MONODROMY AND RATIONAL CURVES ON HOLOMORPHIC SYMPLECTIC VARIETIES

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Let X be an irreducible holomorphic symplectic manifold, i.e., a simply-connected Kähler manifold admitting a nondegenerate holomorphic two-form, unique up to scalar. It is known that $\mathrm{H}^2(X,\mathbb{Z})$ carries a canonical nondegenerate integral quadratic form $(,)$, called the Beauville-Bogomolov form [1]; duality yields an induced $\mathbb{Q}$-valued form on $\mathrm{H}_2(X,\mathbb{Z})$. Let $\mathrm{N}_1(X,\mathbb{Z})$ and $\mathrm{N}^1(X,\mathbb{Z})$ denote curve and divisor classes on X, up to homological equivalence; the monoids of effective classes are denoted $\mathrm{NE}_1(X,\mathbb{Z})$ and $\mathrm{NE}^1(X,\mathbb{Z})$ respectively. The associated closed cones are denoted $\overline{\mathrm{NE}}_1(X) \subset \mathrm{H}_2(X,\mathbb{R})$ and $\overline{\mathrm{NE}}^1(X) \subset \mathrm{H}^2(X,\mathbb{R})$.

Assume that X is deformation equivalent to $S^{[2]}$. Here we have [1, §6]

$$\mathrm{H}^2(S^{[2]},\mathbb{Z})_{(,)} = \mathrm{H}^2(S,\mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta, \quad (\delta,\delta) = -2,$$

where $\mathrm{H}^2(S,\mathbb{Z})$ is endowed with its intersection form and 2δ is the divisor class of the locus of nonreduced subschemes (i.e., the exceptional divisor over the diagonal of the symmetric square). The dual form can be expressed

$$\mathrm{H}_2(S^{[2]},\mathbb{Z})_{(,)} = \mathrm{H}_2(S,\mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta^{\vee}, \quad (\delta^{\vee},\delta^{\vee}) = -1/2.$$

In this paper, we prove conjectures on the cones of effective and ample divisors on X first stated in [6], modulo a conjectural version of the Torelli theorem (Conjecture 9). This completes the intersection-theoretic description of the nef cone proposed there. In [7] we proved that the nef cone was at least as large as conjectured; here we show it is no larger.

Theorem 1. Conditional on Conjecture 9 *Let X be as above with Kähler class κ. Each $C \in \mathrm{N}_1(X,\mathbb{Z})$ with $\kappa \cdot C > 0$ and $-5/2 \leq (C,C) < 0$ is contained in $\mathrm{NE}_1(X,\mathbb{Z})$; furthermore, we have*

$$\mathrm{NE}_1(X,\mathbb{Q}) \supset \{C \in \mathrm{N}_1(X,\mathbb{Z}) : \kappa \cdot C > 0, (C,C) \geq -5/2\}.$$

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Each $D \in \mathrm{N}^1(X,\mathbb{Z})$ with $(\kappa, D) > 0$ and $(D,D) \geq -2$ is effective; we have

$$\mathrm{NE}^1(X,\mathbb{Z}) \supset \{D \in \mathrm{N}^1(X,\mathbb{Z}) : (\kappa, D) > 0, (D,D) \geq -2\}.$$

We use the notation $\langle \cdots \rangle$ to denote the closed cone generated by the enclosed classes. Combining with the main result of [7] and Proposition 5, we obtain:

Corollary 2. Conditional on Conjecture 9 *Let (X, g) be a polarized variety, deformation equivalent to the Hilbert scheme of length-two subschemes of a K3 surface. Then we have*

$$\overline{\mathrm{NE}}_1(X) = \langle C \in \mathrm{N}_1(X,\mathbb{Z}) : g \cdot C > 0, (C,C) \geq -5/2 \rangle$$

and

$$\overline{\mathrm{NE}}^1(X) = \left\langle D \in \mathrm{N}^1(X,\mathbb{Z}) : (g, D) > 0, (D,D) \geq -2 \right\rangle.$$

The proof in [7] relied on a detailed analysis of intersection properties of extremal rays associated with birational contractions of X.

In this paper, the main ingredient is the monodromy representation on the Mukai lattice of a K3 surface. We establish the compatibility of this action with the cones of effective curves and divisors.

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1. Basic results on effective classes

If $D \in \mathrm{N}^1(X,\mathbb{Z})$ satisfies $(D,D) > 0$ then D or $-D$ is big [8, 3.10] and thus in the effective cone. More precisely, we have:

Proposition 3. *Suppose X is deformation equivalent to $S^{[r]}$, the Hilbert scheme of length-r subschemes of a K3 surface S. Let κ denote a Kähler class on X. Each divisor D on X satisfying $(D,D) > 0$ and $(D,\kappa) > 0$ is contained in $\mathrm{NE}^1(X,\mathbb{Z})$. A curve class C on X with $(C,C) > 0$ and $C \cdot \kappa > 0$ is contained in $\mathrm{NE}_1(X,\mathbb{Q})$.*

Proof. We first prove the statement on divisors. Let X' be a small deformation of X such that $\mathrm{N}^1(X,\mathbb{Q}) = \mathbb{Q}D$, which is necessarily ample [8, 3.12]. The Riemann-Roch formula takes the form

$$\chi(\mathcal{O}(D)) = \binom{(D,D)/2 + r + 1}{r};$$

indeed, Fujiki showed [8, 1.11] that the Euler characteristic is a polynomial in the Beauville-Bogomolov form, with coefficients that may be evaluated by direct computation. Kodaira vanishing gives

$$h^0(\mathcal{O}_{X'}(D)) = \chi(\mathcal{O}_{X'}(D)) = \binom{(D,D)/2+r+1}{r} > 0,$$

and semicontinuity implies that $h^0(\mathcal{O}_X(D)) \geq h^0(\mathcal{O}_X(D)) > 0$, i.e., D is effective.

We turn to the statement on curve classes. Using the embedding associated with the Beauville-Bogomolov form

$$\iota : \mathrm{H}^2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}_2(X,\mathbb{Z}),$$

we obtain a divisor D such that $\iota(D) = NC$ for some positive integer N. Repeating the deformation argument above, we obtain a complete-intersection curve

$$C' = H_1 \cap \ldots \cap H_{2r-1} \subset X',$$

where the H_i are general hyperplanes of a projective embedding of X'. The class of C' is nonzero, and thus a positive rational multiple of the class of C. We specialize C' back to an effective curve class in X, which remains proportional to C. □

We are grateful to E. Markman for drawing our attention to the following result, which combines [4, Prop. 1.4] and [3, Thm. 4.5]:

Proposition 4. *Let (X, g) be a polarized irreducible holomorphic symplectic manifold. Suppose that E is an irreducible divisor on X with $(E, E) < 0$. Then there exists a smooth irreducible holomorphic symplectic variety X' birational to X such that the corresponding divisor $E' \subset X'$ is contractible.*

The analysis of divisorial contractions in the proof of [7, Thm. 22] implies:

Proposition 5. *Retain the notation of Proposition 4 and assume that X is deformation equivalent to $S^{[2]}$, where S is a K3 surface. Then*

$$(E, E) = -2 \text{ or } -8,$$

where in the second case $E = 2e$ for some $e \in \mathrm{N}^1(X, \mathbb{Z})$.

Remark 6. In a recent letter, Markman proposed a generalization of this result to deformations of $S^{[r]}$ for $r > 2$.

2. Preliminaries on monodromy and deformation theory

Fix a compact complex manifold X. Given a connected complex manifold B with base point $b \in B$ and a proper family of complex manifolds $f : \mathcal{X} \to B$ with $f^{-1}(b) = X$, we have the monodromy representation

$$\rho_B : \pi(B, b) \to \mathrm{Aut}(\mathrm{H}^*(X, \mathbb{Z})).$$

The *monodromy group* $\mathrm{Mon}(X)$ is the subgroup of $\mathrm{Aut}(\mathrm{H}^*(X, \mathbb{Z}))$ generated by the images of the ρ_B, taken over all families $\mathcal{X} \to B$ as described above; let $\mathrm{Mon}^r(X)$ denote its projection onto $\mathrm{Aut}(\mathrm{H}^r(X, \mathbb{Z}))$.

Let S be a K3 surface and

$$\mathrm{H}^*(S, \mathbb{Z}) = \mathrm{H}^0(S, \mathbb{Z}) \oplus \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathrm{H}^4(S, \mathbb{Z})$$

its Mukai lattice, with associated pairing

$$((r_1, L_1, s_1), (r_2, L_2, s_2)) = -r_1 s_2 + L_1 \cdot L_2 - r_2 s_1.$$

Let $X = S^{[n]}$ for $n > 1$ and $v = (1, 0, n-1) \in \mathrm{H}^*(S, \mathbb{Z})$ the Mukai vector associated with $S^{[n]}$; we have an isomorphism of Hodge structures [12]

$$\mathrm{H}^2(S^{[n]}) = v^\perp.$$

Let Γ_v denote the automorphisms of the Mukai *lattice* fixing v; there is an induced representation

$$\gamma : \Gamma_v \to \mathrm{Aut}(\mathrm{H}^2(S^{[n]}, \mathbb{Z})).$$

There exists a character [11, §4.1]

$$\chi_{cov} : \mathrm{Aut}(\mathrm{H}^*(S, \mathbb{Z})) \to \mu_2$$

arising from a choice of orientation on a maximal positive-definite subspace of the Mukai lattice. We use this to twist γ [11, Eqn. 14]:

$$\begin{array}{rcl} \mu : \Gamma_v & \to & \mathrm{Aut}(\mathrm{H}^2(S^{[n]}, \mathbb{Z})) \\ g & \mapsto & \gamma(g)\chi_{cov}(g) \end{array}$$

Theorem 7. [11, Theorem 1.6,Corollary 1.8] *We have $\mu(\Gamma_v) \subset \mathrm{Mon}^2(X)$. Moreover, $\mu(\Gamma_v)$ contains the group generated by reflections*

$$\mathcal{W} := \left\langle \rho_u : u \in \mathrm{H}^2(S^{[n]}, \mathbb{Z}), (u, u) = \pm 2 \right\rangle \subset \mathrm{Aut}(\mathrm{H}^2(X, \mathbb{Z})),$$

where

$$\rho_u(v) = \frac{-2}{(u, u)} w + (w, u)\, u.$$

Remark 8. [11, Lemma 8.3] Γ_v corresponds to the automorphisms of $\mathrm{H}^2(X, \mathbb{Z}) = v^\perp$ acting trivially on its discriminant group.

3. Generalities on periods

Let X_0 be an irreducible holomorphic symplectic manifold and

$$\psi_0 : \mathrm{H}^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda$$

a marking, where $(\Lambda, (,))$ is an integral lattice. Let $\mathfrak{M}$ denote the connected moduli stack of marked manifolds containing (X_0, ψ_0) [8, 1.18], with universal family $\mathcal{U} \to \mathfrak{M}$. Generally $\mathfrak{M}$ is non-separated, e.g., at K3 surfaces containing (-2)-curves or at higher-dimensional manifolds admitting symplectic modifications. Note that $\mathfrak{M}$ has nontrivial stack structure at a point (X, ψ) precisely when there exists a nontrivial automorphism $\alpha : X \to X$ with $\psi \circ \alpha^* = \psi$. Since $\mathrm{H}^0(X, \mathcal{T}_X) = 0$ for each irreducible holomorphic symplectic manifold X, the stabilizer groups are finite. Furthermore, $\mathfrak{M}$ is smooth [15, 2]. Let M denote the coarse moduli space of $\mathfrak{M}$, which is also smooth as the stabilizers act trivially on the tangent space. Hodge theory shows that the stabilizer of (X, ψ) acts trivially on

$$\mathrm{H}^1(X, \Omega^1_X) \oplus \mathrm{H}^0(X, \Omega^2_X) \subset \mathrm{H}^2(X, \mathbb{C})$$

and thus trivially on $\mathrm{H}^1(X, \mathcal{T}_X)$ as well.

Let $\mathcal{D}'$ denote the period domain for all marked deformations of (X_0, ϕ_0), which we can realize as a (topological) open subset of the quadric hypersurface $Q \subset \mathbb{P}(\mathrm{H}^2(X_0, \mathbb{C}))$ associated with $(,)$. Let

$$\tilde{\tau} : \mathfrak{M} \to \mathcal{D}', \quad \tau : M \to \mathcal{D}'$$

denote the period map and the induced map on the coarse moduli space, which is a local isomorphism and is surjective [8, Thm. 8.1]. The real Lie group generated by the monodromy consists of automorphisms of the quadratic form $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}, (,))$ preserving the two components of the positive cone and the orientation on the positive-definite part.

Fix a class κ_0 in the positive cone of $\mathrm{H}^1(X_0, \Omega^1_{X_0}) \cap \mathrm{H}^2(X_0, \mathbb{R})$ and write

$$\kappa := (\psi_0 \otimes \mathbb{R})(\kappa_0) \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\mathfrak{M}_\kappa \subset \mathfrak{M}$ denote the open subset consisting of pairs (X, ψ) such that $\psi^{-1}(\kappa)$ contains a Kähler class; let M_κ denote its image in M and τ_κ the associated period map.

Conjecture 9. Assume that κ is very general, e.g., orthogonal to each integral class in Λ. Then the induced period mapping

$$\tau_\kappa : M_\kappa \to \mathcal{D}'$$

is a covering map, hence an isomorphism. In particular, M_κ is separated.

Corollary 10. Conditional on Conjecture 9 *Retain the notation above and suppose that $\mathfrak{M}$ has trivial stabilizer at every point. Then there exists a universal family $f : \mathcal{X}' \to \mathcal{D}'$ of marked holomorphic symplectic manifolds deformation-equivalent to X_0 with Kähler class κ.*

In order to apply Corollary 10 to manifolds deformation-equivalent to Hilbert schemes of K3 surfaces, we require the following result (see [10, ff. Prop 1.9] and [9, Cor. 6.9]):

Lemma 11. *Let X be deformation equivalent to $S^{[r]}$, where S is a K3 surface. Then the automorphism group of X admits a faithful representation in the tangent space to the formal deformation space $\mathrm{H}^1(X, \mathcal{T}_X)$. In particular, the natural homomorphism*

$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^2(X, \mathbb{Z}))$$

is injective.

Proof. Suppose $\iota : X \to X$ is a nontrivial automorphism acting trivially on $\mathrm{H}^2(X, \mathbb{Z})$. Any such automorphism preserves the full hyperkähler structure on X. The argument of [9, §6.3] shows that such automorphisms are stable under deformations and specializations of X. In particular, it follows that the Hilbert scheme $S^{[r]}$ admits such an automorphism. However, any automorphism of $S^{[r]}$ fixing the divisor δ lifts to an automorphism of S, which in this case acts trivially on $\mathrm{H}^2(S, \mathbb{Z})$. This contradicts the Torelli theorem.

To complete the proof, it suffices to show there do not exist automorphisms ι acting trivially on

$$\mathrm{H}^1(X, \mathcal{T}_X) = \mathrm{Hom}(\mathrm{H}^0(X, \Omega^2_X), \mathrm{H}^1(X, \Omega^1_X)).$$

The latter group is the tangent space to the period domain $\mathcal{D}'$ at the period associated to a marking of the cohomology of X. The only automorphisms of $\mathrm{H}^2(X, \mathbb{Z})$ acting trivially on this tangent space are $\pm I$. However, $-I$ cannot arise as an automorphism of X, because it exchanges the connected components of the positive cone in $\mathrm{H}^2(X, \mathbb{R})$. □

Remark 12. The main point of [9] is to show this statement *fails* for generalized Kummer manifolds.

Corollary 13. Conditional on Conjecture 9 *Let $\mathcal{D}'$ be the period domain for irreducible holomorphic symplectic varieties deformation-equivalent to Hilbert schemes of points on a K3 surface. Then there exists a universal family*

$$\mathcal{X}' \to \mathcal{D}'$$

parametrizating manifolds with a fixed very general Kähler class κ.

4. Proof of the Theorem

Assume X is deformation equivalent to $S^{[2]}$ where S is a K3 surface. The part of Theorem 1 addressing classes with positive square follows from Proposition 3. We therefore focus our attention on the the curve classes R with $(R,R)<0$. We exhibit a parameter space for deformations of X admitting an algebraic class $R \in \mathrm{N}_1(X,\mathbb{Z})$ with $(R,R)=\epsilon$ where $\epsilon \in \{-1/2,-2,-5/2\}$.

Fix $R \in \mathrm{H}_2(X,\mathbb{Z})$ with the desired self-intersection and consider $Q_R = Q \cap R^{\perp}$, $\mathcal{D}'_R = Q_R \cap \mathcal{D}'$, and restriction of the universal family $\mathcal{X}'_R \to \mathcal{D}'_R$. Note that $\mathcal{D}'_R$ is a homogeneous space isomorphic to $O(3,20)/O(3) \times O(20)$.

Let $\mathrm{Mon}^2(X)_R \subset \mathrm{Mon}^2(X)$ denote the subgroup fixing R; this acts on $\mathcal{D}'_R$, and the stabilizers at each point are discrete subgroups of a compact group, thus finite. Consequently, the quotient $\mathrm{Mon}^2(X)_R \backslash \mathcal{D}'_R$ exists in the category of analytic spaces.

Lemma 14. *The space $\mathcal{D}_\epsilon = \mathrm{Mon}^2(X)_R \backslash \mathcal{D}'_R$ depends only on $\epsilon = (R,R)$.*

To prove this, it suffices to show that $\mathrm{Mon}^2(X)$ acts transitively on the classes in $\mathrm{H}_2(X,\mathbb{Z})$ with self-intersections $-1/2, -2$, and $-5/2$, respectively. This follows from [6, Prop. 2.3] and Theorem 7. Note that since the discriminant group of $\mathrm{H}^2(X,\mathbb{Z})$ equals $\mathbb{Z}/2\mathbb{Z}$, $\mathrm{Aut}(\mathrm{H}^2(X,\mathbb{Z}))$ coincides with $\mathrm{Mon}^2(X)$. (See [11, Lemma 8.3] for discussion of the last statement.)

The group $\mathrm{Mon}^2(X)_R$ acts on $\mathcal{X}'_R$ as well, via

Lemma 15. *There exists a nonempty open subset $U \subset \mathcal{D}_\epsilon$ such that for $[Y] \in U$ there exists a holomorphic embedding $f : \mathbb{P}^1 \hookrightarrow Y$ with $[\mathbb{P}^1] = R$ and normal bundle*

$$\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \text{ or } \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

This is essentially in [6], but we review the construction for completeness. We first show that $U \neq \emptyset$.

- For $\epsilon = -2$, let $Y = S^{[2]}$ where S is a K3 surface containing a smooth rational curve $\mathbb{P}^1 \subset S$. Take $f : \mathbb{P}^1 \to Y$ to be the morphism $f(p) = p + s$ for $s \in S \setminus \mathbb{P}^1$. The divisor $D \subset Y$ corresponding to subschemes intersecting the rational curve has $(D,D) = -2$ and $D \cdot f(\mathbb{P}^1) = -2$. There is an open subset in D isomorphic to $\mathbb{P}^1 \times V$ for some open $V \subset S$, thus $\mathcal{N}_{f(\mathbb{P}^1)/D} \simeq$

$\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Using the exact sequence

$$0 \to \mathcal{N}_{f(\mathbb{P}^1)/D} \to \mathcal{N}_{f(\mathbb{P}^1)/Y)} \to \mathcal{N}_{D/Y}|f(\mathbb{P}^1) \to 0$$

we find that $\mathcal{N}_f$ takes the desired form.

- For $\epsilon = -1/2$, let $Y = S^{[2]}$ where S is an arbitrary K3 surface. Take $f : \mathbb{P}^1 \to Y$ to be the morphism parametrizing the subschemes corresponding to nonreduced subschemes supported at $s \in S$, which are parametrized by $\mathbb{P}(T_s S)$. As we vary s, these sweep out a divisor D with class 2δ, where $(\delta, \delta) = -2$. Again, D has an open subset isomorphic to $\mathbb{P}^1 \times V$ for some open $V \subset S$, and the fibers have class $R = \delta^\vee$ where $2\delta^\vee = \delta$ (via the inclusion $\mathrm{H}^2(Y, \mathbb{Z}) \subset \mathrm{H}_2(Y, \mathbb{Z})$ arising from duality). Thus we have $(\delta^\vee, \delta^\vee) = -1/2$ and $D \cdot \delta^\vee = -2$. The normal bundle analysis of the previous case still applies, so we conclude that $\mathcal{N}_f$ takes the desired form.
- For $\epsilon = -5/2$, let $Y = S^{[2]}$ where $S \to \mathbb{P}^2$ is a double cover branched along a smooth plane sextic. Thus we obtain an embedding $\mathbb{P}^2 \hookrightarrow S^{[2]}$. Let R be the class of a line $\mathbb{P}^1 \subset \mathbb{P}^2$. If f is the class of the degree-two polarization of S (and the induced divisor on $S^{[2]}$) and $D = 2\delta$ is the diagonal class then $f \cdot R = 2$ and $D \cdot R = 6$; we conclude that $R = f - 3\delta^\vee$ and $(R, R) = -5/2$. We have the exact sequence

$$0 \to \mathcal{N}_{f(\mathbb{P}^1)/\mathbb{P}^2} \to \mathcal{N}_{f(\mathbb{P}^1)/Y} \to \mathcal{N}_{\mathbb{P}^2/Y}|f(\mathbb{P}^1) \to 0$$

and the isomorphism $\mathcal{N}_{\mathbb{P}^2/Y} \simeq \Omega^1_{\mathbb{P}^2}$. Thus we obtain an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{N}_f \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to 0,$$

which is necessarily split.

In each case, deformation theory (see [6, §4] for our specific situation or [13] more generally) shows that we can deform the embedding $f : \mathbb{P}^1 \hookrightarrow Y$ as Y deforms, *provided* the class $R = [f(\mathbb{P}^1)]$ remains of type $(3, 3)$. This completes the proof of the Lemma.

Consider the relative Douady spaces parameterizing rational curves in the prescribed classes over our parameter space. Recall that, over an open subset of each special period domain $\mathcal{D}'_R$, the normal bundles $\mathcal{N}_f$ of the rational curves representing R satisfy $h^1(\mathcal{N}_f) = 1$. We use a theorem of Ran [13, Cor. 3.3] asserting that such rational curves deform without obstructions over the locus in the period domain where these classes remain of Hodge type (see [6, §4.1] for a detailed discussion of our particular situation). Results of Fujiki show these are proper for manifolds bimeromorphic to Kähler manifolds [5].

The proper mapping theorem [14, Satz 23] allows us to specialize rational curves over the whole parameter space. This completes the assertion of Theorem 1 concerning effective *curves*.

When $(R, R) = -1/2$ or -2 the curves produced generally deform in two-parameter families, and thus sweep out divisors in the corresponding holomorphic symplectic manifold. (This is Theorem 4.3 of [6].) This proves the part of Theorem 1 on effective divisors.

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