MATH 499 Exercises

November 14, 2007

1. If $f, g \in \mathbb{C}[x]$ are polynomials of positive degree, prove that f and g have a common root in $\mathbb{C}$ if and only if $\operatorname{Res}(f, g, x) = 0$.

   [Hint: Use the fact that $\mathbb{C}$ is algebraically closed; that is, any polynomial $f \in \mathbb{C}[x]$ factors completely into linear factors.]

2. If $f(x) = a_0 x^l + \ldots + a_l \in k[x]$, where $a_0 \neq 0$ and $l > 0$, then the *discriminant* of f is defined to be

$$\operatorname{disc}(f) = \frac{(-1)^{l(l-1)/2}}{a_0} \operatorname{Res}(f, f', x).$$

   Prove that f has a multiple factor (that is, f is divisible by g^2 for some $g \in k[x]$ of positive degree) if and only if $\operatorname{disc}(f) = 0$.

3. Compute the discriminant of the polynomial $f(x) = x^4 + px + q$ for $p, q \in k$.

4. For $f, g \in \mathbb{Z}[x]$, show that $\operatorname{Res}(f, g, x) \in \mathbb{Z}[x]$.

5. Let

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$$

   with $a_m \neq 0$, $b_n \neq 0$. Suppose that the $(m+n) \times (m+n)$ Sylvester matrix has rank $m + n - 1$. Show that f and g share a common linear factor, but not a common quadratic factor.