

Math 465 Lecture Notes

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32 Lecture 32: Applications of projective elimination theory

32.1 Computing the projective elimination ideal

Let $J \subset k[x_0, \dots, x_n, y_1, \dots, y_m]$ be an ideal, homogeneous in x_0, \dots, x_n . How do we compute the projective elimination ideal \hat{J} ?

Method I: By definition,

$$\hat{J} = (J : \mathfrak{m}^N) \cap k[y_1, \dots, y_m]$$

for $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ and $N \gg 0$. For any particular N , we have algorithms for computing the quotient $(J : \mathfrak{m}^N)$.

Method II: For each $i = 0, \dots, n$, let I_i denote the dehomogenization of J with respect to x_i . For each I_i , compute the intersection

$$\hat{I}_i = I_i \cap k[y_1, \dots, y_m].$$

Then the intersection

$$\hat{I} = \hat{I}_0 \cap \dots \cap \hat{I}_n$$

is equal to \hat{J} .

proof of equivalence We sketch a proof for the equality $\hat{I} = \hat{J}$. The direction $\hat{J} \subset \hat{I}$ follows from the definitions. For the reverse inclusion, we recall that if f is homogeneous, g its dehomogenization with respect to x_i , and \hat{f} the rehomogenization, then

$$f = x_i^e \hat{f}$$

for some e . Hence if J is a homogeneous ideal, I_i its dehomogenization with respect to x_i , and J_i the rehomogenization with respect to x_i , then for some N_i

$$J_i = (J : \langle x_i^N \rangle), \text{ for } N \geq N_i.$$

We have

$$\hat{I}_i = I_i \cap k[y_1, \dots, y_m] = J_i \cap k[y_1, \dots, y_m]$$

and

$$\hat{I} = (\bigcap_{i=0}^n J_i) \cap k[y_1, \dots, y_m].$$

On the other hand,

$$\bigcap_{i=0}^n J_i = (J : \mathfrak{m}^N), N \gg 0,$$

so $\hat{I} = \hat{J}$. \square

32.2 Images of projective varieties are universally closed

Theorem 32.1 *Assume k is algebraically closed. Let $X \subset \mathbb{P}^n$ be a projective variety, V an abstract variety, $\phi : X \rightarrow V$ a morphism. Then $\phi(X)$ is closed.*

proof: Let

$$\Gamma_\phi \subset X \times V \subset \mathbb{P}^n \times V$$

be the graph, which is closed. For each affine open $U_i \subset V$, we can realize $U_i \subset k^m$ as a closed subset. The intersection

$$\Gamma_\phi \cap \pi_2^{-1}(U_i) \subset \mathbb{P}^n \times k^m$$

is also closed, and the projective elimination theorem implies

$$\pi_2(\Gamma_\phi \cap \pi_2^{-1}(U_i))$$

is closed, which is the equal to $\pi_2(\Gamma_\phi) \cap U_i = \phi(X) \cap U_i$. Thus the intersection of $\phi(X)$ with each affine open neighborhood is closed in that neighborhood, and $\phi(X)$ is closed in V . \square

Proposition 32.2 *Assume k is algebraically closed. Let $V \subset k^n$ be an affine variety, $\phi : V \rightarrow k^m$ a morphism with graph Γ_ϕ . Let*

$$\overline{\Gamma_\phi} \subset \mathbb{P}^n \times k^m$$

be the projective closure. Then $\pi_2(\overline{\Gamma_\phi}) = \overline{\phi(V)}$.

proof: The Projective Elimination Theorem implies $\pi_2(\overline{\Gamma_\phi})$ is closed, and it contains the image $\phi(V)$, so it also contains $\overline{\phi(V)}$. On the other hand, $\pi_2^{-1}(\overline{\phi(V)})$ contains Γ_ϕ , and also $\overline{\Gamma_\phi}$, so applying ϕ_2 gives the desired result. \square

Remark 32.3 When the image of a morphism of affine varieties $\phi : V \rightarrow k^m$ fails to be closed, the extra points in the closure of the image come from points of V ‘at infinity.’ Symbolically,

$$\overline{\phi(V)} - \phi(V) \subset \pi_2(\overline{\Gamma_\phi} \cap X(x_0)).$$

Let $W \subset k^n \times k^m$ be affine, with projective closure $\overline{W} \subset \mathbb{P}^n \times k^m$. If

$$I(W) \subset k[x_1, \dots, x_n, y_1, \dots, y_m]$$

is the ideal vanishing on W then $J(\overline{W})$ is the homogenization of $I(W)$ with respect to x_0 , i.e., the ideal generated by homogenizations

$$x_0^{\deg_x(f)} f(x_1/x_0, \dots, x_n/x_0, y_1, \dots, y_m), \quad f \in I(W).$$

This is computed by the same algorithm we used in the case where there were no y_i :

Proposition 32.4 *Let $>$ be a monomial order on $k[x_1, \dots, x_n, y_1, \dots, y_m]$ so that $x^\alpha y^\beta > x^\gamma y^\delta$ whenever $|\alpha| > |\gamma|$. If g_1, \dots, g_r is a Gröbner basis of $I(V)$ then the homogenizations of g_1, \dots, g_r generate the homogenization of $I(V)$.*

32.3 Graphs and images of morphisms between projective varieties

Theorem 32.5 *Assume k is algebraically closed. Let $X \subset \mathbb{P}^n$ be a projective variety and $\phi : X \rightarrow \mathbb{P}^m$ a morphism. If*

$$\begin{aligned} I &= \langle y_0 - \phi_0, \dots, y_m - \phi_m, J(X) \rangle \\ J &= I \cap k[y_1, \dots, y_m]. \end{aligned}$$

then $\phi(X) = X(J)$.

proof: Let $C(X) \subset k^{n+1}$ be the *cone* over X , i.e., the subvariety defined by the equations in $J(X)$, or the union of lines $\lambda(x_0, \dots, x_n) \subset k^{n+1}$ for $[x_0, \dots, x_n] \in X$. The polynomials defining ϕ also determine a map of affine varieties

$$\phi' : C(X) \rightarrow k^{m+1},$$

whose graph is defined by the equations in I . It is easy to see that

$$\phi'(C(X)) = C(\phi(X)),$$

which has equations given by the ideal J . \square

Proposition 32.6 *Let $X \subset \mathbb{P}^n$ be projective and $\phi : X \rightarrow \mathbb{P}^m$ a morphism of projective varieties. Then the graph $\Gamma_\phi \subset \mathbb{P}^n \times \mathbb{P}^m$ is given by the bihomogeneous equations*

$$J(\Gamma_\phi) = \langle y_i \phi_j - y_j \phi_i \rangle.$$

Hence over an algebraically closed field, the equations of the image $\phi(X)$ are given by the projective elimination ideal of $J(\Gamma_\phi)$.

proof Let $U_i \subset X$ be the affine open where $\phi_i \neq 0$ and $V_i \subset \mathbb{P}^m$ the affine open where $y_i \neq 0$, so that $\phi_i(U_i) \subset V_i$. On $U_i \times V_i$, $J(\Gamma_\phi)$ dehomogenizes to

$$y_j = \phi_j / \phi_i,$$

the equation of the graph of ϕ . \square