PROBABILISTIC AVERAGES OF JACOBI OPERATORS

HELGE KRÜGER

Abstract. I study the Lyapunov exponent and the integrated density of states for general Jacobi operators. The main result is that questions about these can be reduced to questions about ergodic Jacobi operators. I use this to show that for finite gap Jacobi operators, regularity implies that they are in the Cesàro–Nevai class, proving a conjecture of Barry Simon. Furthermore, I use this to study Jacobi operators with coefficients $a(n) = 1$ and $b(n) = f(n^\rho \pmod 1)$ for $\rho > 0$ not an integer.

1. Introduction

I study discrete Jacobi operators, which are either given on the whole line $\mathbb{Z}$ or on the half line $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. These are operators given by

$$(Hu)(n) = a(n)u(n+1) + b(n)u(n) + a(n-1)u(n-1), \tag{1.1}$$

where we set $u(0) = 0$ in the half line case. We will always assume that there exists a constant $C_0 > 1$ such that the Jacobi coefficients $a(n)$ and $b(n)$ satisfy

$$\frac{1}{C_0} \leq a(n) \leq C_0, \quad |b(n)| \leq C_0. \tag{1.2}$$

In order to distinguish between the half and whole line case, I will use $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ to denote half line operators and $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ to denote whole line operators.

The main reason to consider both half and whole line operators is that I wish to take limit points in the strong operator topology of the translates $H^{(n)}$ of a half line operator H, which are naturally viewed as whole line operators J. Let us denote by $\omega(H)$ the set of all limit points of translates of H. Then results of Last and Simon tell us that [14]

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{J \in \omega(H)} \sigma(J) \tag{1.3}$$

and [13]

$$\sigma_{\mathrm{ac}}(H) \subseteq \bigcap_{J \in \omega(H)} \sigma_{\mathrm{ac}}(J). \tag{1.4}$$

Here $\sigma(H)$, $\sigma_{\mathrm{ess}}(H)$ and $\sigma_{\mathrm{ac}}(H)$ denote the spectrum, essential and absolutely continuous parts of the spectrum of H, respectively. A recent and striking result by Remling [17] even shows that the last inclusion can be improved, since one only has to take the intersection of the reflectionless parts of the spectrum of $J \in \omega(H)$.

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The results of this paper will not be directly concerned with the spectrum of H, but with more statistical quantities like the Lyapunov exponent, which we now introduce. Define for $z \in \mathbb{C}$, the upper Lyapunov exponent $\overline{L}(z, H)$ of the operator H as

$$\overline{L}(z,H) = \limsup_{N\to\infty} \frac{1}{N} \log \left\| \prod_{n=N}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\|. \tag{1.5}$$

Denote by $\mathcal{Z}$ the set where the upper Lyapunov exponent vanishes

$$\mathcal{Z} = \{E : \quad \overline{L}(E,H) = 0\}. \tag{1.6}$$

It follows from subordinacy theory that

$$\sigma_{\mathrm{ac}}(H) \subseteq \overline{\mathcal{Z}}^{\mathrm{ess}}, \tag{1.7}$$

where $\overline{\mathcal{Z}}^{\mathrm{ess}}$ denotes the essential closure of $\mathcal{Z}$. One might ask whether (1.4) might be strengthened to

$$\overline{\mathcal{Z}}^{\mathrm{ess}} \subseteq \bigcap_{J\in\omega(H)} \sigma_{\mathrm{ac}}(J). \tag{1.8}$$

Unfortunately this is not the case, we will give an example following Theorem 2.1. However, our results will imply that

Theorem 1.1. *There exists $J \in \omega(H)$ such that*

$$\overline{\mathcal{Z}}^{\mathrm{ess}} \subseteq \sigma_{\mathrm{ac}}(J). \tag{1.9}$$

The situation is more precise than described by the last theorem, we will even show that 'most' $J \in \omega(H)$ satisfy (1.9). To make the meaning of 'most' precise, introduce $\mathcal{J}$ as the space of all Jacobi operators $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ satisfying (1.2). $\mathcal{J}$ is a compact metric space, where an explicit example of the metric is

$$d(J,\tilde{J}) = \sum_{n\in Z} \frac{1}{2^{|n|}} \big(|a(n) - \tilde{a}(n)| + |b(n) - \tilde{b}(n)|\big). \tag{1.10}$$

By extending the translates $H^{(n)}$ of H by $a \equiv 1$, $b \equiv 0$ to the left, we can view them as elements of $\mathcal{J}$. If K_j, $j = 1, 2$ are two closed sets that satisfy

$$\forall\varepsilon > 0 : \quad \lim_{N\to\infty} \frac{1}{N} \#\{0 \le n \le N-1 : \quad d(H^{(n)}, K_j) > \varepsilon\} = 0, \tag{1.11}$$

then also their intersection $K_1 \cap K_2$ satisfies (1.11). Hence there exists a smallest closed set Ω satisfying (1.11). We are now ready to state the strengthening of Theorem 1.1.

Theorem 1.2. *Let Ω be the smallest closed set in $\mathcal{J}$ that satisfies*

$$\forall\varepsilon > 0 : \quad \lim_{N\to\infty} \frac{1}{N} \#\{0 \le n \le N-1 : \quad d(H^{(n)}, \Omega) > \varepsilon\} = 0. \tag{1.12}$$

Then we have for $\mathcal{Z}$ defined in (1.6) *that*

$$\overline{\mathcal{Z}}^{\mathrm{ess}} = \bigcap_{J\in\Omega} \sigma_{\mathrm{ac}}(J). \tag{1.13}$$

As with (1.4), we can replace the absolutely continuous spectrum on the right hand side by the reflectionless part. Let me recall at this point that J is reflectionless on Σ if

$$\lim_{\varepsilon\downarrow 0} \mathrm{Re}\langle\delta_0, (J - E - \mathrm{i}\varepsilon)^{-1}\delta_0\rangle = 0 \tag{1.14}$$

for almost every $E \in \Sigma$.

It is easy to see that Theorem 1.2 implies Theorem 1.1. In Theorem 2.1, we will not only obtain the above description of the set $\mathcal{Z}$, but even obtain an explicit formula for the Lyapunov exponent in terms of Lyapunov exponents of limit points. However to do so, it is necessary to replace the upper Lyapunov exponent by Lyapunov exponents along subsequences, where the limit exists, and we will only do so in a later section.

I would call condition (1.12) *convergence of* $\{H^{(n)}\}_{n\geq 1}$ *in probability to* Ω. In his related paper [19], Simon has introduced a similar condition

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} d(H^{(n)}, \Omega) = 0, \tag{1.15}$$

which he called *Cesàro–Nevai condition*. One can show that these two conditions are equivalent, see Lemma 2.5.

Since I already mentioned it, let me discuss the relation of this paper to the one of Simon [19]. Let J be a whole line Jacobi operator with periodic coefficients

$$a(n+p) = a(n), \quad b(n+p) = b(n)$$

for some $p \geq 1$. It then follows from Floquet theory that there exists a set $\mathfrak{e}$ consisting of finitely many intervals

$$\mathfrak{e} = [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_{l+1}, \beta_{l+1}] \tag{1.16}$$

such that

$$\sigma_{\mathrm{ess}}(J) = \sigma_{\mathrm{ac}}(J) = \mathfrak{e}. \tag{1.17}$$

Furthermore, it is known that the set $\mathcal{T}(\mathfrak{e})$ consisting of all Jacobi operators, which obey (1.17) and are reflectionless on $\mathfrak{e}$, is topologically a finite dimensional torus. The main result of Simon can be stated in our notation as

Theorem 1.3 (Simon, [19]). *Let $\mathfrak{e}$ be the spectrum of a periodic Jacobi operator. Let $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be a Jacobi operator that satisfies*

$$\overline{L}(E, H) = 0 \tag{1.18}$$

for almost every $E \in \mathfrak{e}$ and

$$\sigma_{\mathrm{ess}}(H) = \mathfrak{e} \tag{1.19}$$

then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} d(H^{(n)}, \mathcal{T}(\mathfrak{e})) = 0. \tag{1.20}$$

Simon did not use the condition (1.18), but he assumed that H is regular. We will show in Appendix B that these two conditions are equivalent. Furthermore Simon conjectured that Theorem 1.3 holds when the spectrum of the periodic Jacobi operator is replaced by any set of the form (1.16), which are called finite gap sets. We will now proceed to prove this conjecture giving an alternative proof of Theorem 1.3.

Theorem 1.4. *Let $\mathfrak{e}$ be a finite gap set and $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be a Jacobi operator that satisfies*

$$\overline{L}(E, H) = 0 \tag{1.21}$$

for almost every $E \in \mathfrak{e}$ and

$$\sigma_{\text{ess}}(H) = \mathfrak{e}. \tag{1.22}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} d(H^{(n)}, \mathcal{T}(\mathfrak{e})) = 0. \tag{1.23}$$

Proof. Let Ω be the set from Theorem 1.2. Then by Theorem 1.2 and the remark following it, every $J \in \Omega$ is reflectionless on $\mathfrak{e}$. Furthermore by (1.3), we have for $J \in \Omega \subseteq \omega(H)$ that $\sigma_{\text{ess}}(J) \subseteq \mathfrak{e}$. Hence

$$\Omega \subseteq \mathcal{T}(\mathfrak{e}).$$

(1.23) now follows since (1.12) implies (1.15). □

This proof is very similar to the proof of Remling [17] of the Denisov-Rakhmanov Theorem. Furthermore, as Remling's proof, this proof extends to more general sets. An interesting examples are homogeneous Cantor sets as treated by Sodin and Yuditskii [21].

Let me now discuss the problem that motivated me to prove Theorem 1.2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and $0 < \rho < 1$. Consider the half line operator

$$(Hu)(n) = u(n + 1) + u(n - 1) + f(n^\rho \pmod 1)u(n), \tag{1.24}$$

where we set $u(0) \equiv 0$. It was shown by Stolz [23] (see also [12]) that

$$\sigma_{\text{ess}}(H) = [-2 + \min(f), 2 + \max(f)] \tag{1.25}$$

and that the Lyapunov exponent vanishes on

$$[-2 + \max(f), 2 + \min(f)], \tag{1.26}$$

at least for f's that extend to a smooth function on the unit circle $\mathbb{R}\backslash\mathbb{Z}$. By using that $(n+k)^\rho = n^\rho + o(1)$ for k in a bounded set and $n \to \infty$, one can show that the set Ω in (1.12) has to be a set of Jacobi operators J with Jacobi coefficients $a \equiv 1$, $b \equiv t$, where t ranges over $[\min(f), \max(f)]$. This implies that

$$\{E : \quad \overline{L}(E, H) = 0\} = [-2 + \max(f), 2 + \min(f)] \tag{1.27}$$

even for continuous f. Here, we used that the Lyapunov exponent of the Jacobi operator with coefficients $a \equiv 1$ and $b \equiv 0$ vanishes exactly on $[-2, 2]$. This is an intriguing fact, since Remling's Oracle Theorem from [17] implies that the absolutely continuous spectrum of H is empty when $f(1) \neq f(0)$.

The study of this problem is not limited to the case when $0 < \rho < 1$, in fact we will prove results for all $\rho > 0$ that are not integers. In particular we will prove integral formulas similar to the ones of Bourgain [1] and Simon–Zhu [20], which relate the Lyapunov exponent at ρ to the Lyapunov exponent of the operator with coefficients $a(n) = 1$ and $b(n) = f(\alpha n^r)$, where $r = \lfloor \rho \rfloor$ and α ranges over $[0, 1]$. These integral formulas will imply that the Lyapunov exponent only depends on the integral part of ρ. This is remarkable, since the numerical computations of Griniasty and Fishman in [8] showed a dependence on $\rho \in (1, 2)$.

Let me now explain the organization of the paper. In Section 2, I will state the main results of this paper and show how they imply Theorem 1.2. In particular, I introduce in this section what I call a *probabilistic average of a Jacobi operator*. Section 3 contains the main technical part of this paper. Various properties of the Lyapunov exponent are derived and the results stated in Section 2 are proven. In Section 3, we will see the full power of the probabilistic averages, since they will imply the existence of the Lyapunov exponent.

Finally Section 4 states all the results for the Jacobi operators with coefficients $a \equiv 1$ and $b(n) = f(n^\rho \pmod 1)$, which follow from the methods of this paper.

Appendix A provides some background on ergodic Jacobi operators. Appendix B makes the connection between the statement of Theorem 1.3 and the one found in Simon's paper [19].

2. Probabilistic Averages and their Consequences

In this section, we will give rigorous statements of our main results. We recall from the introduction that $\mathcal{J}$ denotes the space of all whole line Jacobi operators $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ whose Jacobi parameters obey (1.2). We denote by S the shift map on $\ell^2(\mathbb{Z})$ given by

$$(Su)(n) = u(n-1) \tag{2.1}$$

and by $\hat{S} : \mathcal{J} \to \mathcal{J}$ the map

$$\hat{S}J = S^* J S. \tag{2.2}$$

$\hat{S}$ is a homeomorphism on the compact metric space $\mathcal{J}$. A probability measure μ on $\mathcal{J}$ is called *shift invariant* if

$$\mu(\hat{S}\Omega) = \mu(\Omega) \tag{2.3}$$

for any measurable $\Omega \subseteq \mathcal{J}$. A shift invariant measure β is called *ergodic* if for any decomposition

$$\Omega = \Omega_1 \cup \Omega_2 \tag{2.4}$$

into measurable sets with $\hat{S}\Omega_j = \Omega_j$, $j = 1, 2$, we have

$$\beta(\Omega_j) \in \{0, 1\}, \quad j = 1, 2. \tag{2.5}$$

Given an ergodic measure β, we can define its Lyapunov exponent $\gamma_\beta(z)$ for all $z \in \mathbb{C}$ and its integrated density of states $k_\beta(E)$ in the usual way. We will give some details on this in Appendix A.

Given a half line Jacobi operator $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, we define its extension $\hat{H}$ to the whole line by setting

$$a(n) = 1, \quad b(n) = 0 \tag{2.6}$$

for $n \leq 0$. We will now use the notation $\hat{S}^n \hat{H}$ for what we called in the introduction $H^{(n)}$. This is to emphasize that we are evaluating along the orbit of a shift on $\mathcal{J}$.

The following steps are modeled after the construction of invariant measures for continuous transformations on compact metric spaces. I recommend Section 6.1 and 6.2 in the book of Walters [25] as an introduction to this method. Further information can also be found in Section 4.1 in the book [11] by Katok and Hasselblatt.

We may define a probability measure $A_{N,H}$ on $\mathcal{J}$ by

$$A_{N,H} = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\hat{S}^n \hat{H}}, \tag{2.7}$$

where δ denotes the Dirac measure. The measure $A_{N,H}$ will not be shift invariant for most H, however its limit points in the weak $*$ topology will be. Now, let μ be such a limit point, that is, there exists a sequence $N_j \to \infty$ such that

$$\mu = \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} \delta_{\hat{S}^n \hat{H}}. \tag{2.8}$$

One can check that μ will be a shift-invariant measure.

It follows from Choquet's Theorem that we may write μ as the integral over ergodic measures β, that is,

$$\mu = \int \beta d\alpha(\beta). \tag{2.9}$$

This is called the *ergodic decomposition* of μ. For a proof and discussion of Choquet's Theorem, I can recommend the monograph [15] by Phelps. Let me remark that one has to understand (2.9) in the sense that

$$\int f d\mu = \int \left(\int f d\beta \right) d\alpha(\beta)$$

holds for all continuous $f : \mathcal{J} \to \mathbb{C}$.

If β is an ergodic measure, then for β almost every $J \in \mathcal{J}$, we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\hat{S}^n J} \tag{2.10}$$

exists and is equal to β. This can be shown by the Birkhoff ergodic theorem, see Theorem 4.4 in the book [7] by Glasner for details.

We furthermore need the notion of capacity. For a compact subset $X \subseteq \mathbb{R}$, we define its capacity $C(X)$ as

$$C(X) = \exp\left(-\min\left(\int_{X\times X} \log|x-y|^{-1} d\sigma(x) d\sigma(y) \right) \right), \tag{2.11}$$

where the minimum is taken over all probability measures σ supported on X. We say that $\mathcal{E}$ is a set of capacity 0 if $C(X) = 0$ for any compact set $X \subseteq \mathcal{E}$. Being a set of capacity 0 implies in particular that it has Hausdorff dimension 0 and therefore also Lebesgue measure 0. See Appendix A of the review [18] of Simon.

Theorem 2.1. *Let $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be a half line Jacobi operator. Assume that the limit*

$$\mu = \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N-1} \delta_{\hat{S}^n \hat{H}} \tag{2.12}$$

exists in the weak $*$ topology and is given by a measure with ergodic decomposition

$$\mu = \int \beta d\alpha(\beta). \tag{2.13}$$

Then there exists a set $\mathcal{E} \subseteq \mathbb{R}$ of capacity 0 such that for $z \in \mathbb{C}\backslash\mathcal{E}$, the Lyapunov exponent

$$L(z, \{N_j\}, H) = \lim_{j\to\infty} \left(\frac{1}{N_j} \log \left\| \prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\| \right) \tag{2.14}$$

exists. Furthermore, we have that

$$L(z, \{N_j\}, H) = \int \gamma_\beta(z) d\alpha(\beta) \tag{2.15}$$

for $z \in \mathbb{C}\backslash\mathcal{E}$.

We will prove this theorem in the next section. Furthermore, we note the following corollary of Theorem 2.1, which follows by combining it with Kotani theory.

Corollary 2.2. *Let μ and $L(E, \{N_j\}, H)$ be as in Theorem 2.1. For μ almost every J, we have that*

$$\overline{\{E : \quad L(E, \{N_j\}, H) = 0\}}^{\mathrm{ess}} \subseteq \sigma_{\mathrm{ac}}(J). \tag{2.16}$$

Furthermore such a J is reflectionless on this set.

Proof. By (2.15) and Fubini, it follows that there exists a set $\mathcal{B}$ of ergodic measures such that $\alpha(\mathcal{B}) = 1$ and, for $\beta \in \mathcal{B}$, we have that

$$\gamma_\beta(E) = 0$$

for almost every $E \in \{z : \quad L(z, \{N_j\}, H) = 0\}$. Hence Kotani's Theorem implies that for β almost every J, we have that the conclusions of the corollary hold. Another application of Fubini finishes the proof. □

The (topological) support $\mathrm{supp}(\mu)$ of a probability measure μ is the smallest closed set $K \subseteq \mathcal{J}$ such that $\mu(K) = 1$. The previous corollary clearly implies that there exists a dense set of J in $\mathrm{supp}(\mu)$ such that

$$\overline{\{E : \quad L(E, \{N_j\}, H) = 0\}}^{\mathrm{ess}} \subseteq \sigma_{\mathrm{ac}}(J)$$

holds. Recall the result of Last–Simon from [13], which says that for $J \in \omega(H)$, we have that

$$\sigma_{\mathrm{ac}}(H) \subseteq \sigma_{\mathrm{ac}}(J).$$

We already used this implicitly in (1.4). These two facts imply

Corollary 2.3. *Let μ and $L(E, \{N_j\}, H)$ be as in Theorem 2.1. We have that*

$$\overline{\{E : \quad L(E, \{N_j\}, H) = 0\}}^{\mathrm{ess}} = \bigcap_{J \in \mathrm{supp}(\mu)} \sigma_{\mathrm{ac}}(J). \tag{2.17}$$

Proof. The discussion above shows $\subseteq$. $\supseteq$ follows from (2.15). □

Let me now give an example which shows that (1.8) does not hold. This means we cannot enlarge $\mathrm{supp}(\mu)$ in the previous formula. Define coefficients $a(n)$ and $b(n)$ by

$$a(n) = 1, \quad b(n) = \begin{cases} 10 & n \in [2^k, 2^k + k] \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

It is easy to compute that the measure μ just assigns weight one to the Jacobi operator with coefficients $a \equiv 1$ and $b \equiv 0$ and thus that $\overline{L}(E, H) = 0$ for $E \in [-2, 2]$. However, the Jacobi operator with coefficients $a \equiv 1$, $b \equiv 10$ is also in $\omega(H)$ and thus

$$\bigcap_{J \in \omega(H)} \sigma_{\mathrm{ac}}(J) = \emptyset.$$

Having discussed the Lypapunov exponent, we will now start giving the analog of Theorem 2.1 for the integrated density of states. Given a half line operator $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, we denote by $H_{[1,N]}$ the restriction of H to $\ell^2(\{1, \dots, N\})$. The integrated density of states describes limits of the quantity

$$\frac{1}{N} \mathrm{tr}(P_{(-\infty,E)}(H_{[1,N]})) \tag{2.19}$$

as $N \to \infty$.

Theorem 2.4. *Let $H : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be a half-line Jacobi operator. Assume that*

$$\mu = \lim_{j \to \infty} \frac{1}{N_j} \sum_{n=0}^{N_j - 1} \delta_{\hat{S}^n \hat{H}} \tag{2.20}$$

exists with ergodic decomposition $\mu = \int \beta d\alpha(\beta)$. Then the limit

$$k(E) = \lim_{j \to \infty} \frac{1}{N_j} \mathrm{tr}(P_{(-\infty,E)}(H_{[1,N_j]})) \tag{2.21}$$

exists and is equal to

$$k(E) = \int k_\beta(E) d\alpha(\beta). \tag{2.22}$$

As Theorem 2.1, we will prove this theorem in the next section.

Theorem 2.1 and Theorem 2.4 reduce the computation of the Lyapunov exponent and integrated density of states of a general Jacobi operator to

(i) Understanding the statistical properties of the Jacobi coefficients in the form of understanding the measure μ from (2.12) and (2.20).
(ii) Understanding the Lyapunov exponent and integrated density of states of general ergodic Jacobi operators.

The second step is certainly an ambitious project, but it is less ambitious than if one starts with a general Jacobi operator.

In order to derive Theorem 1.2 from Theorem 2.1, we will need some further preparations. We will first give a proof of the equivalence of (1.12) and (1.15). This follows from the following lemma with $\gamma_n = d(\hat{S}^n \hat{H}, \Omega)$.

Lemma 2.5. *Let α_n be a bounded sequence of nonnegative numbers. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \gamma_n = 0 \tag{2.23}$$

and

$$\forall \varepsilon > 0 : \quad \lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : \quad \gamma_n > \varepsilon\} = 0 \tag{2.24}$$

are equivalent.

Proof. Choose $C_1 > 0$ such that $\gamma_n \le C_1$ for all n. We then have that

$$\varepsilon \#\{1 \le n \le N : \quad \gamma_n > \varepsilon\} \le \sum_{n=1}^{N} \gamma_n \le C_1 \#\{1 \le n \le N : \quad \gamma_n > \varepsilon\} + \varepsilon N,$$

which implies the claim. □

We have the following theorem.

Theorem 2.6. *Assume that*

$$\mu = \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j - 1} \delta_{\hat{S}^n \hat{H}} \tag{2.25}$$

converges in the weak $*$ topology. Then $\mathrm{supp}(\mu)$ *is the smallest closed set K such that*

$$\forall \varepsilon > 0 : \quad \lim_{j\to\infty} \frac{1}{N_j} \#\{0 \le n \le N_j - 1 : \quad d(\hat{S}^n \hat{H}, K) > \varepsilon\} = 0 \tag{2.26}$$

holds.

For the proof, we will need the following result from measure theory, known as Portmanteau-Theorem (see e.g. [6, Theorem VIII.4.10.]).

Theorem 2.7. *The following three conditions are equivalent.*

(i) *$\mu_n \to \mu$ in the weak $*$ topology.*

(ii) *For every Borel set B with $\mu(\partial B) = 0$, we have that*

$$\lim_{n\to\infty} \mu_n(B) = \mu. \tag{2.27}$$

(iii) *For every open Borel set B*

$$\liminf_{n\to\infty} \mu_n(B) \ge \mu(B). \tag{2.28}$$

We now come to

Proof of Theorem 2.6. Given $K \subseteq \mathcal{J}$ and $\varepsilon > 0$, we define $B_{K,\varepsilon}$ as

$$B_{K,\varepsilon} = \{J : \quad \mathrm{dist}(J, K) < \varepsilon\},$$

which is an open set. One then checks that

$$\begin{aligned} A_{N,H}(B_{K,\varepsilon}) &= \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\hat{S}^n \hat{H}}(B_{K,\varepsilon}) \\ &= \frac{1}{N} \#\{0 \le n \le N - 1 : \quad \mathrm{dist}(\hat{S}^n \hat{H}, K) < \varepsilon\}. \end{aligned} \tag{2.29}$$

Letting $K = B_{\mathrm{supp}(\mu),\varepsilon}$, we see that by (2.27)

$$1 = \mu(K) = \lim_{j\to\infty} A_{N_j,H}(K) = \lim_{j\to\infty} \frac{1}{N_j} \#\{0 \le n \le N_j - 1 : \quad \mathrm{dist}(\hat{S}^n \hat{H}, \mathrm{supp}(\mu)) < \varepsilon\},$$

since $\mu(\partial K) = 0$, so that H satisfies (2.26).

Assume now that H satisfies (2.26) for a closed set $K \subseteq \mathcal{J}$. Then for every $\varepsilon > 0$,

$$\begin{aligned} 1 &= \lim_{j\to\infty} \frac{1}{N_j} \#\{0 \le n \le N_j - 1 : \quad \mathrm{dist}(\hat{S}^n \hat{H}, K) < \varepsilon\} \\ &= \lim_{j\to\infty} A_{N_j,H}(B_{K,\varepsilon}) \ge \mu(B_{K,\varepsilon}), \end{aligned}$$

where we used (2.28). Hence, we see that $\mu(B_{K,\varepsilon}) = 1$ and thus

$$\mu(K) = \mu\Big(\bigcap_{k\ge 1} B_{K,\frac{1}{k}}\Big) = 1.$$

This shows $K \supseteq \mathrm{supp}(\mu)$, finishing the proof of Theorem 2.6. □

Now, we come to

Proof of Theorem 1.2. By Theorem 2.6, we obtain that

$$\Omega = \bigcup_{\mu} \mathrm{supp}(\mu),$$

where the union is taken over all weak $*$ limit points of

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{\hat{S}^n \hat{H}}.$$

The claim now follows from Corollary 2.3. □

3. Studying the Lyapunov exponent

In this section, we will study the Lyapunov exponent for general Jacobi operators. The methods used here are not new and can be found for example in Poltoratski–Remling [16]. However, we will work under the assumption that for a sequence $N_j \to \infty$, we have that

$$\mu = \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j - 1} \delta_{\hat{S}^n \hat{H}} \tag{3.1}$$

exists in the weak $*$ topology. This will simplify the treatment, since it will imply the existence of many limits.

Let me furthermore remark that we will not make the dependence on $\{N_j\}$ and H explicit in this section. So we write for example $L(z) = L(z, \{N_j\}, H)$ in the notation of the previous section.

For a whole line Jacobi operator $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, we denote by J_+ its restriction to $\ell^2(\mathbb{Z}_+)$. For $z \in \mathbb{C}_+ = \{z : \quad \mathrm{Im}(z) > 0\}$, introduce the Weyl–Titchmarsh m function by

$$m_+(z, J) = \langle \delta_1, (J_+ - z)^{-1} \delta_1 \rangle. \tag{3.2}$$

We collect its properties:

Lemma 3.1. *We have that*

$$|m_+(z,J)| \leq \frac{1}{\operatorname{Im}(z)} \tag{3.3}$$

and

$$\operatorname{Im}(m_+(z,J)) \geq \operatorname{Im}(z) \inf_{|t|\leq 3C_0} |t-z|^{-2} > 0. \tag{3.4}$$

Furthermore

$$m_+(z,.) : \mathcal{J} \to \mathbb{C}_+ \tag{3.5}$$

is a continuous map.

Proof. The first two inequalities follow by writing $m_+(z,J) = \int \frac{d\sigma(t)}{t-z}$, where σ is the appropriate spectral measure. The last property follows from the resolvent equation. □

We also introduce

$$a_0 : \mathcal{J} \to \mathbb{R}_+, \quad J \mapsto a(0) \tag{3.6}$$

and note that this also a continuous map.

Let us begin by studying the Lyapunov exponent in the upper half plane $\mathbb{C}_+$. Introduce the Weyl solution $u_+(z,n)$ as the solution of

$$Hu_+(z) = zu_+(z) \tag{3.7}$$

except at $n=1$ which is square summable at $+\infty$. A computation shows that

$$\langle \delta_{n+1}, (H_{\{n+1,\dots\}} - z)^{-1}\delta_{n+1}\rangle = -\frac{u_+(z,n+1)}{a(n)u_+(z,n)}, \tag{3.8}$$

where $H_{\{n+1,\dots\}}$ denotes the restriction of H to $\ell^2(\{n+1,n+2,\dots\})$. It is simple to see that

$$\langle \delta_{n+1}, (H_{\{n+1,\dots\}} - z)^{-1}\delta_{n+1}\rangle = m_+(z,\hat{S}^n\hat{H}), \quad a(n) = a_0(\hat{S}^n\hat{H}). \tag{3.9}$$

We have the following lemma

Lemma 3.2. *Assume* (3.1) *and* $z \in \mathbb{C}_+$, *then the following limit also exists*

$$L_+(z) = \lim_{j\to\infty} \left(-\frac{1}{N_j} \log|u_+(z,N_j)| \right). \tag{3.10}$$

Furthermore, we have that $L_+(z) \geq 0$ *and*

$$L_+(z) = \int_{\mathcal{J}} \Big(\log(a_0(J)^{-1}) - \log|m_+(z,J)| \Big) d\mu(J). \tag{3.11}$$

Proof. Since u_+ is square summable and thus $|u_+(z,n)| \to 0$ as $n\to\infty$, we have that $L_+(z) \geq 0$. By (3.8), we obtain that

$$\begin{aligned} -\frac{1}{N_j}\log|u_+(z,N_j)| &= \frac{1}{N_j}\sum_{n=1}^{N_j-1} \Big(\log(a(n)^{-1}) - \log|m_+(z,n)| \Big) + \frac{\log|u_+(z,0)|}{N_j} \\ &= \frac{1}{N_j}\sum_{n=0}^{N_j-1} \Big(\log(a_0(\hat{S}^n\hat{H})^{-1}) - \log|m_+(z,\hat{S}^n\hat{H})| \Big) + o(1). \end{aligned}$$

The claim follows by (3.1), since $a_0, m_+(z,.) : \mathcal{J} \to \mathbb{C}$ are continuous. □

Introduce the cosine type solution $c(z,n)$ as the solution of $Hc = zc$ with initial conditions $c(z,0) = 0$, $c(z,1) = 1$. A computation shows that

$$c(z,n) = \frac{\det(z - H_{[1,n-1]})}{\prod_{j=1}^{n-1} a(j)}, \tag{3.12}$$

which implies by Cramer's rule that

$$m_c(z,n) = \langle \delta_n, (H_{[1,n]} - z)^{-1} \delta_n \rangle \tag{3.13}$$

satisfies

$$\frac{c(z,n+1)}{c(z,n)} = -\frac{1}{a(n) m_c(z,n)}. \tag{3.14}$$

We are now ready to show

Lemma 3.3. *Assume* (3.1) *and* $z \in \mathbb{C}_+$*, then the following limit also exists*

$$L_c(z) = \lim_{j\to\infty} \left(\frac{1}{N_j} \log |c(z, N_j)| \right) = \lim_{j\to\infty} \left(\frac{1}{N_j} \log |c(z, N_j + 1)| \right). \tag{3.15}$$

Furthermore, we have that $L_c(z) = L_+(z)$.

Proof. We prove the claim for $c(z, N_j)$. The proof for $c(z, N_j + 1)$ is similar. The Wronskian

$$W_n(u_+, c) = a(n)(u_+(n+1)c(n) - u_+(n)c(n+1))$$

is constant in n. We may rewrite it as

$$W_n(u_+, c) = u_+(n)c(n) \left(\frac{1}{m_c(z,n)} - m_+(z,n) \right).$$

The claim now follows from the fact that

$$\kappa^{-1} \leq \left| \frac{1}{m_c(z,n)} - m_+(z,n) \right| \leq \kappa$$

for some $\kappa > 1$ independent of n, which one obtains from Lemma 3.1, which is also valid for $m_c(z,n)$. ☐

Define the sine type solution $s(z,n)$ as the solution of $Hs = zs$ satisfying $s(z,0) = 1$ and $s(z,1) = 0$. One can easily check that the above lemma also applies to the sine type solution. Using this, we obtain.

Proposition 3.4. *Assume* (3.1) *and* $z \in \mathbb{C}_+$*, then the following limit also exists*

$$L(z) = \lim_{j\to\infty} \left(\frac{1}{N_j} \log \left\| \prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\| \right). \tag{3.16}$$

Furthermore, we have that $L(z) = L_c(z) = L_+(z)$.

Proof. One can check that

$$\prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} = \begin{pmatrix} c(z, N_j + 1) & s(z, N_j + 1) \\ a(N_j)c(z, N_j) & a(N_j)s(z, N_j) \end{pmatrix}$$

and thus for some $\tilde{C} > 1$

$$\begin{aligned}
&\frac{1}{\tilde{C}} \left\| \prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\| \\
&\leq \max(|c(z, N_j + 1)|, |c(z, N_j)|, |s(z, N_j)|, |s(z, N_j + 1|) \\
&\leq \tilde{C} \left\| \prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\|
\end{aligned}$$

which implies the claim. □

We have now achieved the intermediate goal of a good understanding of the Lyapunov exponent in the upper half plane. The next step is to show that the Lyapunov exponent on the real axis can be studied by taking limits from the upper half plane. To do this, the representation using the cosine type solution will be the most useful.

For the next lemma, we need the following result from potential theory, which can be found as Theorem A.7 in [18]. We will write in the following that a property holds for quasi every $z \in \mathbb{R}$ if there exists a set $\mathcal{E}$ of capacity 0 such that the property holds for $z \in \mathbb{R} \backslash \mathcal{E}$.

Theorem 3.5. *Let ν_n be a sequence of probability measures that converge weakly to a probability measure ν. Assume furthermore that all these measures have their support contained in a fixed compact set $K \subseteq \mathbb{R}$. Then*

$$\limsup_{n\to\infty} \int \log|z - t| d\nu_n(t) \leq \int \log|z - t| d\nu(t) \tag{3.17}$$

for every $z \in \mathbb{C}$ and equality holds for quasi every $z \in \mathbb{R}$.

Define a sequence of measures ν_N by

$$\nu_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{E_n([1,N])}, \tag{3.18}$$

where $E_n([1, N])$ denotes an enumeration of the eigenvalues $H_{[1,N]}$. We furthermore define A as

$$A = \lim_{j\to\infty} \left(\prod_{n=1}^{N_j} a(n) \right)^{1/N_j}, \tag{3.19}$$

which exists if (3.1) is assumed. Now, we are ready for

Lemma 3.6. *Assume* (3.1)*, then*

$$\nu = \lim_{j\to\infty} \nu_{N_j} \tag{3.20}$$

exists, and for $\mathrm{Im}(z) > 0$*,*

$$\lim_{j\to\infty} \frac{1}{N_j} \log|c(z, N_j)| = \log(A^{-1}) + \int \log|t - z| d\nu. \tag{3.21}$$

Furthermore, (3.21) *even holds for quasi every $z \in \mathbb{R}$.*

Proof. By (3.12), we have that

$$\frac{1}{N}\log c(z,N) = \int \log|t-z| d\nu_N(t) - \frac{1}{N}\sum_{n=1}^{N}\log(a(n))$$

for $z \in \mathbb{C}_+$. We have already seen that the limit

$$\log(A^{-1}) = \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=1}^{N_j}\log(a(n)^{-1})$$

exists. Hence the limit

$$\lim_{j\to\infty}\int \log|t-z| d\nu_{N_j}$$

also exists for every $z \in \mathbb{C}_+$ by Lemma 3.3. Now (3.20) follows the fact that the family of functions $t \mapsto \log|t-z|$ for $\mathrm{Im}(z) > 0$ separates points on the real axis. The last statement follows from the previous theorem. □

Since

$$\frac{1}{N}\mathrm{tr}(P_{(-\infty,E)}(H_{[1,N]}) = \nu_N((-\infty,E)) \tag{3.22}$$

we have that (3.20) implies the first part of Theorem 2.4. We will need Theorem 1.1. from [4]:

Theorem 3.7. *If f is a subharmonic function, then*

$$f(z) = \lim_{r\to 0}\frac{1}{\pi r^2}\int_{|\zeta - z|\leq r} f(\zeta)d\zeta, \tag{3.23}$$

and if f is submean, then

$$f(z) \leq \liminf_{r\to 0}\frac{1}{\pi r^2}\int_{|\zeta - z|\leq r} f(\zeta)d\zeta, \tag{3.24}$$

Now, we come to

Lemma 3.8. *For every $E \in \mathbb{R}$, we have that*

$$L(E) \leq \log(A^{-1}) + \int \log|t-E| d\nu \tag{3.25}$$

Proof. First observe that $L(z)$ is a submean function of z, and $z \mapsto \log(A^{-1}) + \int \log|t-z|d\nu$ is subharmonic. This implies the claim by the previous theorem and that equality holds for all $z \notin \mathbb{R}$ by Proposition 3.4 and (3.11). □

We are now ready for

Proposition 3.9. *Assume* (3.1). *Then there exists a set $\mathcal{E} \subseteq \mathbb{R}$ of capacity* 0 *such that the following holds.*

(i) *For $E \in \mathbb{R}\backslash\mathcal{E}$, the limit*

$$L(E) = \lim_{j\to\infty}\left(\frac{1}{N_j}\log\left\|\prod_{n=N_j}^{1}\frac{1}{a(n)}\begin{pmatrix} E-b(n) & -1 \\ a(n)^2 & 0\end{pmatrix}\right\|\right) \tag{3.26}$$

exists.

(ii) *For $E \in \mathbb{R}\backslash\mathcal{E}$, we have that*

$$L(E) = \lim_{\varepsilon\to 0} L(E+i\varepsilon). \tag{3.27}$$

(iii) *For $E \in \mathbb{R}\backslash\mathcal{E}$, the limit*

$$\lim_{j\to\infty} \frac{1}{N_j} \log|c(z, N_j)| \tag{3.28}$$

exists and is equal to $L(E)$.

Proof. We observe

$$|c(E, N_j)| \le \left\| \prod_{n=N_j}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\|$$

for $E \in \mathbb{R}$, which implies

$$\limsup_{j\to\infty} \frac{1}{N_j} \log|c(E, N_j)| \le L(E).$$

Conversely we have by (3.21) and (3.25) for quasi every E that

$$L(E) \le \limsup_{j\to\infty} \frac{1}{N_j} \log|c(E, N_j)|.$$

Combining these two inequalities finishes the proof. □

We clearly have that Proposition 3.4 and 3.9 imply the first part of Theorem 2.1. It remains to see the integral representations in both Theorem 2.1 and Theorem 2.4.

In order to so, we first observe that

Proposition 3.10. *Let $\mathcal{E}$ be as in the last proposition, then for $z \in \mathbb{C}\backslash\mathcal{E}$, we have that*

$$L(z) = \log(A^{-1}) + \int \log|t - z| d\nu(t). \tag{3.29}$$

The above formula is known as *Thouless formula*.

Proof. This follows from part (iii) of the last proposition and (3.21). □

The validity of Thouless formula in both our setting and the ergodic setting implies that we only need to prove the integral representation in Theorem 2.1. We do this now

Proof of Theorem 2.1. We only have to show (2.15). However, this follows from (2.13) and (3.11) for $z \in \mathbb{C}_+$. Now, (ii) of Proposition 3.9 and (A.9) imply that (2.15) even holds for $z \in \mathbb{R}\backslash\mathcal{E}$. This finishes the proof. □

4. The family of potentials

We now continue the discussion of the type of operator described at the end of the introduction. Introduce for a continuous function $f : [0, 1] \to \mathbb{R}$ and $r < \rho < r + 1$, where r is a nonnegative integer, the sequences

$$a(n) = 1, \quad b(n) = f(n^\rho \pmod 1) \tag{4.1}$$

for $n \ge 1$. Denote by H the associated Jacobi operator. Operators of this type were discussed in [12]. In particular it was shown that

$$\sigma_{\text{ess}}(H) = [-2 + \min(f), 2 + \max(f)]. \tag{4.2}$$

For $\alpha \in [0,1]\backslash\mathbb{Q}$, introduce the skew-shift $T_\alpha : [0,1]^r \to [0,1]^r$ by

$$(T_\alpha \omega)_k = \begin{cases} \omega_0 + \alpha & k = 0 \\ \omega_k + \omega_{k-1} & 1 \le k \le r-1. \end{cases} \tag{4.3}$$

It is a known fact that the maps $T_\alpha : [0,1]^r \to [0,1]^r$ are ergodic with respect to the Lebesgue measure when α is irrational. Similarly as in the last part of Appendix A we let β_α be the measure on $\mathcal{J}$ given by the pushforward of the Lebesgue measure on $[0,1]^r$ under

$$[0,1]^r \ni \omega \mapsto \{1, f((T_\alpha^n \omega)_r)\}_{n\in\mathbb{Z}} \in \mathcal{J}. \tag{4.4}$$

We can compute the measure μ from (2.12) explicitly.

Proposition 4.1. *We have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\hat{S}^n \hat{H}} = \int_0^1 \beta_\alpha d\alpha \tag{4.5}$$

in the weak $*$ topology.

We need a little bit of preparation for the proof of this proposition. For $\Lambda \subseteq \mathbb{Z}$, denote by J_Λ the restriction of J to $\ell^2(\Lambda)$. We will call a continuous function $f : \mathcal{J} \to \mathbb{R}$ compactly supported, if there is a finite set $\Lambda \subseteq \mathbb{Z}$ such that

$$f(J) = f(\tilde{J}), \tag{4.6}$$

whenever $J_\Lambda = \tilde{J}_\Lambda$.

Lemma 4.2. *For μ_n, μ probability measures on $\mathcal{J}$, we have*

$$\mu_n \to \mu \tag{4.7}$$

in the weak $*$ topology if and only if for every compactly supported f,

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu. \tag{4.8}$$

Proof. Clearly, weak $*$ convergence implies (4.8). To see the converse, observe that if f is continuous, it is also uniformly continuous as a continuous function on a compact space. So we see that for any $\varepsilon > 0$, we can find an $N > 1$ such that $|f(J) - f(\tilde{J})| \le \varepsilon$ whenever $J_{[-N,N]} = \tilde{J}_{[-N,N]}$. This implies the claim by the triangle inequality. □

Now, we come to the

Proof of Proposition 4.1. The first step of the proof is to reduce the question to a question about the sequence n^ρ (mod 1). By Lemma 4.2 it suffices to check convergence for compactly supported functions g. Since every such g will be a continuous function of $\{b(n)\}_{n=-K}^{K}$ for some $K \ge 1$ it suffices to check that these have the proper distribution. Furthermore, by continuity of f it suffices to show the following: Let $h : [0,1)^{2K+1} \to \mathbb{R}$ be a continuous function. Then the limit of

$$\frac{1}{N} \sum_{n=1}^{N} h(\{(n+j)^\rho\}_{j=0}^{2K+1})$$

as $N \to \infty$ is equal

$$\int_0^1 \int_{[0,1]^r} h(\{T_\alpha^j \omega\}_{j=0}^{2K+1}) d\omega d\alpha.$$

Furthermore, by Lemma 2.1 in [12] and an easy argument we see that both $(n+j)^\rho$ and $(T_\alpha^{n+j}\omega)_r$ are essentially given by degree r polynomials, and thus uniquely determined by

$$\{(n+j)^\rho\}_{j=0}^r \quad \text{and} \quad \{(T_\alpha^{n+j}\omega)_r\}_{j=0}^r.$$

Now Lemma 2.3 in [12] implies that the coefficients of the first polynomial are uniformly distributed. A quick computation shows the same for the skew-shift, finishing the proof. □

This proposition combined with Theorem 2.1 implies the following corollary.

Corollary 4.3. *For almost every E, we have that*

$$\overline{L}(E,H) = \int_0^1 \gamma_{\beta_\alpha}(E)d\alpha. \tag{4.9}$$

This corollary resolves the discrepancy between the numerical and perturbation theoretical computations in [8] and shows in particular that the Lyapunov exponent only depends on the integer part of ρ.

Formulas for the Lyapunov exponent of the above type were first observed by Simon and Zhu in [20] for continuum Schrödinger operators. The case $1 < \rho < 2$ was noted by Bourgain in [1]. We observe further spectral properties in the following result.

Theorem 4.4. *For $0 < \rho < 1$, we have that*

(i) ***Stolz:*** *If f extends to a smooth function on the circle, then J has purely absolutely continuous spectrum in $[-2+\max(f), 2+\min(f)]$.*
(ii) ***Remling's Oracle:*** *If $f(0) \neq f(1)$, then the absolutely continuous spectrum of J is empty.*

Proof. Part (i) is [23]. Part (ii) follows from Remling's Oracle Theorem, which is found in [17]. □

Let us now say some things about the case $1 < \rho < 2$. Then the integral formula for the Lyapunov exponent has on the right hand side an integral over the Lyapunov exponent of quasi-periodic Schrödinger operators. A particular and well understood example is the Almost–Mathieu operator, where

$$f(x) = 2\lambda\cos(2\pi x) \tag{4.10}$$

for some $\lambda > 0$. Let us denote by $\gamma_\alpha(z)$ for $\alpha \in [0,1]\backslash\mathbb{Q}$ the Lyapunov exponent of the operator H_α with coefficients

$$a(n) \equiv 1, \quad b(n) = 2\lambda\cos(2\pi n\alpha). \tag{4.11}$$

Combining results of Herman [10] and Bourgain and Jitomirskaya [2], we have that

Theorem 4.5. *Let α be irrational, then*

$$\gamma_\alpha(z) \geq \max(\log(\lambda), 0) \tag{4.12}$$

with equality exactly when $z \in \sigma(H_\alpha)$.

Furthermore, the following is a consequence of Theorem 2.2 in [9].

Theorem 4.6. *Given $\delta > 0$ there are constants $\kappa_1, \kappa_2 > 0$ and $\lambda_0 > 0$. Let $0 < \lambda < \lambda_0$ and $E \in [-2+\delta, -\delta] \cup [\delta, 2-\delta]$. Then there is a set of α of measure $\kappa_1 \lambda$ such that for each of these α, we have*

$$\sigma(H_\alpha) \cap [E - \lambda\kappa_2, E + \lambda\kappa_2] = \emptyset. \tag{4.13}$$

Combining these two results with (4.9) and the Combes–Thomas estimate, it can be shown that

$$\overline{L}(H, E) \gtrsim \lambda^2 \tag{4.14}$$

for $E \in [-2+\delta, -\delta] \cup [\delta, 2-\delta]$. It is remarkable that this is the same small λ behaviour as in the random case.

Furthermore, it is known that we always have $0 \in \sigma(H_\alpha)$. Hence, we also have by (4.9) that

$$\overline{L}(H, 0) = 0. \tag{4.15}$$

This shows that the set of E above is optimal.

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Appendix A. Families of ergodic Schrödinger operators

This section is a collection of a few things about ergodic Schrödinger operators, which can be found for example in [5] or Section 7 of [18].

Let in the following β be an ergodic measure on $\mathcal{J}$ as defined in Section 2. There is a set $\Sigma(\beta)$ such that

$$\Sigma(\beta) = \sigma(J) \tag{A.1}$$

for β almost every J. We may define the Lyapunov exponent by

$$\gamma_\beta(z) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathcal{J}} \log \left\| \prod_{n=N}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\| d\beta(J). \tag{A.2}$$

This limit is well defined, since the sequence

$$\frac{1}{N} \int_{\mathcal{J}} \log \left\| \prod_{n=N}^{1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix} \right\| d\beta(J)$$

in N is subadditive. Furthermore, we define the integrated density of states by

$$k_\beta(E) = \int_{\mathcal{J}} \langle \delta_0, \chi_{(-\infty,E)}(J)\delta_0 \rangle d\beta(J). \tag{A.3}$$

We note that this quantity is equal to

$$k_\beta(E) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathcal{J}} \mathrm{tr}(P_{(-\infty,E)}(J_{[0,N-1]})) d\beta(J). \tag{A.4}$$

We will need the following result of Kotani theory.

Theorem A.1. *Denote by* $\overline{\mathcal{Z}}^{\mathrm{ess}}$ *the essential closure of the set*

$$\mathcal{Z} = \{E \in \mathbb{R} : \quad \gamma_\beta(E) = 0\} \tag{A.5}$$

then β *almost every* J *has purely absolutely continuous spectrum on* $\mathcal{Z}$ *and is reflectionless there.*

We define

$$\log(A_\beta) = \int_{\mathcal{J}} \log(a_0(J)) d\beta(J). \tag{A.6}$$

Lemma A.2. *We have that*

$$\gamma_\beta(z) = \log(A_\beta^{-1}) - \int_{\mathcal{P}} \log|m_+(z, J)| d\beta(J) \tag{A.7}$$

$$= \log(A_\beta^{-1}) + \int \log|t - z| dk_\beta(z) \tag{A.8}$$

for every $z \in \mathbb{C}$.

(A.8) is known as the *Thouless formula*. It implies that

$$\lim_{\varepsilon\to 0} \gamma_\beta(E + i\varepsilon) = \gamma_\beta(E) \tag{A.9}$$

for every $E \in \mathbb{R}$ by monotone convergence.

We now make the connection to the usual definition of ergodic Jacobi operators (see also Section 2 in [3]). Let (Ω, T, μ) be an ergodic dynamical system, and $a : \Omega \to (0, \infty)$ and $b : \Omega \to \mathbb{R}$ are measurable functions satisfying

$$\frac{1}{C_0} \leq a(\omega) \leq C_0, \quad -C_0 \leq b(\omega) \leq C_0 \tag{A.10}$$

for almost every ω. Then we can define a map

$$f : \Omega \to \mathcal{J} \tag{A.11}$$

by $f(\omega)$ being the Jacobi operator with coefficients

$$\{(a(T^n\omega), b(T^n\omega)\}_{n\in\mathbb{Z}}. \tag{A.12}$$

Introduce a measure β on $\mathcal{J}$ given by

$$\beta(A) = \mu(f^{-1}(A)) \tag{A.13}$$

for Borel subsets $A \subseteq \mathcal{J}$. Then the usual definitions of the Lyapunov exponent and the integrated density of states will just be γ_β and k_β.

Appendix B. Orthogonal Polynomials

In this section, I want to relate my formulation of Theorem 1.3 to the one of Simon in [19]. That means I will relate

$$\overline{L}(E, H) = 0 \tag{B.1}$$

for almost every $E \in \sigma_{\mathrm{ess}}(H)$ and regularity. For background on the notion of regularity, I recommend the article [18] of Simon.

Recall that we introduced the solutions $c(z, n)$ in Section 3. It will be useful to rename these

$$p_n(z) = c(z, n). \tag{B.2}$$

This is related to the fact that $p_n(z)$ form an orthogonal basis in $L^2(d\sigma)$, where σ denotes the spectral measure of H.

We recall that H is called regular if

$$\lim_{N\to\infty}\left(\prod_{n=1}^{N} a(n)\right)^{1/N} = C(\sigma_{\mathrm{ess}}(H)), \tag{B.3}$$

where C denotes the capacity (see (2.11)). In [18] and [22], not H but σ is called regular, since the authors wish to emphasize the dependence on the measure. One has the following characterization of being regular:

Theorem B.1. *The following conditions are equivalent:*

(i) *H is regular.*

(ii) *For quasi every $E \in \sigma(H)$, we have that*

$$\limsup_{n\to\infty} |p_n(E)|^{1/n} = 1. \tag{B.4}$$

If furthermore $\sigma_{\mathrm{ess}}(H)$ is a finite gap set, we have

(iii) *For almost every $E \in \sigma(H)$, we have that*

$$\limsup_{n\to\infty} |p_n(E)|^{1/n} = 1. \tag{B.5}$$

(iv) *For almost every $E \in \sigma(H)$, we have that*

$$\overline{L}(E) = \limsup_{n\to\infty} \frac{1}{N}\log\left\|\prod_{n=N_j}^{1}\frac{1}{a(n)}\begin{pmatrix} z-b(n) & -1\\ a(n)^2 & 0\end{pmatrix}\right\| = 0 \tag{B.6}$$

Proof. The equivalence of (i) and (ii) is [18, Thm.1.10.] or [22, Thm.3.1.1.].

That (ii) implies (iii) is trivial, to see that (iii) implies (i), one can inspect the the proof of Stahl and Totik to see the following. With ω the equilibrium measure of $\sigma_{\mathrm{ess}}(H)$, one has that it suffices to have (B.4) for ω almost every E. Now, it is known that in the case of $\sigma_{\mathrm{ess}}(H)$ being a finite gap set, that ω and the Lebesgue measure are mutually absolutely continuous.

The equivalence of (iii) and (iv) is contained in Proposition 3.9. □

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Department of Mathematics, Rice University, Houston, TX 77005, USA
E-mail address: `helge.krueger@rice.edu`
URL: `http://math.rice.edu/~hk7/`