

OPTIMALITY OF LOG HÖLDER CONTINUITY OF THE INTEGRATED DENSITY OF STATES

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ABSTRACT. We construct examples, that log Hölder continuity of the integrated density of states cannot be improved. Our examples are limit-periodic.

1. INTRODUCTION

We investigate optimality of the log Hölder continuity of the integrated density of states. Let (Ω, μ) be a probability space, $T : \Omega \rightarrow \Omega$ an invertible ergodic transformation, and $f : \Omega \rightarrow \mathbb{R}$ a bounded measurable function. Define a potential $V_\omega(n) = f(T^n \omega)$. The Schrödinger operator $H_\omega : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ is defined by

$$(1.1) \quad H_\omega u(n) = u(n+1) + u(n-1) + V_\omega(n)u(n),$$

and the integrated density of states k by

$$(1.2) \quad k(E) = \lim_{N \rightarrow \infty} \int_{\Omega} \left(\frac{1}{N} \operatorname{tr}(P_{(-\infty, E]}(H_{\omega, [0, N-1]})) \right) d\mu(\omega),$$

where $H_{\omega, [0, N-1]}$ denotes the restriction of H_ω to $\ell^2([0, N-1])$. Using the *Thouless formula*, Craig and Simon showed that

Theorem 1.1 (Craig and Simon, [4]). *There exists a constant $C = C(\|f\|_\infty)$ such that*

$$(1.3) \quad |k(E) - k(\tilde{E})| \leq \frac{C}{\log |E - \tilde{E}|^{-1}}$$

for $|E - \tilde{E}| \leq \frac{1}{2}$.

This is what is well known as *log Hölder continuity*. We will be interested in the optimality of this statement in the sense that $\varepsilon \mapsto \frac{1}{\log(\varepsilon^{-1})}$ cannot be replaced by another function, which goes to zero faster. It was shown by Craig in [3], that the regularity cannot be improved to

$$\varepsilon \mapsto \frac{1}{\log(\varepsilon^{-1}) \log(\log(\varepsilon^{-1}))^\beta},$$

where $\beta > 1$. However, in the case of specific dynamical systems (Ω, μ, T) , there exist many results, which improve the Craig–Simon result. We just mention two. For quasi-periodic Schrödinger operators, Goldstein and Schlag have shown in [7] that the integrated density of states is Hölder continuous and computed the Hölder exponent, and shown that the integrated density of states is almost everywhere

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Lipschitz. For random Schrödinger operators, the integrated density of states is even everywhere Lipschitz. This is known as the Wegner estimate which can be found for example in the exposition of Kirsch in [6].

Our interest in the question of optimality of the Craig–Simon results comes from the importance of the Wegner estimate in multiscale analysis (see the exposition of Kirsch). If one could improve the result to a continuity of the form

$$\varepsilon \mapsto \frac{1}{\log(\varepsilon^{-1})^\beta}$$

for some large enough $\beta > 1$, one would be able to use this for multiscale analysis (see for example Theorem 3.12 in [8]). Already Craig’s result shows that this is impossible, however one could hope that a combination of an improved continuity result and an improvement of multiscale analysis might remove the Wegner estimate assumption. However, we will show that the continuity of integrated density of states cannot be improved for all potentials beyond log Hölder continuity.

A potential $V \in \ell^\infty(\mathbb{Z})$ is called almost-periodic, if the closure Ω of its translates is compact in the ℓ^∞ norm. Furthermore, then Ω can be made into a compact group, with an unique invariant Haar measure. For these our previous definition of the integrated density of states (1.2) can be replaced by

$$(1.4) \quad k_V(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \operatorname{tr}(P_{(-\infty, E)}(H_{[0, N-1]})),$$

where $H_{[0, N-1]}$ denotes now the restriction of $\Delta + V$ to $\ell^2([0, N-1])$ with Δ the discrete Laplacian. This can be found for example as Theorem 2.9 in Avron–Simon [2].

Next, V is called p periodic, if its p -th translate is equal to V . Furthermore, V is limit-periodic if it is the limit in the ℓ^∞ norm of periodic potentials. We denote by $\sigma(\Delta + V)$ the spectrum of the operator $\Delta + V$.

Theorem 1.2. *Given any increasing continuous function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with*

$$(1.5) \quad \lim_{x \rightarrow 0} \varphi(x) = 0$$

and a constant $C_0 > 0$, there is a limit-periodic V satisfying $\|V\|_\infty \leq C_0$ such that its integrated density of states satisfies

$$(1.6) \quad \limsup_{E \rightarrow E_0} \frac{|k_V(E) - k_V(E_0)| \log(|E - E_0|^{-1})}{\varphi(|E - E_0|)} = \infty,$$

for any $E_0 \in \sigma(\Delta + V)$.

This result tells us, that with φ as in the previous theorem, we cannot have

$$|k_V(E) - k_V(E_0)| \leq C \cdot \frac{\varphi(|E - E_0|)}{\log(|E - E_0|^{-1})}$$

for any $C > 0$ and all V . The proof of this theorem essentially happens in two parts. Given a periodic V_0 and $\varepsilon > 0$ satisfying $\|V_0\| \leq C_0 - \varepsilon$, we construct a sequence V_j of periodic potentials, with the following properties

- (i) V_j is p_j -periodic.
- (ii) The Lebesgue measure of $\sigma(\Delta + V_j)$

$$(1.7) \quad \varepsilon_j = |\sigma(\Delta + V_j)|$$

satisfies

$$(1.8) \quad \log(\varepsilon_j^{-1}) \geq p_{j-1} \cdot p_j \cdot \varphi(2\varepsilon_j).$$

(iii) We have that

$$(1.9) \quad \|V_j - V_{j-1}\| \leq \frac{\min(\varepsilon, \varepsilon_1, \dots, \varepsilon_{j-1})}{2^j}.$$

Here $\|\cdot\|$ denotes the ℓ^∞ norm. The construction of these V_j will be given in the next section and uses the tools developed by Avila in [1]. Before proceeding with the proof of Theorem 1.2, recall that positivity of the trace implies that

$$k_W(E - \|V - W\|) \leq k_V(E) \leq k_W(E + \|V - W\|)$$

for any potentials V and W .

Proof of Theorem 1.2. By (1.9), we see that there exists a limiting potential V , such that for each j , we have that

$$\|V_j - V\| \leq \varepsilon_j = |\sigma(\Delta + V_j)|.$$

Furthermore, since $\|V_0 - V\| \leq \varepsilon$, we have that $\|V\| \leq C_0$.

Next, fix $E_0 \in \sigma(\Delta + V)$ and let $j \geq 1$. By the previous equation, we have that there exists $E_1 \in \sigma(\Delta + V_j)$ such that

$$|E_0 - E_1| \leq \varepsilon_j = |\sigma(\Delta + V_j)|.$$

Denote the band of $\sigma(\Delta + V_j)$ containing E_1 by $[E_-, E_+]$. By a general fact about periodic Schrödinger operators, we know that

$$k_{V_j}(E_+) - k_{V_j}(E_-) = \frac{1}{p_j}.$$

We thus get that

$$k_V(E_+ + \varepsilon_j) - k_V(E_- - \varepsilon_j) \geq \frac{1}{p_j}$$

Furthermore, the interval $[E_- - \varepsilon_j, E_+ + \varepsilon_j]$ contains E_0 and we can choose $E_j \in \{E_- - \varepsilon_j, E_+ + \varepsilon_j\}$ such that

$$|k_V(E_0) - k_V(E_j)| \geq \frac{1}{2p_j}$$

and

$$|E_0 - E_j| \leq 2\varepsilon_j.$$

This implies the claim by (1.8), since j was arbitrary. \square

One can slightly improve the above theorem, by for example showing that there is not only one V , that satisfies the conclusion, but that in fact the set is dense in the limit-periodic operators. However, we have not done so, to keep the statement as simple as possible.

2. CONSTRUCTION OF THE PERIODIC POTENTIALS

We will need the machinery developed by Avila in [1], in order to prove our results. In the following, we let Ω be a totally disconnected compact group, known as *Cantor group*. We furthermore let $T : \Omega \rightarrow \Omega$ be a minimal translation on this group. There is a decreasing sequence of Cantor subgroups

$$X_1 \supseteq X_2 \supseteq \dots$$

such that the quotients

$$\Omega/X_k$$

contain p_k elements. We let P_k be the subset of the continuous functions $C(\Omega)$ on Ω , which only depend on Ω/X_k . f is called n -periodic if $f(T^n\omega) = f(\omega)$ for every $\omega \in \Omega$. The elements of P_k will be p_k periodic.

We now fix $\omega \in \Omega$. We have that $\{f(T^n\omega)\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ is limit-periodic, since the periodic f are dense in $C(\Omega)$. For a finite subset F of the periodic potentials $P = \bigcup_{k \geq 1} P_k$, we introduce the averaged Lyapunov exponent $L(E, F)$ as

$$(2.1) \quad L(E, F) = \frac{1}{\#F} \sum_{f \in F} L(E, f),$$

where $\#F$ denotes the number of elements of F (with multiplicities) and

$$L(E, f) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left\| \prod_{n=N}^1 \begin{pmatrix} f(T^n\omega) - E & -1 \\ 1 & 0 \end{pmatrix} \right\|$$

is the Lyapunov exponent of the periodic potential. For $f \in C(\Omega)$, we denote by $\Sigma(f)$ the spectrum of the operator $\Delta + f(T^n\omega)$. We will use the following two lemmas of Avila [1], see also [5].

Lemma 2.1 (Lemma 3.1. in [1]). *Let B be an open ball in $C(\Omega)$, let $F \subset P \cap B$ be finite, and let $0 < \varepsilon < 1$. Then there exists a sequence $F_K \subset P \cap B$ such that*

- (i) $L(E, F_K) > 0$ whenever $E \in \mathbb{R}$,
- (ii) $L(E, F_K) \rightarrow L(E, F)$ uniformly on compacts.

Lemma 2.2 (Lemma 3.2. in [1]). *Let B be an open ball in $C(\Omega)$, and let $F \subset P_k \cap B$ be a finite family of sampling functions. Then for every $N \geq 2$ and K sufficiently large, there exists $F_K \subset P_K \cap B$ such that*

- (i) $L(E, F_K) \rightarrow L(E, F)$ uniformly on compacts,
- (ii) The diameter of F_K is at most p_K^{-10} ,
- (iii) For every $\lambda \in \mathbb{R}$, if

$$(2.2) \quad \inf_{E \in \mathbb{R}} L(E, F) \geq \delta \#F p_k,$$

then for every $f \in F_K$, the spectrum $\Sigma(f)$ has Lebesgue measure at most $e^{-\delta p_K/2}$.

The construction of the V_j will be accomplished by

Proposition 2.3. *Given a continuous function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying*

$$(2.3) \quad \lim_{x \rightarrow 0} \psi(x) = 0,$$

and p -periodic f and $\varepsilon > 0$, then there exists a \tilde{p} -periodic function \tilde{f} , such that

$$(2.4) \quad \|f - \tilde{f}\|_\infty \leq \varepsilon,$$

and

$$(2.5) \quad \log(|\Sigma(\tilde{f})|^{-1}) \geq \tilde{p} \cdot \psi(|\Sigma(\tilde{f})|).$$

Proof. By Lemma 2.1, we can find a finite family of p_1 -periodic potentials F_1 within $B_{\frac{1}{2}\varepsilon}(f)$ such that

$$\delta_1 = \frac{1}{\#F_1 p_1} L(E, F_1) = \frac{1}{\#F_1 p_1} \cdot \frac{1}{\#F_1} \sum_{f \in F_1} L(E, f) > 0.$$

Applying Lemma 2.2 to F_1 , we can get a finite family $F_2 \subset B_\varepsilon(f)$ of \tilde{p} -periodic potentials, where we might require \tilde{p} to be arbitrarily large. Let \tilde{f} be any element of F_2 . By Lemma 2.2 (iii), we have that

$$|\Sigma(f_2)| < e^{-\delta_1 \tilde{p}/2}.$$

Hence, (2.5) turns into

$$\frac{1}{2} \delta_1 \tilde{p} \geq \tilde{p} \psi(e^{-\delta_1 \tilde{p}/2}).$$

The claim now follows from the fact, that $\psi(x) \rightarrow 0$ as $x \rightarrow 0$. \square

It now remains to construct the sequence of potentials V_j . We proceed by induction. By possibly modifying (Ω, T) , we can assume that

$$V_0(n) = f_0(T^n \omega)$$

for some $f_0 \in C(\Omega)$. Assume now that, we are given $V_1 = f_1 \circ T^n, \dots, V_{j-1} = f_{j-1} \circ T^n$ and we wish to construct $V_j = f_j \circ T^n$. We can choose now ε in the previous proposition to be the right hand side of (1.9). We choose $\psi(x) = p_{j-1} \varphi(2x)$ and the claim follows.

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