic version of the complex (3.2). We then have a long exact sequence of cohomology

$$HP_{\bullet-1}(E) \longrightarrow \check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, C_\bullet(E)) \longrightarrow HP_\bullet(E) \xrightarrow{u'} ,$$

where $\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, C_\bullet(E))$ is computed by the total complex of the bicomplex

$$(3.4) \quad \begin{array}{ccccccc} \dots & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & E([1]) & \xrightarrow{\text{id} - \sigma} & E([1]) & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & \dots \\ & & \uparrow b_p & & \uparrow b_p & & \\ \dots & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & E([2]) & \xrightarrow{\text{id} - \sigma} & E([2]) & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & \dots \\ & & \uparrow b_p & & \uparrow b_p & & \\ & & \vdots & & \vdots & & \\ & & \uparrow b_p & & \uparrow b_p & & \\ \dots & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & E([n]) & \xrightarrow{\text{id} - \sigma} & E([n]) & \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} & \dots \\ & & \uparrow b_p & & \uparrow b_p & & \\ & & \vdots & & \vdots & & \end{array}$$

By Lemma 3.1, the connecting differential in the long exact sequence vanishes, so that it suffices to prove that $\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, C_\bullet(E)) = 0$. Since $E([n])$ is free, all the rows of the bicomplex (3.4) are acyclic. But since E is small, $C_\bullet(E)$ is effectively finite; therefore the spectral sequence of the bicomplex (3.4) converges, and we are done. \square

4. Quasi-Frobenius maps

We now fix a perfect base field k of characteristic $p > 0$, and consider an associative algebra A over k . We want to construct a cyclic homology version of the Cartier isomorphism (1.2) for A . In fact, we will construct a version of the inverse isomorphism C^{-1} ; it will be an isomorphism

$$(4.1) \quad C^{-1} : HH_\bullet(A)((u))^{(1)} \longrightarrow HP_\bullet(A),$$

where, as before, $HH_\bullet(A)((u))$ in the left-hand side means ‘‘Laurent power series in one variable u of degree 2 with coefficients in $HH_\bullet(A)$ ’’.

If A is commutative and $X = \text{Spec } A$ is smooth, then $HH_\bullet(A) \cong \Omega^\bullet(X)$, $HP_\bullet(A) \cong H_{DR}^\bullet(X)((u))$, and (4.1) is obtained by inverting (1.2) (and repeating the resulting map infinitely many times, once for every power of the formal variable u).

It is known that the commutative inverse Cartier map is induced by the Frobenius isomorphism; thus to generalize it to non-commutative algebras, it is natural to start the story with the Frobenius map.

At first glance, the story thus started ends immediately: the map $a \mapsto a^p$ is not an algebra endomorphism of A unless A is commutative (in fact, the map is not even additive, $(x + y)^p \neq x^p + y^p$ for general non-commuting x and y). So, there is no Frobenius map in the non-commutative world.

However, to analyze the difficulty, let us decompose the usual Frobenius into two maps:

$$A \xrightarrow{\varphi} A^{\otimes p} \xrightarrow{M} A,$$

where φ is given by $\varphi(a) = a^{\otimes p}$, and M is the multiplication map, $M(a_1 \otimes \cdots \otimes a_p) = a_1 \cdots a_p$. The map φ is very bad (e.g. not additive), but this is the same both in the commutative and in the general associative case. It is the map M which creates the problem: it is an algebra map if and only if A is commutative.

In general, it is not possible to correct M so that it becomes an algebra map. However, even not being an algebra map, it can be made to act on Hochschild homology, and we already saw how: we can take the map (2.7) of Subsection 2.2.

As for the very bad map φ , fortunately, it turns out that it can be perturbed quite a bit. In fact, the only property of this map which is essential is the following one.

LEMMA 4.1. *Let V be a vector space over k , and let the cyclic group $\mathbb{Z}/p\mathbb{Z}$ act on its p -th tensor power $V^{\otimes p}$ by cyclic permutation. Then the map $\varphi : V \rightarrow V^{\otimes p}$, $v \mapsto v^{\otimes p}$ sends V into the kernel of either of the differentials d_+ , d_- of the standard complex (3.3) and induces an isomorphism*

$$V^{(1)} \rightarrow \check{H}_i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

both for odd and even degrees i .

Proof. The map φ is compatible with the multiplication by scalars, and its image is σ -invariant, so that it indeed sends V into the kernel of either of the differentials $d_-, d_+ : V^{\otimes p} \rightarrow V^{\otimes p}$. We claim that it is additive “modulo $\text{Im } d_{\pm}$ ”, and that it induces an isomorphism $V^{(1)} \cong \text{Ker } d_{\pm} / \text{Im } d_{\mp}$. Indeed, choose a basis in V , so that $V \cong k[S]$, the k -linear span of a set S . Then $V^{\otimes p} = k[S^p]$ decomposes as $k[S^p] = k[S] \oplus k[S^p \setminus \Delta]$, where $S \cong \Delta \subset S^p$ is the diagonal. This decomposition is compatible with the differentials d_{\pm} , which actually vanish on the first summand $k[S]$. The map φ , accordingly, decomposes as $\varphi = \varphi_0 \oplus \varphi_1$, $\varphi_0 : V^{(1)} \rightarrow k[S]$, $\varphi_1 : V^{(1)} \rightarrow k[S^p \setminus \Delta]$. The map φ_0 is obviously additive and an isomorphism; therefore it suffices to prove that the second summand of (3.3) is acyclic. Indeed, since the $\mathbb{Z}/p\mathbb{Z}$ -action on $S^p \setminus \Delta$ is free, we have $\check{H}^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k[S^p \setminus \Delta]) = 0$. \square

DEFINITION 4.2. A *quasi-Frobenius map* for an associative unital algebra A over k is a $\mathbb{Z}/p\mathbb{Z}$ -equivariant algebra map $F : A^{(1)} \rightarrow A^{\otimes p}$ which induces the isomorphism $\check{H}^{\bullet}(\mathbb{Z}/p\mathbb{Z}, A^{(1)}) \rightarrow \check{H}^{\bullet}(\mathbb{Z}/p\mathbb{Z}, A^{\otimes p})$ of Lemma 4.1.

Here the $\mathbb{Z}/p\mathbb{Z}$ -action on $A^{(1)}$ is trivial, and the algebra structure on $A^{\otimes p}$ is the obvious one (all the p factors commute). We note that since $\check{H}^i(\mathbb{Z}/p\mathbb{Z}, k) \cong k$ for every i , we have $\check{H}^i(\mathbb{Z}/p\mathbb{Z}, A^{(1)}) \cong A^{(1)}$, so that a quasi-Frobenius map must be injective. Moreover, since the Tate homology $\check{H}^{\bullet}(\mathbb{Z}/p\mathbb{Z}, A^{\otimes p}/A^{(1)})$ vanishes, the cokernel of a quasi-Frobenius map must be a free $k[\mathbb{Z}/p\mathbb{Z}]$ -module.

In this Section, we will construct a Cartier isomorphism (4.1) for algebras which admit a quasi-Frobenius map (and satisfy some additional assumptions). In the interest of full disclosure, we remark right away that quasi-Frobenius maps are very rare—in fact, we know only two examples:

- (i) A is the tensor algebra $T^\bullet V$ of a k -vector space V —it suffices to give F on the generators, where it exists by Lemma 4.1.
- (ii) $A = k[G]$ is the group algebra of a (discrete) group G —a quasi-Frobenius map F is induced by the diagonal embedding $G \subset G^p$.

However, the general construction of the Cartier map given in Section 5 will be essentially the same—it is only the notion of a quasi-Frobenius map that we will modify.

PROPOSITION 4.3. *Assume given an algebra A over k equipped with a quasi-Frobenius map $F : A^{(1)} \rightarrow A^{\otimes p}$, and assume that the category A -bimod of A -bimodules has finite homological dimension. Then there exists a canonical isomorphism*

$$\varphi : HH_\bullet(A)((u)) \cong HP_\bullet(A).$$

Proof. Consider the functors $i, \pi : \Lambda_p \rightarrow \Lambda$ and the restrictions

$$\pi^* A_\#^{(1)}, i^* A_\# \in \text{Fun}(\Lambda_p, k).$$

For any $[n] \in \Lambda_p$, the quasi-Frobenius map $F : A^{(1)} \rightarrow A^{\otimes p}$ induces a map

$$F^{\otimes n} : \pi^* A_\#^{(1)}([n]) = (A^{(1)})^{\otimes n} \rightarrow i^* A_\#([n]) = A^{\otimes pn}.$$

By the definition of a quasi-Frobenius map, these maps commute with the action of the maps $\tau : [n] \rightarrow [n]$ and $m_i^p : [n+1] \rightarrow [n]$, $0 \leq i < n$ (recall that $m_i^p = m_i^{\otimes p}$). Moreover, since $m_0^p \circ \tau = m_{n+1}^p$, $F^{\otimes \bullet}$ also commutes with m_n^p . All in all, the collection of the tensor power maps $F^{\otimes \bullet}$ gives a map $F_\# : \pi^* A_\#^{(1)} \rightarrow i^* A_\#$ of objects in $\text{Fun}(\Lambda_p, k)$. We denote by Φ the induced map

$$\Phi = HP_\bullet(F_\#) : HP_\bullet(\pi^* A_\#^{(1)}) \rightarrow HP_\bullet(\Lambda_p, i^* A_\#).$$

By Lemma 2.5, the right-hand side is precisely $HP_\bullet(A)$. As for the left-hand side, we note that σ is trivial on $\pi^* A_\#([n])$ for every $[n] \in \Lambda_p$; therefore the odd horizontal differentials

$$\begin{aligned} \text{id} + \tau + \dots + \tau^{p^n-1} &= (\text{id} + \tau + \dots + \tau^{n-1}) \circ (\text{id} + \sigma + \dots + \sigma^{p-1}) \\ &= p(\text{id} + \tau + \dots + \tau^{n-1}) = 0 \end{aligned}$$

in (2.10) vanish, and we have

$$HP_\bullet(\pi^* A_\#^{(1)}) \cong HH_\bullet(A^{(1)})(u).$$

Finally, to show that Φ is an isomorphism, we recall that the quasi-Frobenius map F is injective, and its cokernel is a free $k[\mathbb{Z}/p\mathbb{Z}]$ -module. One deduces easily that the same is true for each tensor power $F^{\otimes n}$; thus $F_\#$ is injective, and its cokernel $\text{Coker } F_\#$ is such that $\text{Coker } F_\#([n])$ is a free $k[\mathbb{Z}/p\mathbb{Z}]$ -module for any $[n] \in \Lambda_p$. To finish the proof, use the long exact sequence of cohomology and Proposition 3.3. The only thing left to check is that Proposition 3.3 is applicable—namely, that the p -cyclic object $i^* A_\#$ is small in the sense of Definition 3.2.

To do this, we have to show that the bar complex $C_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$ which computes $HH_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$ is effectively finite. It is here that we need to use the assumption of finite homological dimension on the category A -bimod. Indeed, to compute $HH_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$, we can choose any projective resolution P_\bullet^p of the diagonal $A^{\otimes p}$ -bimodule $A^{\otimes p}$. In particular, we can take any projective resolution P_\bullet of the diagonal A -bimodule A , and use its p -th power. To obtain the bar complex $C_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$, one uses the bar resolution $C'_\bullet(A)$. However, all these projective resolutions P_\bullet are chain homotopic to each other, so that the resulting complexes will be also chain homotopic *as complexes of $k[\mathbb{Z}/p\mathbb{Z}]$ -modules*. By assumption, the diagonal A -bimodule A has a projective resolution P_\bullet of finite length; using it gives a complex of finite length which is chain homotopic to $C_\bullet(A^{\otimes p}, A_\sigma^{\otimes p})$, just as required by Definition 3.2. \square

5. Cartier isomorphism in the general case

5.1. Additivization. We now turn to the general case: we assume given a perfect field k of characteristic $p > 0$ and an associative k -algebra A , and we want to construct a Cartier-type isomorphism (4.1) without assuming that A admits a quasi-Frobenius map in the sense of Definition 4.2.

Consider again the non-additive map $\varphi : A \rightarrow A^{\otimes p}$, $a \mapsto a^{\otimes p}$, and let us change the domain of its definition: instead of A , let φ be defined on the k -vector space $k[A]$ spanned by A (where A is considered as a set). Then φ obviously uniquely extends to a k -linear additive map

$$(5.1) \quad \varphi : k[A] \rightarrow A^{\otimes p}.$$

Taking the k -linear span is a functorial operation: setting $V \mapsto k[V]$ defines a functor Span_k from the category of k -vector spaces to itself. The functor Span_k is non-additive, but it has a tautological surjective map $\text{Span}_k \rightarrow \text{Id}$ onto the identity functor, and one can show that Id is the maximal additive quotient of the functor Span_k . If $V = A$ is an algebra, then $\text{Span}_k(A)$ is also an algebra, and the tautological map $\text{Span}_k(A) \rightarrow A$ is an algebra map.

We note that in both examples (i), (ii) in Section 4 where an algebra A did admit a quasi-Frobenius map, what really happened was that the tautological surjective algebra map $\text{Span}_k(A) \rightarrow A$ admitted a splitting $s : A \rightarrow \text{Span}_k(A)$; the quasi-Frobenius map was obtained by composing this splitting map s with the canonical map (5.1).

Unfortunately, in general the projection $\text{Span}_k(A) \rightarrow A$ does not admit a splitting (or at least, it is not clear how to construct one). In the general case, we will modify both sides of the map (5.1) so that splittings will become easier to come by. To do this, we use the general technique of *additivization* of non-additive functors from the category of k -vector spaces to itself.

Consider the small category $\mathcal{V} = k\text{-Vect}^{\text{fg}}$ of finite-dimensional k -vector spaces, and consider the category $\text{Fun}(\mathcal{V}, k)$ of *all* functors from \mathcal{V} to the category $k\text{-Vect}$ of all k -vector spaces. This is an abelian category. The category $\text{Fun}_{\text{add}}(\mathcal{V}, k)$ of all *additive* functors from \mathcal{V} to $k\text{-Vect}$ is also abelian (in fact, an additive functor is completely defined by its value at the one-dimensional vector space k , so that $\text{Fun}_{\text{add}}(\mathcal{V}, k)$ is equivalent to the category of modules over $k \otimes_{\mathbb{Z}} k$). We have the full embedding $\text{Fun}_{\text{add}}(\mathcal{V}, k) \subset \text{Fun}(\mathcal{V}, k)$, and it admits a left-adjoint functor—in other words, for any functor $F \in \text{Fun}(\mathcal{V}, k)$ there is an additive functor F_{add} and a

map $F \rightarrow F_{add}$ which is universal with respect to maps to additive functors. This “universal additive quotient” is not very interesting. For instance, if F is the p -th tensor power functor, $V \mapsto V^{\otimes p}$, then its universal additive quotient is the trivial functor $V \mapsto 0$.

To obtain a useful version of this procedure, we have to consider the derived category $\mathcal{D}(\mathcal{V}, k)$ of the category $\text{Fun}(\mathcal{V}, k)$ and the full subcategory $\mathcal{D}_{add}(\mathcal{V}, k) \subset \mathcal{D}(\mathcal{V}, k)$ spanned by complexes whose homology objects lie in $\text{Fun}_{add}(\mathcal{V}, k)$.

The category $\mathcal{D}_{add}(\mathcal{V}, k)$ is closed under taking cones, thus triangulated (this has to be checked, but this is not difficult), and it contains the derived category of the abelian category $\text{Fun}_{add}(\text{Fun}_k, k)$. However, $\mathcal{D}_{add}(\mathcal{V}, k)$ is much larger than this derived category. In fact, even for the identity functor $\text{Id} \in \text{Fun}_{add}(\mathcal{V}, k) \subset \text{Fun}(\mathcal{V}, k)$, the natural map

$$\text{Ext}_{\text{Fun}_{add}(\mathcal{V}, k)}^i(\text{Id}, \text{Id}) \rightarrow \text{Ext}_{\text{Fun}(\mathcal{V}, k)}^i(\text{Id}, \text{Id})$$

is an isomorphism only in degrees 0 and 1. Already in degree 2, there appear extension classes which cannot be represented by a complex of additive functors.

Nevertheless, it turns out that just as for abelian categories, the full embedding $\mathcal{D}_{add}(\mathcal{V}, k) \subset \mathcal{D}(\mathcal{V}, k)$ admits a left-adjoint functor. We call it the *additivization functor* and denote by $\text{Add}_\bullet : \mathcal{D}(\mathcal{V}, k) \rightarrow \mathcal{D}_{add}(\mathcal{V}, k)$. For any $F \in \text{Fun}(\mathcal{V}, k)$, $\text{Add}_\bullet(F)$ is a complex of functors from \mathcal{V} to k with additive homology functors.

The construction of the additivization Add_\bullet is relatively technical; we will not reproduce it here and refer the reader to [Kal08, Section 3]. The end result is that first, additivization exists, and second, it can be represented explicitly, by a very elegant “cube construction” introduced fifty years ago by Eilenberg and MacLane. Namely, to any functor $F \in \text{Fun}(\mathcal{V}, k)$ one associates a complex $Q_\bullet(F)$ of functors from \mathcal{V} to k such that the homology of this complex consists of additive functors, and we have an explicit map $F \rightarrow Q_\bullet(F)$ which descends to a universal map in the derived category $\mathcal{D}(\mathcal{V}, k)$. In fact, the complex $Q_\bullet(F)$ is concentrated in non-negative homological degrees, and $Q_0(F)$ simply coincides with F , so that the universal map is the tautological embedding $F = Q_0(F) \rightarrow Q_\bullet(F)$. Moreover, assume that the functor F is *multiplicative* in the following sense: for any $V, W \in \mathcal{V}$, we have a map

$$F(V) \otimes F(W) \rightarrow F(V \otimes W),$$

and these maps are functorial and associative in the obvious sense. Then the complex $Q_\bullet(F)$ is also multiplicative. In particular, if we are given a multiplicative functor F and an associative algebra A , then $F(A)$ is an associative algebra; in this case, $Q_\bullet(F)$ is an associative DG algebra concentrated in non-negative degrees.

5.2. Generalized Cartier map. Consider again the canonical map (5.1). There are two non-additive functors involved: the k -linear span functor $V \mapsto k[V]$, and the p -th tensor power functor $V \mapsto V^{\otimes p}$. Both are multiplicative. We will denote by $Q_\bullet(V)$ the additivization of the k -linear span, and we will denote by $P_\bullet(V)$ the additivization of the p -th tensor power. Since additivization is functorial, the map (5.1) gives a map

$$\varphi : Q_\bullet(V) \rightarrow P_\bullet(V)$$

for any finite-dimensional k -vector space V ; if $A = V$ is an associative algebra, then $Q_\bullet(A)$ and $P_\bullet(A)$ are associative DG algebras, and φ is a DG algebra map. We will need several small refinements of this construction.

- (i) We extend both Q_\bullet and P_\bullet to arbitrary vector spaces and arbitrary algebras by taking the limit over all finite-dimensional subspaces.
- (ii) The p -th power $V^{\otimes p}$ carries the permutation action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$, and the map (5.1) is $\mathbb{Z}/p\mathbb{Z}$ -invariant; by the functoriality of the additivization, $P_\bullet(V)$ also carries an action of $\mathbb{Z}/p\mathbb{Z}$, and the map φ is $\mathbb{Z}/p\mathbb{Z}$ -invariant.
- (iii) The map (5.1), while not additive, respects the multiplication by scalars, up to a Frobenius twist; unfortunately, the additivization procedure ignores this. From now on, we will assume that the perfect field k is actually finite, so that the group k^* of scalars is a finite group whose order is coprime to p . Then k^* acts naturally on $k[V]$, hence also on $Q_\bullet(V)$, and the map φ factors through the space $\overline{Q}_\bullet(V) = Q_\bullet(V)_{k^*}$ of covariants with respect to k^* .

The end result: in the case of a general algebra A , our replacement for a quasi-Frobenius map is the canonical map

$$(5.2) \quad \varphi : \overline{Q}_\bullet(A)^{(1)} \rightarrow P_\bullet(A),$$

which is a $\mathbb{Z}/p\mathbb{Z}$ -invariant DG algebra map. We can now repeat the procedure of Section 4 replacing a quasi-Frobenius map F with this canonical map φ . This gives a canonical map

$$(5.3) \quad \Phi : HH_\bullet(\overline{Q}_\bullet(A)_\#)^{(1)}(u) \rightarrow HP_\bullet(P_\bullet(A)_\#),$$

where $\overline{Q}_\bullet(A)_\#$ in the left-hand side is a complex of cyclic objects, and $P_\bullet(A)_\#$ in the right-hand side is the complex of p -cyclic objects. There is one choice to be made because both complexes are infinite; we agree to interpret the total complex which computes $HP_\bullet(E_\bullet)$ and $HH_\bullet(E_\bullet)$ for an infinite complex E_\bullet of cyclic or p -cyclic objects as the *sum*, not the *product* of the corresponding complexes for the individual terms $HP_\bullet(E_i)$, $HH_\bullet(E_i)$.

To understand what (5.3) has to do with the Cartier map (4.1), we need some information on the structure of DG algebras $P_\bullet(A)$ and $\overline{Q}_\bullet(A)$.

The DG algebra $P_\bullet(A)$ has the following structure: $P_0(A)$ is isomorphic to the p -th tensor power $A^{\otimes p}$ of the algebra A , and all the higher terms $P_i(A)$, $i \geq 1$ are of the form $A^{\otimes p} \otimes W_i$, where W_i is a certain representation of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. The only thing that will matter to us is that all the representations W_i are *free* $k[\mathbb{Z}/p\mathbb{Z}]$ -modules. Consequently, $P_i(A)$ is free over $k[\mathbb{Z}/p\mathbb{Z}]$ for all $i \geq 1$. For the proofs, we refer the reader to [Kal08, Subsection 4.1]. As a corollary, we see that if A is such that $A\text{-bimod}$ has finite homological dimension, then we can apply Proposition 3.3 to all the higher terms in the complex $P_\bullet(A)_\#$ and deduce that the right-hand of (5.3) is actually isomorphic to $HP_\bullet(A)$:

$$HP_\bullet(P_\bullet(A)_\#) \cong HP_\bullet(A).$$

We note that it is here that it matters how we define the periodic cyclic homology of an infinite complex (the complex $P_\bullet(A)$ is actually acyclic, so, were we to take the product and not the sum of individual terms, the result would be 0, not $HP_\bullet(A)$).

The structure of the DG algebra \overline{Q}_\bullet is more interesting. As it turns out, the homology $H_i(\overline{Q}_\bullet(A))$ of this DG algebra in degree i is isomorphic to $A \otimes \text{St}(k)_i$, where $\text{St}(k)_\bullet$ is the dual to the *Steenrod algebra* known in Algebraic Topology—more precisely, $\text{St}(k)_i^*$ is the algebra of stable cohomology operations with coefficients in k

of degree i . The proof of this is contained in [Kal08, Section 3]; [Kal08, Subsection 3.1] contains a semi-informal discussion of why this should be so, and what is the topological interpretation of all the constructions in this Section. The topological part of the story is quite large and well-developed—among other things, it includes the notions of *Topological Hochschild Homology* and *Topological Cyclic Homology* which have been the focus of much attention in Algebraic Topology in the last fifteen years. A reader who really wants to understand what is going on should definitely consult the sources, some of which are indicated in [Kal08]. However, within the scope of the present lectures, we will leave this subject completely alone. The only topological fact that we will need is the following description of the Steenrod algebra in low degrees:

$$(5.4) \quad \text{St}_i(k) = \begin{cases} k, & i = 0, 1, \\ 0, & 1 < i < 2p - 2. \end{cases}$$

The proof can be easily found in any algebraic topology textbook.

Thus in particular, the 0-th homology of $\overline{Q}_\bullet(A)$ is isomorphic to A itself, so that we have an augmentation map $\overline{Q}_\bullet(A) \rightarrow A$ (this is actually induced by the tautological map $Q_0(A) = k[A] \rightarrow A$). However, there is also non-trivial homology in higher degrees. Because of this, the left-hand side of (5.3) is larger than the left-hand side of (4.1), and the canonical map Φ of (5.3) has no chance of being an isomorphism (for a topological interpretation of the left-hand side of (5.3), see [Kal08, Subsection 3.1]).

In order to get an isomorphism (4.1), we have to resort to splittings again, and it would seem that we gained nothing, since splitting the projection $\overline{Q}_\bullet(A) \rightarrow A$ is the same as splitting the projection $\overline{Q}_0(A) = k[A]_{k^*} \rightarrow A$. Fortunately, in the world of DG algebras we can get away with something less than a full splitting map. We note the following obvious fact: any *quasi-isomorphism* $f : A_\bullet \rightarrow B_\bullet$ of DG algebras induces an isomorphism $HH_\bullet(A_\bullet) \rightarrow HH_\bullet(B_\bullet)$ of their Hochschild homology. Because of this, it suffices to split the projection $\overline{Q}_\bullet(A) \rightarrow A$ “up to a quasi-isomorphism”. More precisely, we introduce the following.

DEFINITION 5.1. A *DG splitting* $\langle \overline{A}_\bullet, s \rangle$ of a DG algebra map $f : \tilde{A}_\bullet \rightarrow A_\bullet$ is a pair of a DG algebra \overline{A}_\bullet and a DG algebra map $s : \overline{A}_\bullet \rightarrow A_\bullet$ such that the composition $f \circ s : \overline{A}_\bullet \rightarrow A_\bullet$ is a quasi-isomorphism.

LEMMA 5.2. *Assume that the associative algebra A is such that A -bimod has finite homological dimension. For any DG splitting $\langle A_\bullet, s \rangle$ of the projection $\overline{Q}_\bullet(A) \rightarrow A$, the composition map*

$$\begin{aligned} \Phi \circ s : HH_\bullet(A)^{(1)}((u)) &\cong HH_\bullet(A_\bullet)^{(1)}((u)) \rightarrow \\ &\rightarrow HH_\bullet(\overline{Q}_\bullet(A))^{(1)}((u)) \rightarrow HP_\bullet(P_\bullet(A)_\#) \cong HP_\bullet(A) \end{aligned}$$

is an isomorphism in all degrees.

The proof is not completely trivial but very straightforward; we leave it as an exercise (or see [Kal08, Subsection 4.1]). By virtue of this lemma, all we have to do to construct a Cartier-type isomorphism (4.1) is to find a DG splitting of the projection $\overline{Q}_\bullet(A) \rightarrow A$.

5.3. DG splittings. To construct DG splittings, we use obstruction theory for DG algebras, which turns out to be pretty much parallel to the usual obstruction theory for associative algebras. A skeleton theory sufficient for our purposes is given in [Kal08, Subsection 4.3]. Here are the main points.

- (i) Given a DG algebra A_\bullet and a DG A_\bullet -bimodule M_\bullet , one defines *Hochschild cohomology* $HH_{\mathcal{D}}^\bullet(A_\bullet, M_\bullet)$ as

$$HH_{\mathcal{D}}^\bullet(A_\bullet, M_\bullet) = \text{Ext}_{\mathcal{D}}^\bullet(A_\bullet, M_\bullet),$$

where A_\bullet in the right-hand side is the diagonal A_\bullet -bimodule, and $\text{Ext}_{\mathcal{D}}^\bullet$ are the spaces of maps in the “triangulated category of A_\bullet -bimodules”—that is, the derived category of the abelian category of DG A_\bullet -bimodules localized with respect to quasi-isomorphisms. Explicitly, $HH^\bullet(A_\bullet, M_\bullet)$ can be computed by using the bar resolution of the diagonal bimodule A_\bullet . This gives a complex with terms $\text{Hom}(A_\bullet^{\otimes n}, M_\bullet)$, where $n \geq 0$ is a non-negative integer, and a certain differential $\delta : \text{Hom}(A_\bullet^{\otimes n}, M_\bullet) \rightarrow \text{Hom}(A_\bullet^{\otimes n+1}, M_\bullet)$; the groups $HH^\bullet(A_\bullet, M_\bullet)$ are computed by the total complex of the bicomplex

$$M_\bullet \xrightarrow{\delta} \text{Hom}(A_\bullet, M_\bullet) \xrightarrow{\delta} \dots \xrightarrow{\delta} \text{Hom}(A_\bullet^{\otimes \dots}, M_\bullet) \xrightarrow{\delta} \dots$$

- (ii) By a *square-zero extension* of a DG algebra A_\bullet by a DG A_\bullet -bimodule we understand a DG algebra \widetilde{A}_\bullet equipped with a surjective map $\widetilde{A}_\bullet \rightarrow A_\bullet$ whose kernel is identified with M_\bullet (in particular, the induced \widetilde{A}_\bullet -bimodule structure on the kernel factors through the map $\widetilde{A}_\bullet \rightarrow A_\bullet$). Then square-zero extensions are classified up to a quasi-isomorphism by elements in the Hochschild cohomology group $HH_{\mathcal{D}}^2(A_\bullet, M_\bullet)$. A square-zero extension admits a DG splitting if and only if the corresponding class in $HH_{\mathcal{D}}^2(A_\bullet, M_\bullet)$ is trivial.

To apply this machinery to the augmentation map $\overline{Q}_\bullet(A) \rightarrow A$, we consider the *canonical filtration* $\overline{Q}_\bullet(A)_{\geq \bullet}$ on $\overline{Q}_\bullet(A)$ defined, as usual, by

$$\overline{Q}_i(A)_{\geq j} = \begin{cases} 0, & i \leq j, \\ \text{Ker } d, & i = j + 1, \\ \overline{Q}_i(A), & i > j + 1, \end{cases}$$

where d is the differential in the complex $\overline{Q}_\bullet(A)$. We denote the quotients by $\overline{Q}_\bullet(A)_{\leq j}(A) = \overline{Q}_\bullet(A) / \overline{Q}_\bullet(A)_{\geq j}$, and we note that for any $j \geq 1$, $\overline{Q}_\bullet(A)_{\leq j}$ is a square-zero extension of $\overline{Q}_\bullet(A)_{\leq j-1}$ by a DG bimodule quasi-isomorphic to $A \otimes \text{St}_j(k)[j]$ (here $\text{St}_j(k)$ is the corresponding term of the dual Steenrod algebra, and $[j]$ means the degree shift). We use induction on j and construct a collection $\langle A_\bullet^j, s \rangle$ of compatible DG splittings of the surjections $\overline{Q}_\bullet(A)_{\leq j} \rightarrow A$. There are three steps.

Step 1. For $j = 0$, there is nothing to do: the projection $\overline{Q}_\bullet(A)_{\leq 0} \rightarrow A$ is a quasi-isomorphism.

Step 2. For $j = 1$, it turns out that the projection $\overline{Q}_\bullet(A)_{\leq 1} \rightarrow A$ admits a DG splitting if and only if the k -algebra A admits a lifting to a flat algebra \widetilde{A} over the ring $W_2(k)$ of second Witt vectors of the field k . In fact, even more is true: DG splittings are in some sense in a functorial one-to-one correspondence with such liftings; the reader will find precise statements and explicit detailed proofs in [Kal08, Subsection 4.2].

Step 3. We then proceed by induction. Assume given a DG splitting A_\bullet^j , $s : A_\bullet^j \rightarrow \overline{Q}_\bullet(A)_{\leq j}$ of the projection $\overline{Q}_\bullet(A)_{\leq j} \rightarrow A$. Form the “Baer sum” \overline{A}_\bullet^j of the map s with the square-zero extension $p : \overline{Q}_\bullet(A)_{\leq j+1} \rightarrow \overline{Q}_\bullet(A)_{\leq j}$ —that is, let

$$\overline{A}_\bullet^j \subset \overline{Q}_\bullet(A)_{\leq j+1} \oplus A_\bullet^j$$

be the subalgebra obtained as the kernel of the map

$$\overline{Q}_\bullet(A)_{\leq j+1} \oplus A_\bullet^j \xrightarrow{p \oplus (-s)} \overline{Q}_\bullet(A)_{\leq j}.$$

Then \overline{A}_\bullet^j is a square-zero extension of A_\bullet^j by a DG A_\bullet^j -bimodule $\text{Ker } p$ which is quasi-isomorphic to $A \otimes \text{St}_j(k)[j]$. Since A_\bullet^j is quasi-isomorphic to A , these are classified by elements in the Hochschild cohomology group

$$HH^2(A_\bullet^j, \text{Ker } p) \cong HH^{3+j}(A, A) \otimes \text{St}_{j+1}(k).$$

If $j < 2p - 3$, this group is trivial by (5.4), so that a DG splitting A_\bullet^{j+1} exists. In higher degrees, we have to impose conditions on the algebra A . Here is the end result.

PROPOSITION 5.3. *Assume given an associative algebra A over a finite field k of characteristic p such that*

- (i) *A lifts to a flat algebra over the ring $W_2(k)$ of second Witt vectors, and*
- (ii) *A -bimod has finite homological dimension, and moreover, we have that $HH^j(A, A) = 0$ whenever $j \geq 2p$.*

Then there exists a DG splitting A_\bullet , $s : A_\bullet \rightarrow \overline{Q}_\bullet(A)$ of the augmentation map $\overline{Q}_\bullet(A) \rightarrow A$.

Proof. Construct a compatible system of DG splittings A_\bullet^j as described above, and let $A_\bullet = \lim_{\leftarrow} A_\bullet^j$. □

THEOREM 5.4. *Assume given an associative algebra A over a finite field k of characteristic p which satisfies the assumptions (i), (ii) of Proposition 5.3. Then there exists an isomorphism*

$$C^{-1} : HH_\bullet(A)((u))^{(1)} \longrightarrow HP_\bullet(A),$$

as in (4.1).

Proof. Combine Proposition 5.3 and Lemma 5.2. □

This is our generalized Cartier map. We note that the conditions (i), (ii) that we have to impose on the algebra A are completely parallel to the conditions (i), (ii) on page 539 which appear in the commutative case: (i) is literally the same, and as for (ii), note that if A is commutative, then the category of A -bimodules is equivalent to the category of quasicohherent sheaves on $X \times X$, where $X = \text{Spec } A$. By a famous theorem of Serre, this category has finite homological dimension if and only if X is smooth, and this dimension is equal to $\dim(X \times X) = 2 \dim X$.

6. Applications to Hodge Theory

To finish the paper, we return to the original problem discussed in the Introduction: the degeneration of the Hodge-to-de Rham spectral sequence. On the surface of it, Theorem 5.4 is strong enough so that one can apply the method of Deligne and Illusie in the non-commutative setting. However, it has one fault. While in the commutative case we are dealing with an *algebraic variety* X , Theorem 5.4 is only valid for an associative *algebra*. In particular, were we to try to deduce the classical Cartier isomorphism (1.2) from Theorem 5.4, we would only get it for affine algebraic varieties. In itself, it might not be completely meaningless. However, the commutative Hodge-to-de Rham degeneration is only true for a *smooth* and *proper* algebraic variety X —and a variety of dimension ≥ 1 cannot be proper and affine at the same time. The general non-commutative degeneration statement also requires some versions of properness, and in the affine setting, this reduces to requiring that the algebra A is finite-dimensional over the base field. A degeneration statement for such algebras, while not as completely trivial as its commutative version, is not, nevertheless, very exciting.

Fortunately, the way out of this difficulty has been known for some time; roughly speaking, one should pass to the level of *derived categories*—after which all varieties, commutative and non-commutative, proper or not, become essentially affine.

More precisely, one first notices that Hochschild homology of an associative algebra A is *Morita-invariant*—that is, if B is a different associative algebra such that the category $B\text{-mod}$ of B -modules is equivalent to the category $A\text{-mod}$ of A -modules, then $HH_{\bullet}(A) \cong HH_{\bullet}(B)$. The same is true for cyclic and periodic cyclic homology, and for Hochschild cohomology $HH^*(A)$. In fact, B. Keller has shown in [Kel99] how to construct $HC_{\bullet}(A)$ and $HH_{\bullet}(A)$ starting directly from the abelian category $A\text{-mod}$, without using the algebra A at all.

Moreover, Morita-invariance holds on the level of derived categories: if there exists a left-exact functor $F : A\text{-mod} \rightarrow B\text{-mod}$ such that its derived functor is an equivalence of the derived categories $\mathcal{D}(A\text{-mod}) \cong \mathcal{D}(B\text{-mod})$, then $HH_{\bullet}(A) \cong HH_{\bullet}(B)$, and the same is true for $HC_{\bullet}(-)$, $HP_{\bullet}(-)$, and $HH^*(-)$.

Unfortunately, one cannot recover $HH_{\bullet}(A)$ and other homological invariants directly from the derived category $\mathcal{D}(A\text{-mod})$ considered as a triangulated category—the notion of a triangulated category is too weak. One has to fix some “enhancement” of the triangulated category structure. At present, it is not clear what is the most convenient choice among several competing approaches. In practice, however, every “natural” way to construct a triangulated category \mathcal{D} also allows one to equip it with all possible enhancements, so that the Hochschild homology $HH_{\bullet}(\mathcal{D})$ and other homological invariants can be defined.

As long as we work over a fixed field, probably the most convenient of those “natural” ways is provided by the DG algebra techniques. For every associative DG algebra A^{\bullet} over a field k , one defines $HH_{\bullet}(A^{\bullet})$, $HC_{\bullet}(A^{\bullet})$, $HP_{\bullet}(A^{\bullet})$, and $HH^*(A^{\bullet})$ in the obvious way, and one shows that if two DG algebras A^{\bullet} , B^{\bullet} have equivalent triangulated categories $\mathcal{D}(A^{\bullet}\text{-mod})$, $\mathcal{D}(B^{\bullet}\text{-mod})$ of DG modules, then all their homological invariants such as $HH_{\bullet}(-)$ are isomorphic. Moreover, the DG algebra approach is versatile enough to cover the case of non-affine schemes. Namely, one can show that for every quasiprojective variety X over a field k , there

exists a DG algebra A^\bullet over k such that $\mathcal{D}(A^\bullet\text{-mod})$ is equivalent to the derived category of coherent sheaves on X . Then $HH_\bullet(A^\bullet)$ is the same as the Hochschild homology of the category of coherent sheaves on X , and the same is true for the other homological invariants—in particular, if X is smooth, we have

$$HH_i(A^\bullet) \cong \bigoplus_j H^j(X, \Omega_X^{i+j}),$$

and $HC_\bullet(A^\bullet)$ is similarly expressed in terms of the de Rham cohomology groups of X . It is in this sense that all the varieties become affine in the “derived non-commutative” world. We note that in general, although X is the usual commutative algebraic variety, one cannot ensure that the algebra A^\bullet which appears in this construction is also commutative.

Thus for our statement on the Hodge-to-de Rham degeneration, we use the language of associative DG algebras. The formalism we use is mostly due to B. Toën; the reader will find a good overview in [TV05, Section 2], and also in B. Keller’s talk [Kel06] at ICM Madrid.

DEFINITION 6.1. Assume given a DG algebra A^\bullet over a field k .

- (i) A^\bullet is *compact* if it is perfect as a complex of k -vector spaces.
- (ii) A^\bullet is *smooth* if it is perfect as the diagonal DG bimodule over itself.

By definition, a DG B^\bullet -module M_\bullet over a DG algebra B^\bullet is perfect if it is a compact object of the triangulated category $\mathcal{D}(B^\bullet)$ in the sense of category theory—that is, we have

$$\text{Hom}(M_\bullet, \varinjlim N_\bullet) = \varinjlim \text{Hom}(M_\bullet, N_\bullet)$$

for any filtered inductive system $N_\bullet \in \mathcal{D}(B^\bullet)$. It is an easy exercise to check that compact objects in the category $k\text{-Vect}$ are precisely the finite-dimensional vector spaces, so that a complex of k -vector spaces is perfect if and only if its homology is trivial outside of a finite range of degrees, and all the non-trivial homology groups are finite-dimensional k -vector spaces. In general, there is a theorem which says that a DG module M_\bullet is perfect if and only if it is a retract—that is, the image of a projector—of a DG module M'_\bullet which becomes a free finitely-generated B^\bullet -module if we forget the differential. We refer the reader to [TV05] for exact statements and proofs. We note only that if a DG algebra A^\bullet describes an algebraic variety X —that is, $\mathcal{D}(A^\bullet) \cong \mathcal{D}(X)$ —that A^\bullet is compact if and only if X is proper, and A^\bullet is smooth if and only if X is smooth (for smoothness, one uses Serre’s Theorem mentioned in the end of Section 5).

THEOREM 6.2. Assume given an associative DG algebra A^\bullet over a field K of characteristic 0. Assume that A^\bullet is smooth and compact. Moreover, assume that A^\bullet is concentrated in non-negative degrees. Then the Hodge-to-de Rham spectral sequence

$$HH_\bullet(A^\bullet)[u] \Rightarrow HC_\bullet(A^\bullet)$$

of (1.5) degenerates at first term.

In this theorem, we have to require that A^\bullet is concentrated in non-negative degrees. This is unfortunate but inevitable in our approach to the Cartier map, which in the end boils down to Lemma 4.1—whose statement is obviously incompatible with any grading one might wish to put on the vector space V . Thus our construction of the Cartier isomorphism does not work at all for DG algebras. We

circumvent this difficulty by passing from DG algebras to *cosimplicial* algebras—that is, associative algebras $\mathcal{A} \in \text{Fun}(\Delta, K)$ in the tensor category $\text{Fun}(\Delta, K)$ —for which one can construct the Cartier map “pointwise” (it is the passage from DG to cosimplicial algebras which forces us to require $A^i = 0$ for negative i). This occupies the larger part of [Kal08, Subsection 5.2], to which we refer the reader. Here we will only quote the end result.

PROPOSITION 6.3. *Assume given a smooth and compact DG algebra A^\bullet over a finite field k of characteristic $p = \text{char } k$. Assume that A^\bullet is concentrated in non-negative degrees, and that, moreover,*

- (i) A^\bullet can be lifted to a flat DG algebra over the ring $W_2(k)$ of second Witt vectors of the field k , and
- (ii) $HH^i(A, A) = 0$ when $i \geq 2p$.

Then there exists an isomorphism

$$C^{-1} : HH_\bullet(A^\bullet)((u)) \cong HP_\bullet(A^\bullet),$$

and the Hodge-to-de Rham spectral sequence (1.5) for the DG algebra A^\bullet degenerates at first term.

As in the commutative case of [DI87], degeneration follows immediately from the existence of the Cartier isomorphism C^{-1} for dimension reasons. The construction of the map C^{-1} essentially repeats what we did in Section 5 in the framework of cosimplicial algebras, with a lot of technical nuisance because of the need to ensure the convergence of various spectral sequences, see [Kal08, Subsection 5.3]. To deduce Theorem 6.2, one uses the standard technique of the reduction to positive characteristic, just as in the commutative case; this is made possible by the following beautiful theorem due to B. Toën [Toë08].

THEOREM 6.4 ([Toë08]). *Assume given a smooth and compact DG algebra A^\bullet over a field K . Then there exists a finitely generated subring $R \subset K$ and a DG algebra A_R^\bullet , smooth and compact over R , such that $A^\bullet \cong A^\bullet \otimes_R K$.*

We note that this result does not require the algebra A^\bullet to be concentrated in non-negative degrees. We expect that neither does our Theorem 6.2, but so far, we could not prove it—the technical difficulties seem to be much too severe.

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