

MULTISCALE ANALYSIS FOR ERGODIC SCHRÖDINGER OPERATORS AND POSITIVITY OF LYAPUNOV EXPONENTS

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ABSTRACT. A variant of multiscale analysis for ergodic Schrödinger operators is developed. This enables us to prove positivity of Lyapunov exponents given initial scale estimates and an initial Wegner estimate. This is then applied to high dimensional skew-shifts at small coupling, where initial conditions are checked using the Pastur–Figotin formalism. Furthermore, it is shown that for potentials generated by the doubling map one has positive Lyapunov exponent except in a superpolynomially small set.

1. INTRODUCTION

The discrete one dimensional Schrödinger operator is one of the simplest models in quantum mechanics. It describes the motion of a particle in an one dimensional medium. Of particular interest is the case of ergodic potentials, where the potential V is given by

$$(1.1) \quad V_\omega(n) = f(T^n \omega)$$

for (Ω, μ) a probability space, $f : \Omega \rightarrow \mathbb{R}$ a real-valued and bounded function, $T : \Omega \rightarrow \Omega$ an invertible and ergodic transformation, and $\omega \in \Omega$. Then the Schrödinger operator is given by

$$(1.2) \quad \begin{aligned} H_\omega : \ell^2(\mathbb{Z}) &\rightarrow \ell^2(\mathbb{Z}) \\ H_\omega u(n) &= u(n+1) + u(n-1) + V_\omega(n)u(n). \end{aligned}$$

If one considers $H_\omega u = Eu$ as a formal difference equation, the Lyapunov exponent $L(E)$ describes the exponential behavior of solutions for almost every ω . It follows from Kotani theory, that the essential closure of the set $\{E : L(E) = 0\}$ is the absolutely continuous spectrum of H_ω for almost every ω . Furthermore, in the presence of uniform lower bounds $L(E) \geq \gamma > 0$, there has been a considerable development of machinery around the turn of the last millennium that implies localization.

In [31], Schlag has posed the following two open problems (and others)

- (i) Positivity of the Lyapunov exponent for small disorders for the skew-shift $(T(x, y) = (x + \alpha, y + x) \pmod{1})$, $\alpha \notin \mathbb{Q}$.
- (ii) Positivity of the Lyapunov exponent and Anderson localization for all positive disorders with $Tx = 2x \pmod{1}$.

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These two problems have also been noted by Bourgain in [8], Chulaevsky and Spencer [15], Damanik [18], and Goldstein and Schlag [22].

My original goal was to make a progress on the first problem, and it turned out that I needed to enlarge the torus on which the skew-shift is acting to obtain results, see Theorem 2.7 and 2.8. However, I was fortunate enough that my methods also applied to the second problem, for which I can extend the range, where positive Lyapunov exponent holds from a region of small disorders to also include the region of large disorders, see Theorem 2.3.

Most of the methods developed in this paper are independent of the underlying ergodic transformation, and I will present these in Section 3. However, since the already mentioned transformations are of special importance, I have decided to state the results for them in the next section, while reviewing parts of the current knowledge on them.

2. THE DOUBLING MAP AND THE SKEW-SHIFT

The plan for this section, is as follows, we will first review the current knowledge for the doubling map, and then state my new result, and then repeat this for the skew-shift. I wish to remark here, that random and quasi-periodic Schrödinger operators are reasonably well understood (see e.g. [8], [21]). In order to keep this paper at a reasonable length, I have decided not to discuss these two examples.

One of the most prototypical examples of a deterministic map, which behaves close to random, is the doubling map. Let $\Omega = \mathbb{T} \cong [0, 1)$ be the unit circle, and introduce $T : \Omega \rightarrow \Omega$ by

$$(2.1) \quad T\omega = 2\omega \pmod{1}.$$

It is well-known, that T is ergodic with respect to the Lebesgue measure. Furthermore, if we consider the dyadic expansion of ω

$$\omega = \sum_{j=1}^{\infty} \frac{\omega_j}{2^j}, \quad \omega_j \in \{0, 1\}$$

then the action of T is conjugated to the left shift

$$\{\omega_j\}_{j=1}^{\infty} \mapsto \{\omega_{j+1}\}_{j=1}^{\infty}$$

on the space $\{0, 1\}^{\mathbb{N}}$. Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function, and introduce for $\omega \in \Omega$ the potential

$$(2.2) \quad V_{\omega}(n) = f(T^n \omega), \quad n \geq 0.$$

Denote by Δ the discrete Laplacian. It seems natural to expect the Schrödinger operator

$$(2.3) \quad H_{\omega} = \Delta + V_{\omega}$$

to behave like a random Schrödinger operator (see [13]), and in particular have positive Lyapunov exponent for all energies. In order to state results, it turns out convenient to introduce a coupling constant $\lambda > 0$, so that

$$(2.4) \quad H_{\omega, \lambda} = \Delta + \lambda V_{\omega}$$

Denote by $L_{\lambda}(E)$ the Lyapunov exponent of this model. One has that

Theorem 2.1 (Chulaevsky–Spencer [15]). *Let $\delta > 0$ and $\lambda > 0$ small enough, then*

$$(2.5) \quad L_\lambda(E) \geq c_0 \lambda^2$$

for some $c_0 > 0$ and

$$(2.6) \quad E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta].$$

The approach of Chulaevsky and Spencer was then exploited by Bourgain and Schlag in [13] to prove Anderson localization and Hölder continuity of the integrated density of states. The restriction (2.6) was removed by Avila and Damanik [3] and by Sadel and Schulz-Baldes in [30].

All these results have been for small coupling constants λ . At my best knowledge, there are currently no results for large coupling (except [20]), but there is work in progress by Avila and Damanik [3] on it. In order to state my result, I introduce the following class of functions

Definition 2.2. *Let (Ω, μ) be a probability space, and $f : \Omega \rightarrow \mathbb{R}$ a measurable function. We call f non-degenerate, if there are $F, \alpha > 0$ such that for every $E \in \mathbb{R}$ and $\varepsilon > 0$, we have that*

$$(2.7) \quad \mu(\{\omega : |f(\omega) - E| \leq \varepsilon\}) \leq F\varepsilon^\alpha.$$

It follows from the Łojasiewicz inequality (see [27] or Theorem IV.4.1. in [28]), that if $\Omega = \mathbb{T}^K$ and $f : \Omega \rightarrow \mathbb{R}$ is real analytic, then f is non-degenerate in the above sense (see also Lemma 7.3. in [8]). It is necessary that (Ω, μ) contains no atoms, such that a function $f : \Omega \rightarrow \mathbb{R}$ can be non-degenerate. We will prove that

Theorem 2.3. *Let $f : \Omega \rightarrow \mathbb{R}$ be non-degenerate. There are constants $\lambda_0 = \lambda_0(f) > 0$ and $\kappa = \kappa(f) > 0$ such that for $\lambda > \lambda_0$ there is a set \mathcal{E}_b of measure*

$$(2.8) \quad |\mathcal{E}_b| \leq e^{-\lambda^{\alpha/2}}$$

and for $E \notin \mathcal{E}_b$, we have

$$(2.9) \quad L_\lambda(E) \geq \kappa \log(\lambda).$$

This question partially answers the question of positivity of the Lyapunov exponent. Furthermore, if one assumes a Wegner type estimate for this model, one can show that the lower bound actually holds for all energies.

One should note here that the claim is non-trivial, since the set of energies excluded in (2.8) is small compared to the expected size of the spectrum, which is $|\sigma(H_\lambda)| \gtrsim \lambda$ as $\lambda \rightarrow \infty$, see [19].

It should furthermore be remarked, that the proof of the theorem is done by an iteration of the resolvent identity combined with an energy elimination mechanisms. Both of these ideas are not restricted to the one dimensional setting.

We now turn our attention to the skew-shift. Let $\Omega = \mathbb{T}^2$ and introduce for an irrational number α the map $T_\alpha : \Omega \rightarrow \Omega$ by

$$(2.10) \quad T_\alpha(x, y) = (x + \alpha, x + y).$$

It turns out that $T_\alpha : \Omega \rightarrow \Omega$ is uniquely ergodic and minimal. Understanding the skew-shift is of physical importance, since it relates to the quantum kicked

rotor problem in the theory of quantum chaos. This problem asks to describe the behavior of solutions of

$$(2.11) \quad i\dot{\psi}(x, t) = a\psi''(x, t) + ib\psi'(x, t) + \kappa \cos(2\pi x) \left(\sum_{n \in \mathbb{Z}} \delta(t - n) \right) \psi(x, t),$$

where ψ is a 1 periodic function in x and a, b, κ are parameters, see Chapter 16 in [8]. Localization for operators¹ generated by skew-shift corresponds to quasi-periodic behavior of solutions of (2.11) (see [7]).

For a function $f : \Omega \rightarrow \mathbb{R}$ and $\lambda > 0$, introduce the potential by

$$(2.12) \quad V_{\lambda, \alpha, \omega}(n) = \lambda f(T_\alpha^n \omega).$$

Denote by $H_{\lambda, \alpha, \omega} = \Delta + V_{\lambda, \alpha, \omega}$ the associated Schrödinger operator and by $L_{\lambda, \alpha}(E)$ its Lyapunov exponent. In difference, to the doubling map the situation for large coupling has some understanding. In particular, there is the following result

Theorem 2.4 (Bourgain–Goldstein–Schlag [12], Bourgain [7]). *Let $f : \Omega \rightarrow \mathbb{R}$ be a real analytic. Given $\varepsilon > 0$, there is $\lambda_0 = \lambda_0(\varepsilon, f) > 0$ such that for all α except for a set of measure at most ε , we have that*

$$(2.13) \quad L_{\lambda, \alpha}(E) \geq c_0 \log \lambda$$

for all E and $\lambda > \lambda_0$. Furthermore, Anderson localization holds for all ω not in the exceptional set.

We will be able to prove a variant of this, with again eliminating energies, if we do not assume a Wegner type estimate.

Theorem 2.5. *Let $f : \Omega \rightarrow \mathbb{R}$ be non-degenerate. There are constants $\lambda_0 = \lambda_0(f) > 0$ and $\kappa = \kappa(f) > 0$ such that for $\lambda \geq \lambda_0$ there is a set $\mathcal{E}_b = \mathcal{E}_b(\alpha, \lambda)$ of measure*

$$(2.14) \quad |\mathcal{E}_b| \leq e^{-\lambda^{\alpha/2}}$$

and for $E \notin \mathcal{E}_b$, we have

$$(2.15) \quad L_{\lambda, \alpha}(E) \geq \kappa \log(\lambda).$$

In distinction to the previous result, my result eliminates energies E and not frequencies α .

The current knowledge at small coupling is far from satisfactory. At my best knowledge the current results are by Bourgain in [5], [6], which prove a statement of the following form.

Theorem 2.6 (Bourgain [5], [6]). *Let $f(x, y) = 2 \cos(2\pi y)$, then for each $\lambda > 0$ small enough, we may choose $\alpha(\lambda)$ from a set of positive measure such that*

$$(2.16) \quad \lim_{\lambda \rightarrow 0} |\{E : L_{\lambda, \alpha(\lambda)}(E) = 0\}| = 0.$$

In order to prove this theorem, Bourgain uses approximation of the skew-shift by rotation, for which he needs α to be small. In fact, $\alpha(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. Furthermore, Bourgain does not compute a quantitative lower bound for the Lyapunov exponent.

¹One needs to consider more general Toeplitz operators here.

Instead of using α as a perturbation parameter, we will use the dimension K of the torus, on which the skew-shift acts. For $K \geq 2$ and an irrational α , introduce the K skew-shift $T_{\alpha,K} : \mathbb{T}^K \rightarrow \mathbb{T}^K$ by

$$(2.17) \quad (T_{\alpha,K}\omega)_k = \begin{cases} \omega_1 + \alpha & k = 1 \\ \omega_k + \omega_{k-1} & k > 1. \end{cases}$$

We will furthermore, assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a 1 bounded, non-constant function of mean zero. For $\lambda > 0$, α irrational, $\omega \in \mathbb{T}^K$, we introduce the potential

$$(2.18) \quad V_{\lambda,\alpha,\omega}(n) = \lambda f((T_{\alpha,K}^n \omega)_K).$$

We will show that

Theorem 2.7. *Let $\delta > 0$. There is a constant $\kappa = \kappa(f) > 0$. Let $K \geq 1$ be large enough. There are $\lambda_1 = \lambda_1(\delta, f, K) < \lambda_2 = \lambda_2(\delta, f)$ with $\lim_{K \rightarrow \infty} \lambda_1 = 0$ such that for*

$$(2.19) \quad \lambda_1 \leq \lambda \leq \lambda_2$$

we have that

$$(2.20) \quad L_{\lambda,\alpha,K}(E) \geq \kappa \lambda^2$$

for $E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta]$ except in a set of small measure.

This result has a major drawback to the one of Bourgain by not applying to the case of $K = 2$, but needing K large. However, it has other advantages, like applying to all irrational α , and providing an explicit lower bound. The restriction of the energy region has similar reasons as the restriction in the result of Chulaevsky and Spencer, since the method to verify the initial condition is similar.

It should be pointed out that one can again drop eliminating energies, if one assumes a Wegner type estimate. This estimate can be verified explicitly in the case, when

$$(2.21) \quad f(x) = 2 \left(x - \frac{1}{2} \right).$$

Then we obtain

Theorem 2.8. *Let the quantities be as in the previous theorem, and f as above, then*

$$(2.22) \quad L_{\lambda,\alpha,K}(E) \geq \kappa \lambda^2$$

for $E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta]$ and $\lambda_1 \leq \lambda \leq \lambda_2$.

I believe that it should be possible to extend Theorem 2.7 to a result holding at all energies using the methods of Bourgain, Goldstein, and Schlag found in [7] and [12]. However, without further developments these results would be perturbative in the frequency α and in particular require Diophantine conditions on it.

A much greater challenge would be to establish a result that holds at all energies and for all frequencies α . This has only been achieved recently for rotations on high dimensional tori by Bourgain [10] at large coupling. The corresponding result is still unknown for the skew-shift, but has been announced by Bourgain [9] and Jitomirskya [24].

3. STATEMENT OF THE RESULTS

We will now begin stating the main results of this article. First, recall that H denotes a discrete one-dimensional Schrödinger operator given by its action on $u \in \ell^2(\mathbb{Z})$ by

$$(3.1) \quad (Hu)(n) = u(n+1) + u(n-1) + V(n)u(n),$$

where $V : \mathbb{Z} \rightarrow \mathbb{R}$ is a bounded sequence known as the potential. We will usually not make the dependance of the operator H and its associated potential explicit.

For $\Lambda \subseteq \mathbb{Z}$ an interval, we denote by H_Λ the restriction of H to $\ell^2(\Lambda)$. We furthermore denote by e_x the standard basis of $\ell^2(\mathbb{Z})$, that is

$$e_x(n) = \begin{cases} 1 & n = x \\ 0 & \text{otherwise.} \end{cases}$$

For $E \notin \sigma(H_\Lambda)$ and $x, y \in \Lambda$, we denote by $G_\Lambda(E, x, y)$ the Green's function, defined by

$$(3.2) \quad G_\Lambda(E, x, y) = \langle e_x, (H_\Lambda - E)^{-1} e_y \rangle.$$

The resolvent equation implies that if $x \in [a, b] \subseteq \Lambda \subseteq \mathbb{Z}$ and $y \in \Lambda \setminus [a, b]$, then

$$(3.3) \quad G_\Lambda(E, x, y) = -G_{[a,b]}(E, x, a)G_\Lambda(E, a-1, y) - G_{[a,b]}(E, x, b)G_\Lambda(E, b+1, y)$$

as long as $E \notin \sigma(H_\Lambda) \cup \sigma(H_{[a,b]})$. This formula is a key ingredient in multiscale schemes, since it enables us to go from decay on small intervals $[a, b]$ to decay on a large interval Λ . We will quantify the decay of the Green's function using the following notion.

Definition 3.1. For $a \in \mathbb{Z}$ and $K \geq 1$. $[a - K, a + K]$ is called (γ, \mathcal{E}) -good if

$$(3.4) \quad |G_{[a-K, a+K]}(E, a, a \pm K)| \leq \frac{1}{2} e^{-\gamma K}$$

for $E \in \mathcal{E}$. Otherwise, $[a - K, a + K]$ is called (γ, \mathcal{E}) -bad.

We are now ready to state our first result.

Theorem 3.2. Given $0 < \sigma \leq \frac{1}{4}$, $K \geq 1$, $\gamma > 0$, $L \geq 1$, and $\mathcal{E} \subseteq \mathbb{R}$ an interval. Assume that

$$(3.5) \quad \#\{1 \leq l \leq L : [(l-1)K+1, (l+1)K-1] \text{ is } (\gamma, K, \mathcal{E})\text{-bad}\} \leq \sigma L,$$

and the following inequalities hold

$$(3.6) \quad \gamma \cdot K \geq \max\left(\frac{1}{\sigma}, \frac{25}{\sigma} \ln(|\mathcal{E}|^{-1})\right)$$

$$(3.7) \quad \frac{1}{K^3} e^{\frac{8}{75}\sigma\gamma K} \geq \frac{2^{17}e^3}{\sigma^4}.$$

Then, there is $\mathcal{E}_0 \subseteq \mathcal{E}$ such that

$$(3.8) \quad |\mathcal{E}_0| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|$$

and for $E \in \mathcal{E}_0$, we have that

$$(3.9) \quad \frac{1}{LK} \log \left\| \prod_{n=1}^{LK} \begin{pmatrix} V(LK-n) - E & -1 \\ 1 & 0 \end{pmatrix} \right\| \geq e^{-8\sigma} e^{-\frac{1}{99}\gamma} - \frac{\sqrt{2}}{LK}$$

The proof of this theorem is our most basic implementation of multiscale analysis without a Wegner estimate. The main idea is that for a scale $K_1 \gg K$, one has that the set of energies, where Wegner type estimates fail has small measure for each of the sets of the form $lK_1 + [-K_1 + 1, K_1 - 1]$. Thus one may achieve that the Wegner estimate holds for most l outside a set of small measure by using Markov's inequality. The implementation of this can be found in Section 6. Then Section 7 finishes the proof of this theorem. I furthermore wish to point out that similar methods of proof have been used by Bourgain in [5].

We remark that the requirement $|\mathcal{E}| \geq e^{-\frac{1}{25}\gamma\sigma K}$ is essential to our approach, since it ensures that the bad set of energies is smaller than the one, we start with. Furthermore, it is a non-trivial condition, since by perturbing the energy in the Green's function will give only a set of smaller measure. However, one can still use this approach to obtain this condition, by then decreasing the estimate on the Green's function.

In order to state the next theorem, we will need to phrase it in the ergodic setting. For this let (Ω, μ) be a probability space and $f : \Omega \rightarrow \mathbb{R}$ a bounded real-valued function. For $\omega \in \Omega$, we introduce a potential by

$$(3.10) \quad V_\omega(n) = f(T^n\omega), \quad n \in \mathbb{Z}.$$

We will now write H_ω for $\Delta + V_\omega$. It follows from the ergodic theorem, that (3.5) is roughly equivalent to

$$(3.11) \quad \mu(\{\omega : [1, 2K - 1] \text{ is } (\gamma, \mathcal{E})\text{-bad for } H_\omega\}) \leq \sigma.$$

In particular, this condition is now independent of N . Thus, one can hope to obtain the conclusion of the previous theorem for all sufficiently large N . In order to exploit this, we will now introduce the Lyapunov exponent $L(E)$ by

$$(3.12) \quad L(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\Omega} \log \left\| \prod_{n=1}^N \begin{pmatrix} V_\omega(N-n) - E & -1 \\ 1 & 0 \end{pmatrix} \right\| d\mu(\omega),$$

where the limit exists for all E and defines a subharmonic function.

Theorem 3.3. *Given $0 < \sigma \leq \frac{1}{4}$, $K \geq 1$, $\gamma > 0$, $L \geq 1$, and $\mathcal{E} \subseteq \mathbb{R}$ an interval. Assume the inequalities (3.6), (3.7), and the initial condition (3.11). Then, there is $\mathcal{E}_0 \subseteq \mathcal{E}$ such that*

$$(3.13) \quad |\mathcal{E}_0| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|$$

and for $E \in \mathcal{E}_0$, we have that

$$(3.14) \quad L(E) \geq e^{-8\sigma} e^{-\frac{1}{99}\gamma}.$$

This theorem is a combination of the last theorem and properties of ergodic Schrödinger operators. The new parts of the proof depend on ideas from ergodic theory discussed in Section 4 and about the Lyapunov exponent for ergodic Schrödinger operators discussed in Section 5. The proof is then given in Section 9.

Given this criterion for positive Lyapunov exponent, it seems a natural question if the assumption (3.11), can be checked. It is classical, that (3.11) holds at large coupling, that is if we consider the family of potentials

$$(3.15) \quad V_{\omega,\lambda}(n) = \lambda f(T^n\omega)$$

for $\lambda > 0$ and f is a nice enough function. More precisely, we have that.

Proposition 3.4. *Assume that f is non-degenerate in the sense of Definition 2.2. Let $E_0 \in \mathbb{R}$, $\sigma > 0$, and introduce*

$$(3.16) \quad K = \left\lfloor \frac{\sigma \lambda^{\alpha/2}}{F} \right\rfloor$$

$$(3.17) \quad \gamma = \frac{1}{5} \log(\lambda)$$

$$(3.18) \quad \mathcal{E} = [E_0 - 1, E_0 + 1].$$

Assume that λ is sufficiently large. Then (3.11) holds.

For the convenience of the reader, I have included a proof in Section 10. With this proposition, we are ready for the proof of Theorem 2.3 and 2.5. Instead of proving them, we will instead proof the following more abstract version.

Theorem 3.5. *Let (Ω, μ) be a probability space, and f a non-degenerate function, with $|f(\omega)| \leq 1$. There are constants $\kappa = \kappa(f) > 0$, $\lambda_0 = \lambda_0(f) > 0$. For any $T : \Omega \rightarrow \Omega$ ergodic and $\lambda > \lambda_0$, there exists a set $\mathcal{E}_b = \mathcal{E}_b(T, \lambda)$ of measure*

$$(3.19) \quad |\mathcal{E}_b| \leq e^{-\lambda^{\alpha/2}},$$

such that for $E \notin \mathcal{E}_b$

$$(3.20) \quad L_{T, \lambda}(E) \geq \kappa \log(\lambda).$$

Proof. It follows from the Combes–Thomas estimate (see Lemma 10.1) that the estimate on the Lyapunov exponent holds for $|E| \geq 3\lambda$ and $\lambda > 1$. Next, observe that we can cover the interval $[-3\lambda, 3\lambda]$ with 3λ intervals of length 2 as described in the previous proposition.

For one of these intervals \mathcal{E} , we can the apply Theorem 3.3 with $\sigma = \frac{1}{4}$, $K = \lceil (4F)^{-1} \lambda^{\alpha/2} \rceil$, and $\lambda = \frac{1}{5} \log \lambda$. In particular, we see that the arithmetic conditions hold for large enough λ , and the estimate on the size of \mathcal{E}_b follows, since the bad set $\mathcal{E} \setminus \mathcal{E}_0$ has measure behaving like $e^{-c \log(\lambda) \cdot \lambda^{\alpha/2}}$ for some $c > 0$. This finishes the proof. \square

The somewhat surprising thing is, that the largeness of the Lyapunov exponent does not depend on the ergodic transformation T in this theorem.

Using the Pastur–Figotin [29] formalism combined with with large deviation estimates of Bourgain and Schlag [13], we are able to obtain an initial condition at small coupling. In order to state these results, we will need to introduce a bit of notation about random Schrödinger operators. For an integer $N \geq 1$, $\lambda > 0$, and $\underline{V} \in [-1, 1]^N$ introduce the operator $H_{\underline{V}, \lambda, [0, N-1]}$ acting on $\ell^2([0, N-1])$ by

$$(3.21) \quad H_{\underline{V}, \lambda, [0, N-1]} u(n) = \begin{cases} u(1) + \lambda V(0)u(0) & n = 0 \\ u(n+1) + u(n-1) + \lambda V(n)u(n) & 1 \leq n \leq N-2 \\ u(N-2) + \lambda V(N-1)u(N-1) & n = N-1. \end{cases}$$

We will show

Proposition 3.6. *Let ν be a probability measure with support in $[-1, 1]$ and mean zero. Introduce*

$$(3.22) \quad \sigma_2 = \int x^2 d\nu, \quad \sigma_4 = \int (x^2 - \sigma_2)^2 d\nu.$$

Let

$$(3.23) \quad E = 2 \cos(\kappa) \in (-2, 0) \cup (0, 2).$$

Let $A = \min(1, E^2 - 2)$, assume the inequalities

$$(3.24) \quad K \geq \max\left(\frac{2800 \cdot \sigma_2}{|E| \cdot \sqrt{4 - E^2} \cdot A}, \frac{4608\sigma_4}{(\sigma_2)^2}\right)$$

$$(3.25) \quad \lambda \leq \frac{\sqrt{4 - E^2}}{2} \min\left(\frac{\sigma_2}{7000}, \frac{|E| \cdot \sqrt{4 - E^2} \cdot A}{4400 \cdot \sigma_2}\right)$$

$$(3.26) \quad \lambda^2 K \geq 150000 \frac{4 - E^2}{\sigma_2}$$

Let

$$(3.27) \quad \gamma = \frac{\lambda^2 \sigma_2}{4(4 - E^2)}.$$

Then there exists a set \mathcal{V} satisfying

$$(3.28) \quad \nu^{\otimes 2K}(\mathcal{V}) \geq 1 - \frac{1}{16}.$$

For each $\underline{V} \in \mathcal{V}$, there is $M = M(\underline{V}) \in \{2K - 3, 2K - 2\}$, such that the following estimates

$$(3.29) \quad |G_{\underline{V}, \lambda, [1, M]}(E, 1, K)| \leq \frac{\sqrt{1 - \frac{|E|}{2}}}{2} e^{-\gamma K}$$

$$(3.30) \quad |G_{\underline{V}, \lambda, [1, M]}(E, M, K)| \leq \frac{\sqrt{1 - \frac{|E|}{2}}}{2} e^{-\gamma K}$$

$$(3.31) \quad \|(H_{\underline{V}, \lambda, [1, M]} - E)^{-1}\| \leq \sqrt{1 - \frac{|E|}{2}} 2K e^{(\frac{10}{3}\gamma + \log(6))K}$$

hold.

We note the following corollary.

Corollary 3.7. *Under the assumptions of the previous proposition, we have for \tilde{E} in the set*

$$(3.32) \quad \tilde{E} \in \mathcal{E} = [E - \varepsilon, E + \varepsilon], \quad \varepsilon = \frac{e^{-(\tilde{\gamma} + (\frac{10}{3}\gamma + \log(6)))K}}{16\sqrt{1 - \frac{|E|}{2}}K}$$

for $\tilde{\gamma} = \gamma - \frac{1}{K}(\frac{1}{2} \log(1 - \frac{|E|}{2}) + \log(2))$ that

$$(3.33) \quad |G_{\underline{V}, \lambda, [1, M]}(\tilde{E}, 1, K)| \leq \frac{1}{2} e^{-\tilde{\gamma} K}$$

$$(3.34) \quad |G_{\underline{V}, \lambda, [1, M]}(\tilde{E}, M, K)| \leq \frac{1}{2} e^{-\tilde{\gamma} K}$$

$$(3.35)$$

where $\underline{V} \in \mathcal{V}$ and $M = M(\underline{V}) \in \{2K - 3, 2K - 2\}$.

Proof. From the resolvent formula, one obtains that

$$\begin{aligned} (H_{\underline{V},\lambda,[1,M]} - \tilde{E})^{-1} &= (H_{\underline{V},\lambda,[1,M]} - E)^{-1} \\ &+ (H_{\underline{V},\lambda,[1,M]} - E)^{-1} \cdot \left(\sum_{n=1}^{\infty} \left((\tilde{E} - E)(H_{\underline{V},\lambda,[1,M]} - E)^{-1} \right)^n \right). \end{aligned}$$

Hence, we obtain the estimate

$$|G_{\underline{V},\lambda,[1,M]}(\tilde{E}, 1, K)| \leq \frac{\sqrt{1 - \frac{|E|}{2}}}{2} e^{-\gamma K} + \frac{\varepsilon \|(H_{\underline{V},\lambda,[1,M]} - E)^{-1}\|^2}{1 - \varepsilon \|(H_{\underline{V},\lambda,[1,M]} - E)^{-1}\|}.$$

A quick computation now finishes the proof. \square

We will need the following variant of Definition 3.1.

Definition 3.8. *Given $K \geq 1$, $\mathcal{E} \subseteq \mathbb{R}$ an interval, $\gamma > 0$. $\omega \in \Omega$ is called (K, \mathcal{E}, γ) -good, there is $M \in \{2K - 3, 2K - 2\}$ such that for $x \in \{K - 1, K\}$*

$$(3.36) \quad |G_{\omega,[1,M]}(E, 1, x)|, |G_{\omega,[1,M]}(E, 1, M)| \leq \frac{1}{2} e^{-\gamma K}$$

for $E \in \mathcal{E}$.

We observe that the previous proposition implies, that we are good in this sense. One can adapt the proof of Theorem 3.3 in order to only require to be good in the sense of Definition 3.8 instead of Definition 3.1.

We now return to the investigation of ergodic Schrödinger operators, and start by introducing K -independence, which will allow us to apply the tools from random Schrödinger operators.

Definition 3.9. *Let (Ω, μ) be a probability space, $T : \Omega \rightarrow \Omega$ an ergodic transformation, and $f : \Omega \rightarrow \mathbb{R}$ bounded and measurable. (Ω, μ, T, f) is called K -independent if there exists a probability measure ν on \mathbb{R} such that*

$$(3.37) \quad \nu^{\otimes K}(\{(f(\omega), f(T\omega), \dots, f(T^{K-1}\omega)) : \omega \in A\}) = \mu(A)$$

for all $A \subseteq \Omega$ measurable.

In the case of random variables $\Omega = I^{\mathbb{Z}}$, T the left shift, and $f(\omega) = \omega_0$, one clearly has that the system is K independent for any $K \geq 1$. We furthermore note, the following lemma which shows how independent the K skew-shift is.

Lemma 3.10. *Let $g : \mathbb{T} \rightarrow \mathbb{R}$ be a bounded function, define $f : \mathbb{T}^K \rightarrow \mathbb{R}$ by $f(\underline{\omega}) = g(\omega_K)$. Let T_α be the K skew-shift, then $(\mathbb{T}^K, \text{Lebesgue}, T_\alpha, f)$ is K independent.*

Proof. One can check that

$$\begin{pmatrix} (\underline{\omega})_K \\ (T\underline{\omega})_K \\ \vdots \\ (T^{K-1}\underline{\omega})_K \end{pmatrix} = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_K \end{pmatrix},$$

where $*$ denotes a non zero number. This implies the claim by the transformation formula for integrals. \square

We now come

Theorem 3.11. *Let (Ω, μ, T, f) be K -independent. Given $\delta > 0$, there is a $\kappa = \kappa(f, \delta) > 0$. Furthermore, there is $\lambda_1 = \lambda_1(\delta) > 0$ and for $K \geq 1$ a $\lambda_2 = \lambda_2(\delta, K) > 0$ with $\lambda_2 \rightarrow 0$ as $K \rightarrow \infty$, such that one has for*

$$(3.38) \quad \lambda_2 \leq \lambda \leq \lambda_1,$$

that

$$(3.39) \quad L(E) \geq \kappa \lambda^2,$$

for $E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta]$ except in a set, whose measure goes to 0 as $K \rightarrow \infty$.

Proof. In order to ensure that $|\mathcal{E}|$ from the previous corollary is large enough, just decrease $\tilde{\gamma}$. This finishes the proof by an application of Theorem 3.3. \square

Of course these results have still a major drawback: the need to eliminate energies. This can be eliminated by assuming Wegner type estimates, as they are common in the theory of random Schrödinger operators. For this, we will denote by $\sigma(H_{\omega, \Lambda})$ the spectrum of $H_{\omega, \Lambda}$ given $M \geq 1$, an energy E , and $\varepsilon > 0$, we will be interested in the probability

$$(3.40) \quad \mu(\{\omega : \text{dist}(\sigma(H_{\omega, [0, M-1]}), E) \leq \varepsilon\}),$$

which we will need to assume to be small. The most convenient form of this estimate for us, will be that

$$(3.41) \quad \mu(\{\omega : \text{dist}(\sigma(H_{\omega, [0, M-1]}), E) \leq \varepsilon\}) \leq C \cdot \frac{M^\beta}{|\log(\varepsilon)|^\rho},$$

where $C > 0$, $\beta \geq 0$ and $\rho \geq 1$. One has to restrict here to $0 < \varepsilon \leq \frac{1}{2}$, so one does not run into problems, when the logarithm becomes 0.

In the theory of random Schrödinger operators, one has as already mentioned that $V(n)$ are independent identically distributed random variables. If one assumes, that the density is a bounded function, one can obtain the following estimate, which is known as a Wegner estimate

$$(3.42) \quad \mu(\{\omega : \text{dist}(\sigma(H_{\omega, [0, M-1]}), E) \leq \varepsilon\}) \leq C \cdot M \cdot \varepsilon,$$

where $C > 0$ is a constant. We will follow the ideas of the proof and show in Section 16 that a similar estimate holds for the skew-shift model.

Assuming (3.41), we are able to remove the assumption of removing energies from our theorems, and obtain.

Theorem 3.12. *Assume the initial length scale (3.11), the Wegner type estimate (3.41),*

$$(3.43) \quad 3\beta + 3 - \rho \leq 0,$$

and

$$(3.44) \quad \gamma^\rho K^{\rho-\beta} \sigma^{\rho-1} \geq 4 \cdot 2^{\beta+\rho} \cdot e^{(\beta+1)(4\sigma+\frac{1}{99})} C.$$

Then

$$(3.45) \quad L(E) \geq e^{-\frac{1}{99}} e^{-4\sigma} \gamma.$$

This theorem gives a satisfying criterion for positivity of Lyapunov exponents, where the conditions exactly correspond to the ones necessary for localization in the theory of random Schrödinger operators.

Of course (3.41) is not a simple estimate to check, since it involves information at all scales. We are thus only able to check it in the special case of $f(x) = x - \frac{1}{2}$. This then allows us to prove the following theorem for the skew-shift at small coupling.

For $\lambda > 0$, we introduce the potential

$$(3.46) \quad V_{\lambda, \alpha, \underline{\omega}}(n) = \lambda f(T_\alpha^n \underline{\omega}).$$

We will show in Section 16 the following proposition, which shows that (3.41) holds for this family.

Proposition 3.13. *Let $H_{\lambda, \alpha, \underline{\omega}} = \Delta + V_{\lambda, \alpha, \underline{\omega}}$. Given $\rho \geq 1$, we have for any $E \in \mathbb{R}$ and $M \geq 10$ that*

$$(3.47) \quad \mu(\{\underline{\omega} : \text{dist}(\sigma(H_{\lambda, \alpha, \underline{\omega}, [0, M-1]}), E) \leq \varepsilon\}) \leq 14 \cdot \max(1, \frac{1}{\lambda}) \frac{\rho^\rho \cdot M^4}{|\log(\varepsilon)|^\rho}.$$

Combining this proposition with Proposition 3.6 and Theorem 3.12, we can show the following theorem.

Theorem 3.14. *Given $\varepsilon, \delta > 0$, let*

$$(3.48) \quad E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta].$$

There are constants $C_1 = C_1(\varepsilon, \delta), C_2 = C_2(\delta), \gamma_0 = \gamma_0(\delta) > 0$ such that for

$$(3.49) \quad \frac{C_1}{K^{\frac{1}{2}-\varepsilon}} \leq \lambda \leq C_2,$$

and α irrational, we have

$$(3.50) \quad L_{\lambda, \alpha}(E) \geq \gamma_0 \lambda^2.$$

Proof. We can assume $0 < \lambda < 1$, so by Proposition 3.13, we may take

$$\beta = 4, \quad C = \frac{14\rho^\rho}{\lambda}$$

for any $\rho \geq 15$ in (3.41) ($\rho \geq 15$ such that (3.43) holds). We furthermore, see that $d\nu = \chi_{[-1,1]} dx$ and thus

$$\sigma_2 = \frac{2}{3}, \quad \sigma_4 = \frac{2}{5}.$$

We can thus choose $\gamma = \frac{\lambda^2}{48 \sin(\kappa)}$. We may choose $\sigma = \frac{1}{4}$, and thus (3.44) becomes

$$\frac{\lambda^{2\rho+1}}{\sin(\kappa)^\rho} K^{\rho-3} \geq C_1 (384\rho)^\rho$$

for some constant $C_1 > 0$. This finishes the proof by applying Theorem 3.12 to the initial condition obtained by Proposition 3.6. \square

Proving positive Lyapunov exponent is not the only problem concerning ergodic Schrödinger operators. There is probably an even larger literature as how to go from positive Lyapunov exponent to Anderson localization (see for example [23] and [11] in the case of rotations). However, one cannot expect Anderson localization to hold in the generality discussed in this paper, since for example the results of Avron and Simon in [4] show, that if $T : \Omega \rightarrow \Omega$ is well approximated by periodic

transformation, then the spectrum of H is purely continuous, and hence Anderson localization cannot hold.

We now give an overview, of what happens in the following sections. Section 4 derives some consequences of the ergodic theorem, which will be needed in the following. Section 5 discusses properties of the Lyapunov exponent, which will be needed. Section 6, 7, and 8 contain the proof of Theorem 3.2. Then Theorem 3.3 is proven in Section 9. Proposition 3.4 is proven in Section 10 and Proposition 3.6 in Sections 11 and 12. Sections 13 and 14 contain the proof of Theorem 3.12. Finally Proposition 3.13 is proven in Section 16.

4. ERGODIC THEORY

In this section, we review the notions of ergodic theory, we will use. As usual, we denote by (Ω, μ) a probability space and by $T : \Omega \rightarrow \Omega$ an ergodic transformation, that is if $A \subseteq \Omega$ satisfies $T^{-1}A = A$ almost everywhere, then $\mu(A) \in \{0, 1\}$. We recall that the mean ergodic theorem tells us, that if f is a function in $L^2(\Omega, \mu)$, then its averages

$$(4.1) \quad f_N(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n \omega)$$

converge to $\int_{\Omega} f(\omega) d\mu(\omega)$ in $L^2(\Omega, \mu)$. This result will be the mean ingredient of ergodic theory, we will use. However, some of the results from ergodic Schrödinger operators, we are using depend on the somewhat different Birkhoff ergodic theorem, saying that one has pointwise convergence almost everywhere.

We will be interested in the following question: Given the good set $\Omega_g \subseteq \Omega$ and an integer $K \geq 1$, can we choose a large set of ω such that, we have

$$T^{lK} \omega \in \Omega_g$$

for a set of l with density close to $\mu(\Omega_g)$. The following lemma does exactly this.

Lemma 4.1. *Given $\Omega_g \subseteq \Omega$, $0 < \kappa < 1$, $K \geq 1$. Then, there exists $\Omega_0 \subseteq \Omega$ such that for $\omega \in \Omega_0$, there is a sequence $L_t = L_t(\omega) \rightarrow \infty$ such that*

$$(4.2) \quad \frac{1}{L_t} \#\{0 \leq l \leq L_t - 1 : T^{lK} \omega \in \Omega_g\} \geq \kappa \mu(\Omega_g)$$

and $\mu(\Omega_0) > 0$.

Proof. Letting $f = \chi_{\Omega_g}$ in the mean ergodic theorem, we find that

$$\lim_{N \rightarrow \infty} \int_{\Omega} \left| \frac{1}{N} \#\{0 \leq n \leq N - 1 : T^n \omega \in \Omega_g\} - \mu(\Omega_g) \right|^2 d\mu(\omega) = 0.$$

Thus, we obtain in particular

$$\lim_{N \rightarrow \infty} \mu(\{\omega : \frac{1}{N} \#\{0 \leq n \leq N - 1 : T^n \omega \in \Omega_g\} < \kappa \mu(\Omega_g)\}) = 0.$$

We thus may find a set Ω_1 of positive measure, such that for each $\omega \in \Omega_1$, there is a sequence $N_t = N_t(\omega)$ going to ∞ such that

$$\frac{1}{N_t} \#\{0 \leq n \leq N_t - 1 : T^n \omega \in \Omega_g\} \geq \kappa \mu(\Omega_g).$$

For each $\omega \in \Omega_1$, we may find an $0 \leq s = s(\omega) \leq K - 1$ such that $N_t \pmod{K} = s$ for infinitely many t . Introduce

$$\Omega_0 = \{T^{-s(\omega)}\omega : \omega \in \Omega_1\},$$

and choose for $\omega \in \Omega_0$ the sequence $L_t = \frac{N_t}{K}$, for the N_t with $N_t \pmod{K} = s$. The claim now follows by construction. \square

Furthermore recall that a transformation $T : \Omega \rightarrow \Omega$ is called totally ergodic, if for every $n \geq 1$ the transformation $T^n : \Omega \rightarrow \Omega$ is ergodic. Total ergodicity allows us to not need the step of passing to a subsequence in the proof of the last lemma. Thus, we may conclude that

Lemma 4.2. *Suppose that $T : \Omega \rightarrow \Omega$ is totally ergodic. Given $\Omega_g \subseteq \Omega$, $0 < \kappa < 1$, $0 < \tau < 1$, and $K \geq 1$. There is $\Omega_0 \subseteq \Omega$ such that for $\omega \in \Omega_0$ and L large enough*

$$(4.3) \quad \frac{1}{L} \#\{0 \leq l \leq L - 1 : T^{lK}\omega \in \Omega_g\} \geq \kappa\mu(\Omega_g)$$

and

$$(4.4) \quad \mu(\Omega_0) \geq \tau.$$

5. THE LYAPUNOV EXPONENT

We let again (Ω, μ) be a probability space, $f : \Omega \rightarrow \mathbb{R}$ a bounded measurable function, $T : \Omega \rightarrow \Omega$ an invertible ergodic transformation, and set $V_\omega(n) = f(T^n\omega)$ for $\omega \in \Omega$ and $n \in \mathbb{Z}$. Introduce the N step transfer matrix $A_\omega(E, N)$ by

$$(5.1) \quad A_\omega(E, N) = \prod_{n=1}^N \begin{pmatrix} E - V_\omega(N - n) & -1 \\ 1 & 0 \end{pmatrix}.$$

Let u be a solution of $H_\omega u = Eu$ interpreted as a difference equation. Then we have that

$$(5.2) \quad \begin{pmatrix} u(N+1) \\ u(N) \end{pmatrix} = A_\omega(E, N) \cdot \begin{pmatrix} u(1) \\ u(0) \end{pmatrix},$$

explaining the name. Define the Lyapunov exponent by

$$(5.3) \quad L(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\Omega} \log \left\| \prod_{n=1}^N \begin{pmatrix} V_\omega(N - n) - E & -1 \\ 1 & 0 \end{pmatrix} \right\| d\mu(\omega),$$

where the limit exists because of submultiplicativity of the matrix norm, which implies that the sequence

$$\frac{1}{N} \int_{\Omega} \log \left\| \prod_{n=1}^N \begin{pmatrix} V_\omega(N - n) - E & -1 \\ 1 & 0 \end{pmatrix} \right\| d\mu(\omega)$$

is subadditive. Furthermore, the following lemma was shown by Craig and Simon in [17].

Lemma 5.1. *The function $L(E)$ is subharmonic in E .*

We will mainly use the upper semicontinuity provided by this result. The next result will allow us to go from Green's function estimates to estimates for the Lyapunov exponent.

Lemma 5.2. *If*

$$(5.4) \quad |G_{\omega,\Lambda}(E, k, N)| \leq e^{-\gamma N}$$

for $\Lambda \in \{[0, N], [1, N]\}$, $k \in \{k_0 - 1, k_0\}$, then

$$(5.5) \quad \frac{1}{N} \log \|A_\omega(E, N)\| \geq \gamma - \frac{\log \sqrt{2}}{N}.$$

Proof. We first observe, that

$$A_\omega(E, n) = \begin{pmatrix} c_{\omega,E}(n) & s_{\omega,E}(n) \\ c_{\omega,E}(n-1) & s_{\omega,E}(n-1) \end{pmatrix},$$

where these solve

$$H_\omega c_{\omega,E} = E c_{\omega,E}, \quad H_\omega s_{\omega,E} = E s_{\omega,E},$$

with initial conditions

$$\begin{pmatrix} c_{\omega,E}(0) & s_{\omega,E}(0) \\ c_{\omega,E}(-1) & s_{\omega,E}(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We let $u_{\omega,E}$ be the solution of $H_\omega u_{\omega,E} = E u_{\omega,E}$, that satisfies $u_{\omega,E}(N) = 1$ and $u_{\omega,E}(N+1) = 0$. We then find for $x \leq y$, that

$$G_{\omega,[0,N]}(E, x, y) = \frac{c_{\omega,E}(x)u_{\omega,E}(y)}{W(c_{\omega,E}, u_{\omega,E})}, \quad G_{\omega,[1,N]}(E, x, y) = \frac{s_{\omega,E}(x)u_{\omega,E}(y)}{W(s_{\omega,E}, u_{\omega,E})},$$

where $W(u, v) = u(n+1)v(n) - u(n)v(n+1)$ is the Wronskian. One can check that $W(u, v)$ is independent of n if u and v solve $H_\omega u = E u$, $H_\omega v = E v$. Evaluating the Wronskian at N yields

$$W(c_{\omega,E}, u_{\omega,E}) = -c_{\omega,E}(N+1), \quad W(s_{\omega,E}, u_{\omega,E}) = -s_{\omega,E}(N+1).$$

Hence, we obtain the formulas

$$|G_{\omega,[0,N]}(E, x, N)| = \left| \frac{c_{\omega,E}(x)}{c_{\omega,E}(N+1)} \right|, \quad |G_{\omega,[1,N]}(E, x, N)| = \left| \frac{s_{\omega,E}(x)}{s_{\omega,E}(N+1)} \right|,$$

Since $\det(A_\omega(E, k_0)) = 1$, it follows that

$$\min(|c_{\omega,E}(k_0)|, |c_{\omega,E}(k_0-1)|, |s_{\omega,E}(k_0)|, |s_{\omega,E}(k_0-1)|) \geq \frac{1}{\sqrt{2}}.$$

Hence, we see that

$$\begin{aligned} \min_{k \in k_0-1, k_0, a \in \{0,1\}} |G_{\omega,[a,N]}(E, k, N)| &\geq \frac{1}{\sqrt{2}} \cdot \min \left(\frac{1}{|c_{\omega,E}(N+1)|}, \frac{1}{|s_{\omega,E}(N+1)|} \right) \\ &\geq \frac{1}{\sqrt{2}} \cdot \|A_\omega(E, N)\| \end{aligned}$$

taking logarithms and dividing by N implies the result. \square

This lemma will allow us to go from estimates on the Green's function to estimates on the Lyapunov exponent. One should furthermore observe, that in order to conclude in the general setting, that $L(E) > 0$, one would need information for all large N . However, in the ergodic setting one can relax this a little bit. By a Theorem of Craig and Simon, we have that

Theorem 5.3. *Introduce*

$$(5.6) \quad \bar{L}(E, \omega) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A_\omega(E, n)\|.$$

Then there exists $\Omega_{CS} \subseteq \Omega$ of measure $\mu(\Omega_{CS}) = 1$, such that

$$(5.7) \quad \bar{L}(E, \omega) \leq L(E)$$

for $\omega \in \Omega_{CS}$.

Proof. This is Theorem 2.3 in [17]. \square

We will call Ω_{CS} the Craig–Simon set. We note the following consequence

Lemma 5.4. *Suppose, we are given $\gamma > 0$, $\omega \in \Omega_{CS}$ and for $k \geq 1$ integers $n_k \rightarrow \infty$ such that*

$$(5.8) \quad |G_{\Lambda, \omega}(E, x, y)| \leq e^{-\gamma n_k}$$

for $\Lambda \in \{[0, n_k], [1, n_k]\}$, $x \in \{x_0, x_0 + 1\}$, some x_0 and $y \in \partial\Lambda$. Then

$$(5.9) \quad L(E) \geq \gamma.$$

Proof. By Lemma 5.2, we have that (5.8) implies that $\bar{L}(E) \geq \gamma$. Now, the claim follows from Theorem 5.3. \square

6. THE MULTISCALE STEP

In this section, we will begin with the exposition of our adaptation of multiscale analysis. For this, we will not work with an ergodic potential, but will assume that $\{V(n)\}_{n=0}^{N-1}$ is any real valued sequence of N numbers. We then define H as the corresponding Schrödinger operators on $\ell^2([0, N-1])$ and denote by H_Λ the restrictions to intervals $\Lambda \subseteq [0, N-1]$. This generality is mainly used to simplify the notation, and to make clear, when ergodicity enters.

Furthermore, since we do not make quantitative assumptions on the recurrence properties of $T : \Omega \rightarrow \Omega$, it is necessary to work in this section with intervals of varying length. However, this does not create major technical difficulties, since their boundary still consists of only two points.

We now start by defining our basic notion of a good sequence $\{V(n)\}_{n=0}^{N-1}$.

Definition 6.1. *Let $\delta > 0$, $0 < \sigma \leq \frac{1}{4}$, $\mathcal{E} \subseteq \mathbb{R}$ an interval, and $L \geq 1$.*

A sequence $\{V(n)\}_{n=0}^{N-1}$ is called $(\delta, \sigma, L, \mathcal{E})$ -critical, if there are integers

$$(6.1) \quad 0 \leq k_0 < k_1 < k_2 < k_3 < \cdots < k_L < k_{L+1} \leq N,$$

and a set $\mathcal{L} \subseteq [1, L]$ such that

$$(6.2) \quad \frac{\#\mathcal{L}}{L} \leq \sigma.$$

And for $l \notin \mathcal{L}$, we have that

$$(6.3) \quad |G_{[k_{l-1}+1, k_{l+1}-1]}(E, k_l, k_{l \pm 1} \mp 1)| \leq \frac{1}{2} e^{-\delta}$$

for $E \in \mathcal{E}$.

In order to state the next theorem, we have to explain a division of $\mathcal{E} = [E_0, E_1]$ into Q intervals of length $\approx e^{-\sigma\delta}$. Introduce $Q = \lceil (E_1 - E_0)e^{\sigma\delta} \rceil$, and

$$(6.4) \quad \mathcal{E}_q = \left[E_0 + q \frac{E_1 - E_0}{Q}, E_0 + (q+1) \frac{E_1 - E_0}{Q} \right],$$

for $q = 0, \dots, Q-1$. If

$$(6.5) \quad E_1 - E_0 \geq e^{-\sigma\delta}$$

holds, we have that

$$(6.6) \quad (E_1 - E_0)e^{\sigma\delta} \leq Q \leq 2(E_1 - E_0)e^{\sigma\delta}$$

and for all q

$$(6.7) \quad \frac{1}{2}e^{-\sigma\delta} \leq |\mathcal{E}_q| \leq e^{-\sigma\delta}.$$

The main result of this section will be

Theorem 6.2. *Assume that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$ -critical, $M \geq 3$,*

$$(6.8) \quad \frac{\sigma L}{M} \geq 2,$$

and $\sigma \leq \frac{1}{4}$. Introduce

$$(6.9) \quad \tilde{\sigma} = \frac{1}{2}\sigma$$

and

$$(6.10) \quad \tilde{\delta} = (1 - 2\sigma)M\delta.$$

Then there exists a set $\mathcal{Q} \subseteq [0, Q-1]$ and $\tilde{L} \geq 1$ such that

$$(6.11) \quad \#\mathcal{Q} \leq \frac{2^{15}}{\tilde{\sigma}} \left(\frac{(M+1)}{\sigma} \cdot \frac{N}{L} \right)^3$$

and

$$(6.12) \quad (1 - 2\sigma) \frac{L}{M+1} \leq \tilde{L} \leq \frac{L}{M+1}$$

and for $q \notin \mathcal{Q}$, we have that $\{V(n)\}_{n=0}^{N-1}$ is also $(\tilde{\delta}, \tilde{\sigma}, \tilde{L}, \mathcal{E}_q)$ -critical.

We observe that in our case $L \gtrsim N$, so (6.8) will be satisfied for all large enough N . The rest of this section is spent proving the above theorem.

We will now describe how we choose the sequence \tilde{k}_l given the integer $M \geq 1$ from Theorem 6.2. This will be the sequence, we check Definition 6.1 with. First pick

$$(6.13) \quad \tilde{k}_0 = k_0.$$

Now assume that we are given $\tilde{k}_s = k_{l_s}$ for $0 \leq s \leq j$, then we choose $\tilde{k}_{j+1} = k_{l_{j+1}}$ so that

$$(6.14) \quad \#\{l \notin \mathcal{L} : \tilde{k}_j < k_l < \tilde{k}_{j+1}\} = M.$$

This procedure stops once, we would have to choose $\tilde{k}_{j+1} > N$. We will call the maximal l so that \tilde{k}_{l+1} is defined \tilde{L} . This means that we have now defined

$$0 \leq \tilde{k}_0 < \tilde{k}_1 < \dots < \tilde{k}_{\tilde{L}} < \tilde{k}_{\tilde{L}+1} \leq N-1.$$

We have the following lemma

Lemma 6.3. *Assume $\sigma \frac{L}{M} \geq 2$, that is (6.8), then we have that*

$$(6.15) \quad \tilde{L} \geq (1 - 2\sigma) \frac{L}{M+1}.$$

Proof. By (6.2), we have that

$$\#([1, L] \setminus \mathcal{L}) \geq (1 - \sigma)L.$$

We observe now, that $l_{j+1} - l_j \geq M + 1$, and even

$$l_{j+1} - l_j = M + 1 + \#\{l \in \mathcal{L} : \tilde{k}_j < k_l < \tilde{k}_{j+1}\}.$$

Hence, we may choose

$$\tilde{L} \geq (1 - \sigma) \frac{L}{M+1} - 2$$

the claim now follows by $2 \leq \sigma \frac{L}{M+1}$. \square

We furthermore have the following estimate

Lemma 6.4. *Assume $\sigma \leq \frac{1}{4}$. Let*

$$(6.16) \quad \tilde{\mathcal{L}}_0 = \left\{ l : \tilde{k}_{l+1} - \tilde{k}_{l-1} \geq \frac{16N(M+1)}{\sigma L} \right\}$$

then, we have that

$$(6.17) \quad \frac{\#\tilde{\mathcal{L}}_0}{\tilde{L}} \leq \frac{1}{2} \tilde{\sigma}.$$

Proof. Since $0 \leq \tilde{k}_0 \leq \tilde{k}_{\tilde{L}+1} \leq N$, we have that

$$\sum_{l=1}^{\tilde{L}} (\tilde{k}_{l+1} - \tilde{k}_{l-1}) = \tilde{k}_{\tilde{L}+1} - \tilde{k}_0 + \tilde{k}_{\tilde{L}} - \tilde{k}_1 \leq 2N.$$

Now, Markov's inequality implies that

$$\#\tilde{\mathcal{L}}_0 \leq \left(\frac{1}{2} \cdot \frac{\sigma}{2} \right) \cdot \left(\frac{L}{2(M+1)} \right).$$

By (6.15) and $\sigma \leq \frac{1}{4}$, we have that $\frac{1}{L} \leq \frac{2(M+1)}{L}$. Now, the claim follows from $\tilde{\sigma} = \frac{\sigma}{2}$ and the above equation. \square

Before coming to the next lemma, we will first introduce the notion of non-resonance.

Definition 6.5. *Given an interval $I \subseteq [0, N-1]$, an energy interval \mathcal{E} , and $\varepsilon > 0$. $\{V(n)\}_{n=0}^{N-1}$ is called $(I, \mathcal{E}, \varepsilon)$ non-resonant, if for every $\Lambda \subseteq I$, we have that*

$$(6.18) \quad \text{dist}(E, \sigma(H_\Lambda)) \geq \varepsilon$$

for all $E \in \mathcal{E}$. Otherwise, $\{V(n)\}_{n=0}^{N-1}$ is called $(I, \mathcal{E}, \varepsilon)$ resonant.

Introduce the set \mathfrak{L}_q for $0 \leq q \leq Q$ by

$$(6.19) \quad \mathfrak{L}_q = \{1 \leq l \leq \tilde{L} : \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2e^{-\sigma\delta}) \text{ resonant}\}.$$

We will now discuss the size of this set.

Lemma 6.6. *There is a set \mathcal{Q} such that*

$$(6.20) \quad \#\mathcal{Q} \leq \frac{2^{15}}{\tilde{\sigma}} \left(\frac{N(M+1)}{\sigma L} \right)^3$$

and for $q \notin \mathcal{Q}$, we have that

$$(6.21) \quad \frac{\#\mathfrak{L}_q}{\tilde{L}} \leq \tilde{\sigma}.$$

The estimate on $\#\mathcal{Q}$ is not sharp. By a more careful analysis, the power in $\left(\frac{N(M+1)}{\sigma L}\right)^3$ could be lowered to $\left(\frac{N(M+1)}{\sigma L}\right)$. However, we have decided not to pursue this, since the overall improvement is minor. In order to achieve this, one has to make explicit in Lemma 6.7 for which intervals the non-resonance condition is being used, and only assume it for them.

Proof of Lemma 6.6. For l introduce

$$g(l) = \#\{q : \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2e^{-\sigma\delta}) \text{ resonant}\}.$$

We will now derive an upper bound on $g(l)$. First note that $\sigma(H_\Lambda)$ consists of $\#\Lambda$ elements, so

$$\bigcup_{\Lambda \subseteq [\tilde{k}_{l-1}, \tilde{k}_{l+1}]} \sigma(H_\Lambda)$$

consists of at most $(\tilde{k}_{l+1} - \tilde{k}_{l-1})^3$ elements. For each E in the above set, we have that its $2e^{-\sigma\delta}$ neighborhood can intersect at most 8 of the \mathcal{E}_q intervals. Thus, we have that

$$g(l) \leq 8(\tilde{k}_{l+1} - \tilde{k}_{l-1})^3.$$

In particular for $l \notin \tilde{\mathcal{L}}_0$, we have by (6.16) that

$$g(l) \leq 2^{15} \left(\frac{N(M+1)}{\sigma L} \right)^3.$$

Let $h(q) = \#\mathfrak{L}_q$, so that

$$h(q) \leq \#\{l \notin \tilde{\mathcal{L}}_0 : \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2e^{-\sigma\delta}) \text{ resonant}\}.$$

We obtain

$$\sum_{q=0}^{Q-1} h(q) \leq \sum_{l \notin \tilde{\mathcal{L}}_0} g(l) \leq 2^{15} \tilde{L} \left(\frac{N(M+1)}{\sigma L} \right)^3.$$

Let \mathcal{Q} be the set

$$\mathcal{Q} = \{q : h(q) \geq \tilde{\sigma} \tilde{L}\},$$

now the claim follows from Markov's inequality. \square

We observe that (6.18) implies that

$$(6.22) \quad \|(H_\Lambda - E)^{-1}\| \leq \frac{1}{2} e^{\sigma\delta}.$$

Lemma 6.7. *Assume for (l, q) that $\{V(n)\}_{n=0}^{N-1}$ is $([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2e^{-\sigma\delta})$ non-resonant, then*

$$(6.23) \quad |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, \tilde{k}_{l\pm 1} \mp 1)| \leq \frac{1}{2} e^{-\tilde{\delta}}$$

for $E \in \mathcal{E}_q$.

Proof. Let $x = \tilde{k}_{l\pm 1}$ (one of the two). Since (6.18), we have that

$$|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, x)| \leq \frac{1}{2}e^{\sigma\delta}.$$

By construction of \tilde{k}_l , we have sets \mathcal{J}_\pm such that for $j \in \mathcal{J}_\pm$ we have $[k_{j-1}, k_{j+1}] \subseteq [\tilde{k}_l, \tilde{k}_{l\pm 1}] \cup [\tilde{k}_{l\pm 1}, \tilde{k}_l]$. Furthermore, for $j \in \mathcal{J}_+ \cup \mathcal{J}_-$, we have that

$$|G_{[k_{j-1}+1, k_{j+1}-1]}(E, k_j, k_{j\pm 1} \mp 1)| \leq \frac{1}{2}e^{-\delta}$$

for $E \in \mathcal{E}_q \subseteq \mathcal{E}$.

By the resolvent equation, we find that

$$\begin{aligned} |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, x)| &\leq \frac{1}{2}e^{\sigma\delta} \left(|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_-}, x)| \right. \\ &\quad \left. + |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_+}, x)| \right), \end{aligned}$$

where $j_+ = \max(\mathcal{J}_+)$ and $j_- = \min(\mathcal{J}_-)$. Now, by the decay of the Green's function, we know that

$$\begin{aligned} |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, x)| &\leq \frac{1}{4}e^{-(1-\sigma)\delta} \left(|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_- - 1}, x)| \right. \\ &\quad + |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_- + 1}, x)| \\ &\quad + |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_+ - 1}, x)| \\ &\quad \left. + |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_+ + 1}, x)| \right). \end{aligned}$$

We may iterate this procedure $M = \#\mathcal{J}_+ = \#\mathcal{J}_-$ many times, proving the proposition by our choice of $\tilde{\delta}$.

We remark that if we have to estimate $|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j\pm 1} \mp 1, x)|$ with $j \notin \mathcal{L}$, we use (6.18) on the interval $[k_{j_0} - 1, k_{j_1} + 1]$, where $j_0 \leq j$, $j_1 \geq j$ are chosen minimal such that $j_0, j_1 \notin \mathcal{L}$. This is similar to the treatment of the original $[k_{j_-} + 1, k_{j_+} - 1]$ interval. \square

Proof of Theorem 6.2. We are essentially done. We observe, that for $q \notin \mathcal{Q}$, we can choose $\mathfrak{L} = \mathfrak{L}_q$, which satisfies

$$\frac{\#\mathfrak{L}}{\tilde{L}} \leq \tilde{\sigma},$$

by (6.21). Furthermore, we then have the estimate on the Green's function on $[\tilde{k}_{l-1}, \tilde{k}_{l+1}]$ by the last lemma for $l \notin \mathfrak{L}$. This finishes the proof that $\{V(n)\}_{n=0}^{N-1}$ is $(\tilde{\delta}, \tilde{\sigma}, \tilde{L}, \mathfrak{E}_q)$ -critical. \square

7. INDUCTIVE USE OF THE MULTISCALE STEP

In this section, we develop an inductive way to apply Theorem 6.2. This will lead in the following section to the proof of Theorem 3.2. A major part of this section is taken up by checking inequalities between various numerical quantities, necessary to show that everything converges.

Given numbers $\delta > 0$ and $0 < \sigma \leq \frac{1}{4}$, we will first introduce δ_j , σ_j , and M_j . Introduce $\delta_0 = \delta$ and

$$(7.1) \quad M_j = 100^{j+1}$$

$$(7.2) \quad \sigma_j = \frac{1}{2^j} \sigma$$

$$(7.3) \quad \delta_{j+1} = (1 - 2\sigma_j) M_j \delta_j.$$

This choice is motivated by (6.9) and (6.10). We first observe that

Lemma 7.1. *We have that*

$$(7.4) \quad \prod_{k=0}^j M_k = 10^{(j+1)(j+2)} = 10^{j^2} \cdot 1000^j \cdot 100$$

$$(7.5) \quad \delta_j \geq e^{-4\sigma} 10^{(j+1)(j+2)} \delta$$

$$(7.6) \quad \sigma_j \delta_j \geq e^{-4\sigma} 10^{j^2} 500^j 100 \sigma \delta.$$

Proof. For (7.4), observe that

$$\prod_{k=0}^j M_k = 100^{\sum_{k=0}^j (k+1)}$$

and $\sum_{k=0}^j (k+1) = \frac{(j+1)(j+2)}{2}$.

For (7.5), we have that $\delta_{j+1} = \prod_{k=1}^j (1 - \frac{\sigma}{2^k}) M_k \cdot \delta$, and since $\prod_{k=1}^j (1 - 2\frac{\sigma}{2^k}) \geq \prod_{k=1}^{\infty} (1 - 2\frac{\sigma}{2^k})$, we have that

$$\prod_{k=1}^j (1 - 2\frac{\sigma}{2^k}) \geq \exp\left(\sum_{k=1}^{\infty} \log(1 - 2\frac{\sigma}{2^k})\right).$$

Now using that $\log(1-x) \geq -2x$ for $0 < x < 1/2$, we have that $\sum_{j=1}^{\infty} \log(1 - 2\frac{\sigma}{2^j}) \geq -4\sigma \sum_{k=1}^{\infty} \frac{1}{2^k} = -4\sigma$ and thus the inequalities follow. \square

We let L_j be a sequence of numbers, that satisfies

$$(7.7) \quad (1 - 2\sigma_j) \frac{L_j}{M_j} \leq L_{j+1} \leq \frac{L_j}{M_j}.$$

This is motivated by (6.12).

Lemma 7.2. *The L_j satisfy*

$$(7.8) \quad e^{-4\sigma} e^{-\frac{1}{99}} L 10^{-(j+1)(j+2)} \leq L_{j+1} \leq L 10^{-(j+1)(j+2)}.$$

Proof. Recall from the last lemma that $\prod_{k=1}^j (1 - 2\sigma_k) \geq e^{-4\sigma}$. An iteration of (7.7) shows

$$\prod_{k=1}^j \frac{1 - 2\sigma_k}{M_k + 1} L_0 \leq L_{j+1} \leq \prod_{k=1}^j \frac{1}{M_k + 1} L_0.$$

Since

$$1 \geq \prod_{k=1}^j \frac{M_k}{M_k + 1} = \exp\left(-\sum_{k=1}^j \log\left(1 + \frac{1}{100^k}\right)\right) \geq \exp\left(-\frac{1}{99}\right),$$

we have that (7.4) implies the claim. \square

We define j_{max} by being the maximal j such that

$$(7.9) \quad \sigma_{j_{max}} L_{j_{max}} \geq 2M_{j_{max}}$$

holds. This is needed in order that we can satisfy (6.8) in Theorem 6.2. We have that

Lemma 7.3. *If σ stays fixed, then $\delta_{j_{max}} \rightarrow \infty$ as $L \rightarrow \infty$. Furthermore,*

$$(7.10) \quad \delta_{j_{max}} L_{j_{max}} \geq e^{-8\sigma} e^{-\frac{1}{99}} L\delta$$

Proof. We observe that (7.9) only depends on σ and L . Furthermore, if L becomes large, the restriction becomes less and less restrictive.

The second claim follows by (7.5) and (7.8) \square

We will now start by exploiting the multiscale step stated in Theorem 6.2. We will show

Theorem 7.4. *Assume that*

$$(7.11) \quad \frac{\sigma L}{M} \geq 2$$

$$(7.12) \quad |\mathcal{E}| \geq e^{-\frac{1}{25}\sigma\delta}$$

$$(7.13) \quad \frac{2^{17}e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \leq e^{\frac{8}{25}e^{-4\sigma}\sigma\delta}$$

hold and that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$ -critical, then there is $\mathcal{E}_0 \subseteq \mathcal{E}$ satisfying

$$(7.14) \quad \frac{|\mathcal{E}_0|}{|\mathcal{E}|} \geq \exp\left(-\frac{25}{4} \frac{e^{-\frac{8}{25}\sigma\delta}}{\sigma\delta \ln(50)}\right)$$

such that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta_{j_{max}}, \sigma_{j_{max}}, L_{j_{max}}, \mathcal{E}_0)$ -critical.

We will now start the proof of this theorem. The proof is based on induction. First observe, that by the assumption that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$ -critical, we have that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta_0, \sigma_0, L_0, \mathcal{E})$ -critical. This means that the base case is taken care of. The main problem with applying induction is that the interval \mathcal{E} will shrink with the induction procedure, that is why we will need to do something slightly more sophisticated. This motivates the following definition:

Definition 7.5. *Given $\{V(n)\}_{n=0}^{N-1}$. A collection of intervals $\{\mathcal{E}_q\}_{q=0}^Q$ is called (σ, δ, L) -acceptable if*

- (i) *For each q , we have that $\{V(n)\}_{n=0}^{N-1}$ is $(\sigma, \delta, L, \mathcal{E}_q)$ -critical*
- (ii) *For q, \tilde{q} , we have that $|\mathcal{E}_q| = |\mathcal{E}_{\tilde{q}}|$.*
- (iii) *We have that*

$$(7.15) \quad |\mathcal{E}_q| \geq e^{-\frac{1}{25}\sigma\delta}$$

for each q .

We first observe that $\{\mathcal{E}\}$ is $(\sigma_0, \delta_0, L_0)$ -acceptable, since we assume criticality and (7.12). This implies the following consequence of Theorem 6.2.

Lemma 7.6. *Given $\{V(n)\}_{n=0}^{N-1}$ and a collection of $(\sigma_j, \delta_j, L_j)$ -acceptable intervals $\{\mathcal{E}_q^j\}_{q=0}^{Q_j}$, then there exists a collection of intervals $\{\mathcal{E}_q^{j+1}\}_{q=0}^{Q_{j+1}}$ that is $(\sigma_{j+1}, \delta_{j+1}, L_{j+1})$ -acceptable.*

Proof. All but condition (iii) of Definition 7.5 are direct consequences of Theorem 6.2. For (iii) observe that (6.7) implies that

$$|\mathcal{E}_q^{j+1}| \geq e^{-\sigma_j \delta_j}$$

for any q . Now, observe that since $0 < \sigma_j \leq \frac{1}{4}$ and $M_j \geq 100$, we have that

$$\sigma_{j+1} \delta_{j+1} = \frac{1}{2} \sigma_j (1 - 2\sigma_j) M_j \delta_j \leq 25 \sigma_j \delta_j.$$

So the claim follows. \square

It remains to compare the size of

$$\bigcup_{q=0}^{Q_j} \mathcal{E}_q^j \text{ and } \bigcup_{q=0}^{Q_{j+1}} \mathcal{E}_q^{j+1}.$$

For this, we will first need the following lemma.

Lemma 7.7. *Assume (7.13), then we have that*

$$(7.16) \quad 10^{3(j+1)(j+2)} \leq e^{\frac{8}{25} \sigma_j \delta_j}$$

$$(7.17) \quad \frac{2^{17} e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \leq e^{\frac{8}{25} \sigma_j \delta_j}.$$

Proof. Since $(j+1)(j+2) \leq 50^j$, these inequalities follow from

$$10^3 \leq e^{\frac{8}{25} \sigma \delta e^{-4\sigma}} \text{ and } \frac{2^{17} e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \leq e^{\frac{8}{25} e^{-4\sigma} \sigma \delta}$$

By $N \geq L$ and $0 < \sigma \leq \frac{1}{4}$, we have that

$$10^3 \leq 2^{25} \leq \frac{2^{17} e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3$$

so both of the above equations follow from (7.13). \square

The next lemma, will allow us to compare the size of an interval \mathcal{E}_q^j to the size of the intervals \mathcal{E}_p^{j+1} contained in \mathcal{E}_q^j .

Lemma 7.8. *We have that*

$$(7.18) \quad \frac{1}{|\mathcal{E}_q^j|} \cdot \left| \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| \geq 1 - e^{-\frac{8}{25} \sigma_j \delta_j}.$$

Proof. By (6.11), we have that

$$\left| \mathcal{E}_q^j \setminus \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| \leq \frac{2^{17} 100^3 N^3}{\sigma_j^4 L_j^3} \cdot |\mathcal{E}_s^{j+1}|.$$

By construction, we have that (6.7) holds, that is $|\mathcal{E}_s^{j+1}| \leq e^{-\sigma_j \delta_j}$. Hence, we obtain that

$$\left| \mathcal{E}_q^j \setminus \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| \leq \frac{2^{17} e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \cdot 10^{3(j+1)(j+2)} \cdot e^{-\sigma_j \delta_j}.$$

Since, we have that $|\mathcal{E}_q^j| \geq e^{-\frac{1}{25}\sigma_j\delta_j}$, we obtain that

$$\begin{aligned} \frac{1}{|\mathcal{E}_q^j|} \cdot \left| \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| &\geq 1 - \frac{2^{17}e^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \cdot 10^{3(j+1)(j+2)} \cdot e^{-\frac{24}{25}\sigma_j\delta_j} \\ &\geq 1 - e^{-\frac{8}{25}\sigma_j\delta_j}, \end{aligned}$$

where we used (7.16) and (7.17). This finishes the proof. \square

We now come to

Lemma 7.9. *We have that*

$$(7.19) \quad \left| \bigcup_{q=0}^{Q_{j+1}} \mathcal{E}_q^{j+1} \right| \geq \left| \bigcup_{q=0}^{Q_j} \mathcal{E}_q^j \right| \cdot (1 - e^{-\frac{8}{25}\sigma_j\delta_j}).$$

Proof. This is a consequence of the last lemma. \square

Proof of Theorem 7.4. By the previous discussion, we can choose \mathcal{E}_0 such that

$$|\mathcal{E}_0| \geq \prod_{j=1}^{\infty} (1 - e^{-\frac{8}{25}\sigma_j\delta_j}) |\mathcal{E}|.$$

Using (7.6) and $\log(1-x) \geq -2x$, we find that

$$|\mathcal{E}_0| \geq \exp\left(-2 \sum_{j=1}^{\infty} e^{-\frac{8}{25}\sigma\delta} e^{-4\sigma} 50^j\right) \geq \exp\left(-2 \frac{e^{-\frac{8}{25}\sigma\delta}}{\frac{8}{25}\sigma\delta \ln(50)}\right),$$

since $\sum_{j=1}^{\infty} e^{-ta^j} \leq \frac{e^{-t}}{\ln(a)t}$. \square

8. PROOF OF THEOREM 3.2

We begin by observing that (3.5) implies that, for L large enough $\{V(n)\}_{n=0}^{LK-1}$ is $(\delta, \sigma, L, \mathcal{E})$ -critical, $\delta = \gamma K$ in the sense of Definition 6.1. To see this, choose $k_j = jK$, and \mathcal{L} as the complement of the set in (3.5). The rest follows. We now use the mechanism of the last two sections to improve the estimate.

Lemma 8.1. *$\{V(n)\}_{n=0}^{LK-1}$ will be $(\hat{\delta}, \hat{\sigma}, \hat{L}, \hat{\mathcal{E}})$ -critical, where $\hat{\mathcal{E}} \subseteq \mathcal{E}$ satisfies*

$$(8.1) \quad \frac{|\hat{\mathcal{E}}|}{|\mathcal{E}|} \geq \exp\left(-\frac{25}{4} \frac{e^{-\frac{8}{25}\sigma\delta}}{\sigma\delta \ln(50)}\right)$$

and by Lemma 7.3, we have that

$$(8.2) \quad \hat{\delta}\hat{L} \geq e^{-8\sigma - \frac{1}{95}\gamma K} \cdot L.$$

Proof. Since $\{V_\omega(n)\}_{n=0}^{N_t-1}$ is $(\delta, \sigma, L_t, \mathcal{E})$ -critical, we now wish to apply Theorem 7.4 to improve this estimate. In order to do this, we still have to ensure that (7.11), (7.12) (7.13) hold. (3.6) implies (7.12). (7.13) is implied by (3.7). For (7.11) observe that it is satisfied if L is large enough. \square

We now come to

Lemma 8.2. *We may choose the set $\widehat{\mathcal{E}}$ even so, that we have for every $\Lambda \subseteq [0, LK - 1]$, that for every $E \in \widehat{\mathcal{E}}$, we have*

$$(8.3) \quad \text{dist}(H, \sigma(H_\Lambda)) \geq e^{-\widehat{\sigma}\widehat{\delta}}.$$

Proof. This follows by an inspection of the argument of the last section. \square

Now repeating the argument to obtain Green's function estimates as done in Lemma 6.7, we obtain the estimates required by Lemma 5.2. Hence, we obtain that

$$(8.4) \quad \frac{1}{LK} \log \|A(E, LK)\| \geq e^{-8\sigma} e^{-\frac{1}{99}\gamma} - \frac{\sqrt{2}}{LK}$$

for $E \in \widehat{\mathcal{E}}$. This finishes the proof of Theorem 3.2, using that $e^{-x} \geq 1 - x$ for $x \geq 1$.

9. PROOF OF THEOREM 3.3

We first need the following observation.

Lemma 9.1. *There exists $\omega \in \Omega$, such that the following properties hold*

(i) *We have that*

$$(9.1) \quad L(E) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A_\omega(E, n)\|$$

for all E .

(ii) *There are sequences $N_t, L_t \rightarrow \infty$ such that $\{V_\omega(n)\}_{n=0}^{N_t-1}$ is $(\gamma K, \sigma, L_t, \mathcal{E})$ -critical and*

$$(9.2) \quad \lim_{t \rightarrow \infty} \frac{N_t}{L_t} = K.$$

Proof. We let Ω_{CS} be the set from Theorem 5.3. This implies that property (i) holds as long as $\omega \in \Omega_{CS}$. Furthermore, we have that $\mu(\Omega_{CS}) = 1$.

We let Ω_g be the complement of the set in (3.11). By Lemma 4.1, we can find a set $\widetilde{\Omega}$ with $\mu(\widetilde{\Omega}) > 0$, and for each $\omega \in \widetilde{\Omega}$ sequences $N_t, L_t \rightarrow \infty$ such that property (ii) holds.

So we have that $\Omega_{CS} \cap \widetilde{\Omega}$ is non-empty and by choosing $\omega \in \Omega_{CS} \cap \widetilde{\Omega}$, we are done. \square

We now fix ω as in the last lemma, and abbreviate

$$(9.3) \quad V(n) = V_\omega(n).$$

The claim now follows by repeating the arguments of the last section of the proof of Theorem 3.2. Giving more details, we obtain a sequence of sets \mathcal{E}_t , satisfying

$$|\mathcal{E}_t| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|$$

and for $E \in \mathcal{E}_t$, we have

$$\frac{1}{N_t} \log \|A(E, N_t)\| \geq e^{-8\sigma} e^{-\frac{1}{99}\gamma} + o(1)$$

as $t \rightarrow \infty$. Hence, we have that

$$L(E) \geq e^{-8\sigma} e^{-\frac{1}{99}\gamma}$$

for

$$E \in \mathfrak{E} = \bigcap_{s \geq 1} \bigcup_{t \geq s} \mathcal{E}_t.$$

We have that

Lemma 9.2. *The set $\mathfrak{E} = \bigcap_{s \geq 1} \bigcup_{t \geq s} \mathcal{E}_t$ has measure*

$$(9.4) \quad |\mathfrak{E}| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|.$$

Proof. Let $\mathfrak{E}_s = \bigcup_{t \geq s} \mathcal{E}_t$. We have that $\mathfrak{E}_{s+1} \subseteq \mathfrak{E}_s$ and $|\mathfrak{E}_s| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|$. This implies the claim, since $\mathfrak{E}_s \subseteq \mathcal{E}$ with $|\mathcal{E}| < \infty$. \square

This finishes the proof of Theorem 3.3.

10. THE INITIAL CONDITION AT LARGE COUPLING: PROOF OF PROPOSITION 3.4

In this section, we will discuss how our initial conditions can be verified for large λ . We let (Ω, μ) be a probability space and $T : \Omega \rightarrow \Omega$ an ergodic transformation (measure preserving is enough for the purpose of this section). Given a function $f : \Omega \rightarrow \mathbb{R}$ and $\lambda > 0$, we introduce our potential by

$$(10.1) \quad V_{\omega, \lambda}(n) = \lambda f(T^n \omega),$$

where $\omega \in \Omega$. We will assume that $f : \Omega \rightarrow \mathbb{R}$ is non-degenerate in the sense of Definition 2.2. That is, there are $F, \alpha > 0$ such that for all $E \in \mathbb{R}$

$$(10.2) \quad \mu(\{\omega \in \Omega : |f(x) - E| \leq \varepsilon\}) \leq F\varepsilon^\alpha.$$

Before coming to the proof of Proposition 3.4, We first recall the Combes–Thomas estimate (see [16])

Lemma 10.1. *Let $\Lambda \subseteq \mathbb{Z}$, $V : \Lambda \rightarrow \mathbb{R}$ be a bounded sequence, and $H : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ be defined by its action on $u \in \ell^2(\Lambda)$ by*

$$(10.3) \quad Hu(n) = u(n+1) + u(n-1) + V(n)u(n)$$

for $n \in \Lambda$ (where we set $u(n) = 0$ for $n \notin \Lambda$). Assume that $\text{dist}(\sigma(H), E) > \delta$. Let

$$(10.4) \quad \gamma = \frac{1}{2} \log(1 + \frac{\delta}{4}), \quad K = \frac{1}{\gamma} \log(\frac{4}{3\delta}).$$

Then for $k, l \in \Lambda$, $|k - l| \geq K$, the estimate

$$(10.5) \quad |G(E, k, l)| \leq \frac{1}{2} e^{-\gamma|k-l|}$$

holds.

We start by observing the following lemma.

Lemma 10.2. *Let f be a non-degenerate function, $K \geq 1$, $B > 0$. Then for $E \in \mathbb{R}$, the set*

$$(10.6) \quad A_{K, B}(E) = \{\omega \in \Omega : |f(T^k \omega) - E| \geq B, k = 0, \dots, K-1\}$$

has measure

$$(10.7) \quad \mu(A_{K, B}(E)) \geq 1 - B^\alpha FK.$$

Proof. By (10.2), the set

$$A_B(E) = \{\omega \in \Omega : |f(\omega) - E| < B\}$$

has measure $\mu(A_B(E)) \leq B^\alpha F$. Since

$$A = \Omega \setminus \left(\bigcup_{k=0}^{K-1} T^{-k} A_B(E) \right)$$

the claim follows and $T : \Omega \rightarrow \Omega$ being measure preserving. \square

This implies

Lemma 10.3. *Let (Ω, μ, T, f) be as above. Let $E_0 \in \mathbb{R}$ and $\sigma > 0$. Introduce*

$$(10.8) \quad K(\lambda) = \left\lfloor \frac{\sigma \lambda^{\alpha/2}}{F} \right\rfloor.$$

Then there is a set A of measure $\mu(A) \geq 1 - \frac{1}{2}\sigma$ such that for $\omega \in A$, we have that

$$(10.9) \quad |\lambda f(T^k \omega) - E_0| > \sqrt{\lambda},$$

for $k = 0, \dots, K(\lambda)$.

Proof. Letting $B = \frac{1}{\sqrt{\lambda}}$ in the last lemma, we obtain that the set $A_{K,B}(\frac{1}{\lambda}E_0)$ has measure $\mu(A_{K,B}(E)) \geq 1 - \frac{FK}{\lambda^{\alpha/2}}$. We have $\mu(A_{K,B}(E)) \geq 1 - \frac{1}{2}\sigma$ as long as $\frac{FK}{\lambda^{\alpha/2}} \leq \frac{1}{2}\sigma$. Hence the claim follows. \square

We are now ready for

Proof of Proposition 3.4. By Lemma 10.3, we obtain $A \subseteq \Omega$ of measure $\mu(A) \geq 1 - \frac{1}{2}\sigma$ and such that

$$\text{dist}(\mathcal{E}, \sigma(H_{\omega, [0, M-1]})) \geq \sqrt{\lambda} - 3 > \frac{1}{2}\sqrt{\lambda}$$

for $\omega \in A$ (Here we used $\lambda > 36$). We choose $M = 2K - 2$. We may thus apply the Combes–Thomas estimate (Lemma 10.1) to obtain that

$$|G_{T^{-1}\omega, [1, 2K-2]}(E, K, l)| \leq \frac{1}{2}e^{-\gamma M}$$

for $l \in \{1, 2K - 1\}$. Hence, we see that $[1, 2K - 2]$ is (γ, \mathcal{E}) -good for $H_{T^{-1}\omega}$ in the sense of Definition 3.1. This finishes the proof. \square

11. THE PASTUR–FIGOTIN FORMALISM AND PROOF OF PROPOSITION 3.6

In this section, we will prove Proposition 3.6. For this we develop the Pastur–Figotin formalism from [29] as it was improved by Chulaevsky and Spencer in [15] and later in Bourgain and Schlag [13], and then use it to prove large deviation estimates for matrix elements of the Green’s function. We will denote by H the operator defined in (3.21)

We will begin by introducing Prüfer variables. Define $\rho(n)$, $\varphi(n)$ for a solution u of $Hu = 2 \cos(\kappa)u$ by

$$(11.1) \quad \begin{aligned} \rho(n) \sin(\varphi(n)) &= \sin(\kappa)u(n-1) \\ \rho(n) \cos(\varphi(n)) &= u(n) - \cos(\kappa)u(n-1). \end{aligned}$$

This implies the following lemma, after a bit of computation.

Lemma 11.1. *We let u be the solution of $Hu = 2 \cos(\kappa)u$, with initial conditions*

$$(11.2) \quad u(0) = \frac{\sin(\theta)}{\sin(\kappa)}\rho(1), \quad u(1) = \cos(\theta) - \frac{\cos(\kappa)}{\sin(\kappa)}\sin(\theta)\rho(1).$$

We have that

$$(11.3) \quad \min(|u(n-1)|, |u(n)|) \leq \frac{1}{\sqrt{1 - |\cos(\kappa)|}} \rho_\theta(n)^2,$$

$$(11.4) \quad \max(|u(n-1)|, |u(n)|) \geq \frac{1}{2} \rho_\theta(n).$$

In the following, we will fix $\kappa \in (0, \pi) \setminus \{\pi/2\}$ and let $\rho_\theta, \varphi_\theta$ denote the Prüfer variables with initial condition (11.2). We will prove the following proposition in the next section.

Proposition 11.2. *Assume the following inequalities*

$$(11.5) \quad N \geq \frac{344 \cdot \sigma_2}{|\sin(\kappa) \cos(\kappa)| \cdot \min(1, 2|\cos(\kappa)^2 - \sin(\kappa)^2|)}$$

$$(11.6) \quad \lambda \leq |\sin(\kappa)| \min\left(\frac{\sigma_2}{7000}, \frac{|\sin(\kappa) \cos(\kappa)| \cdot \min(1, 2|\cos(\kappa)^2 - \sin(\kappa)^2|)}{1032 \cdot \sigma_2}\right)$$

Introduce

$$(11.7) \quad \gamma_1 = \frac{\sigma_2 \lambda^2}{8 \sin(\kappa)^2}.$$

We have that

$$(11.8) \quad \nu^{\otimes N} \left(\left\{ \underline{V} \mid \frac{1}{N} \log(\rho_\theta(N)) - \gamma_1 \geq \frac{1}{6} \gamma_1 \right\} \right) \leq \frac{2400}{N} \cdot \frac{\sigma_4}{(\sigma_2)^2} + 3e^{-\frac{\gamma_1^2 N}{80000}}.$$

We now begin deriving consequences of the last proposition.

Lemma 11.3. *Assume (3.24), (3.25), and (3.26) then*

$$(11.9) \quad \nu^{\otimes 2K} \left(\left\{ \underline{V} : \sup_{M \in \{2K-3, 2K-2\}} |\det(H_{\underline{V}, [1, M]} - E)| \leq \frac{e^{\frac{5}{3} \gamma_1 K}}{\sqrt{1 - |\cos(\kappa)|}} \right\} \right) \leq \frac{1}{48}$$

$$(11.10) \quad \nu^{\otimes 2K} \left(\left\{ \underline{V} : |\det(H_{\underline{V}, [1, K-1]} - E)| \geq \frac{1}{2} e^{\frac{7}{6} \gamma_1 K} \right\} \right) \leq \frac{1}{48}$$

$$(11.11) \quad \nu^{\otimes 2K} \left(\left\{ \underline{V} : \sup_{M \in \{2K-3, 2K-2\}} |\det(H_{\underline{V}, [K+1, M]} - E)| \geq \frac{1}{2} e^{\frac{7}{6} \gamma_1 K} \right\} \right) \leq \frac{1}{48}.$$

hold.

Proof. Observe that (3.24) implies (11.5) with $N = K/2$. We need to make a few observations. First, if we choose θ depending on v_0 , we can still apply the above estimates to \underline{V}' such that $\underline{V} = (v_0, \underline{V}')$. Next, we may choose $\theta = \theta(v_0)$ in such a way that

$$u(n) = \det(H_{\underline{V}, [1, n]} - E)$$

for $n \geq 1$. We do this and obtain by (11.3) that

$$\sup(|\det(H_{\underline{V},[1,2K-3]} - E)|, |\det(H_{\underline{V},[1,2K-2]} - E)|) \leq \frac{1}{\sqrt{2(1 - |\cos(\kappa)|)}} \rho_\theta(2K-2).$$

Hence, we apply (11.8) with $N = 2K - 3$ for (11.9). The claim now follows by a sequence of computations. (11.10) and (11.11) are similar, but we need $N = K - 3$. So since, we assume $K \geq 6$, we have $N \geq K/2$, which is exactly our assumption. \square

We need the following lemma

Lemma 11.4. *Assume that the potential $V(n)$ is bounded by $C > 0$ and $H_{[0,M-1]}$ acts on $\ell^2([0, M-1])$, then for $|E| \leq 2 + C$ we have that*

$$(11.12) \quad \|(H_{[0,M]} - E)^{-1}\|_{\text{HS}} \leq \frac{M(4 + 2C)^{M/2}}{|\det(H_{[0,M]} - E)|}.$$

Proof. By Cramer's rule, we have that

$$\|(H_{[0,M]} - E)\|_{\text{HS}}^2 = \frac{2}{|\det(H_{[0,M]} - E)|^2} \left(\sum_{0 \leq j < k \leq M} |\det(H_{[0,j-1]} - E)|^2 \cdot |\det(H_{[k+1,M]} - E)|^2 \right).$$

By Hadamard's inequality, we have $|\det(H_{[x,y]} - E)|^2 \leq \prod_{i=x}^y (2 + |V(i) - E|^2) \leq (4 + 2C)^{y-x+1}$. Thus

$$\|(H_{[0,M]} - E)^{-1}\|_{\text{HS}}^2 = \frac{M^2(4 + 2C)^M}{|\det(H_{[0,M]} - E)|^2}.$$

This implies the claim. \square

Now we are ready for

Proof of Proposition 3.6. By (11.9), we can choose $M \in \{2K - 3, 2K - 2\}$ and \underline{V} in a set of measure $1 - \frac{1}{48}$ such that

$$|\det(H_{\underline{V},[1,M]} - E)| \geq \frac{1}{\sqrt{1 - |\cos(\kappa)|}} e^{\frac{5}{3}\gamma_1 \cdot K}.$$

By Cramer's rule, we have that

$$|G_{\underline{V},[1,M]}(E, 1, K)| = \frac{|\det(H_{\underline{V},[K+1,M]} - E)|}{|\det(H_{\underline{V},[1,M]} - E)|}$$

and

$$|G_{\underline{V},[1,M]}(E, M, K)| = \frac{|\det(H_{\underline{V},[1,K-1]} - E)|}{|\det(H_{\underline{V},[1,M]} - E)|}.$$

These imply the first two inequalities. The third follows from (11.12). \square

12. PROOF OF PROPOSITION 11.2

Let $\varphi(n)$ and $\rho(n)$ be as defined in (11.1). Introduce

$$(12.1) \quad \zeta(n) = e^{2i\varphi(n)}, \quad \mu = e^{2i\kappa}.$$

We have that (see [25], [26])

Lemma 12.1. *The next equations hold*

$$(12.2) \quad \zeta(n+1) = \mu\zeta(n) + \frac{i\lambda}{2} \frac{V(n)}{\sin(\kappa)} \frac{(\mu\zeta(n) - 1)^2}{1 - \frac{i\lambda}{2} \frac{V(n)}{\sin(\kappa)} (\mu\zeta(n) - 1)},$$

$$(12.3) \quad \frac{\rho(n+1)^2}{\rho(n)^2} = 1 + \frac{\lambda}{2} \frac{V(n)}{\sin(\kappa)} (\zeta(n)\mu - \overline{\zeta(n)\mu}) \\ + \frac{\lambda^2}{2} \left(\frac{V(n)}{\sin(\kappa)} \right)^2 (\zeta(n)\mu - 2 + \overline{\zeta(n)\mu}).$$

Here \bar{z} denotes the complex conjugate.

We start by verifying an inequality

Lemma 12.2. *Assume the inequalities (11.5) and (11.6), then*

$$(12.4) \quad \frac{\sigma_2}{\min(|1-\mu|, |1-\mu^2|)} \left(\frac{2}{N} + \frac{6\lambda}{|\sin(\kappa)|} \right) \leq \frac{1}{172}$$

holds.

Proof. Observe that

$$|1-\mu| \geq |\operatorname{Im}(\mu)| = |\sin(2\kappa)| = 2|\sin(\kappa)||\cos(\kappa)| \\ |1-\mu^2| \geq |\operatorname{Im}(\mu^2)| = |\sin(4\kappa)| = 4|\sin(\kappa)||\cos(\kappa)||\cos(\kappa)^2 - \sin(\kappa)^2|.$$

Now the claim is a quick computation. \square

We are now ready for

Lemma 12.3. *Assume (11.6) and (11.5), then*

$$(12.5) \quad \left| \sum_{n=1}^N \zeta(n) \right| \leq \frac{1}{172} \frac{N}{\sigma_2}, \quad \left| \sum_{n=1}^N \zeta(n)^2 \right| \leq \frac{1}{172} \frac{N}{\sigma_2}$$

hold.

Proof. First, (12.2) implies that $|\zeta(n+1) - \mu\zeta(n)| \leq 3 \frac{\lambda}{|\sin(\kappa)|}$, and since

$$\zeta(n+1)^2 - \mu^2\zeta(n)^2 = \zeta(n+1)(\zeta(n+1) - \mu\zeta(n)) + \mu\zeta(n)(\zeta(n+1) - \mu\zeta(n)),$$

also $|\zeta(n+1)^2 - \mu^2\zeta(n)^2| \leq 6 \frac{\lambda}{|\sin(\kappa)|}$. Hence from $\sum_{n=1}^N \zeta(n) = \zeta(1) + \sum_{n=1}^{N-1} \zeta(n+1)$, we obtain

$$\left| (1-\mu) \sum_{n=1}^N \zeta(n) + \zeta(1) + \mu\zeta(N) \right| \leq 3N \frac{\lambda}{|\sin \kappa|}$$

This implies (12.5) by the last lemma. \square

We will now suppose that for some $\theta \in [0, \pi)$, we consider the solution to (12.2) and (12.3) satisfying the initial conditions

$$(12.6) \quad \zeta(0) = e^{2i\theta}, \quad \rho(0) = 1.$$

In order to highlight the dependence on θ , we will sometimes write $\zeta_\theta(n)$ and $\rho_\theta(n)$. Introduce the following terms

$$(12.7) \quad \mathcal{F}_1(\theta, \underline{V}, N) = \frac{\lambda^2}{8N \sin(\kappa)^2} \sum_{n=1}^N V(n)^2$$

$$(12.8) \quad \mathcal{F}_2(\theta, \underline{V}, N) = \frac{\lambda}{4N \sin(\kappa)} \sum_{n=1}^N V(n) (\zeta_\theta(n)\mu - \overline{\zeta_\theta(n)\mu})$$

$$(12.9) \quad \mathcal{F}_3(\theta, \underline{V}, N) = -\frac{\lambda^2}{8N \sin(\kappa)^2} \sum_{n=1}^N V(n)^2 (\zeta_\theta(n)\mu + \overline{\zeta_\theta(n)\mu})$$

$$(12.10) \quad \mathcal{F}_4(\theta, \underline{V}, N) = \frac{\lambda^2}{16N \sin(\kappa)^2} \sum_{n=1}^N V(n)^2 ((\zeta_\theta(n)\mu)^2 + (\overline{\zeta_\theta(n)\mu})^2).$$

We furthermore introduce

$$(12.11) \quad \mathcal{F}(\theta, \underline{V}, N) = \mathcal{F}_1(\theta, \underline{V}, N) + \dots + \mathcal{F}_4(\theta, \underline{V}, N).$$

We obtain the following lemma

Lemma 12.4. *Assume (11.6). For any $\theta \in [0, \pi)$, we have that*

$$(12.12) \quad \left| \frac{1}{N} \log(\rho_\theta(N)) - \mathcal{F}(\theta, \underline{V}, N) \right| \leq \frac{\gamma_1}{12}$$

Proof. Let

$$x(n) = \frac{\lambda V(n)}{2 \sin(\kappa)} (\zeta(n)\mu - \overline{\zeta(n)\mu}) + \frac{(\lambda V(n))^2}{2 \sin(\kappa)^2} (\zeta(n)\mu - 2 + \overline{\zeta(n)\mu}),$$

so $|x(n)| \leq \frac{3\lambda}{|\sin(\kappa)|} \leq \frac{1}{2}$ and by (12.3) $\frac{\rho(n+1)^2}{\rho(n)^2} = 1 + x(n)$. Since $\rho(1) = 1$, we have that $\log(\rho_N(\theta)) = \sum_{n=1}^N \log(1 + x(n))$. Using that $|\log(1 + x) - x + \frac{x^2}{2}| \leq \frac{|x|^3}{3(1-x)^3}$, and $|x(n)| \leq \frac{1}{2}$, we find

$$\left| \log(1 + x) - x + \frac{x^2}{2} \right| \leq \frac{8}{3} |x|^3,$$

and the claim follows, by expanding the terms and comparing them. \square

We next have that

Lemma 12.5. *We have that*

$$(12.13) \quad \nu^{\otimes N}(\{\underline{V} : |\mathcal{F}_1 - \gamma_1| \geq \frac{1}{48} \gamma_1\}) \leq \frac{2400}{N} \cdot \frac{\sigma_4}{(\sigma_2)^2}.$$

Proof. One can compute that $\int \mathcal{F}_1 d\nu^{\otimes N} = \frac{\lambda^2 \sigma_2}{8 \sin(\kappa)^2}$ and

$$\int \left(\mathcal{F}_1 - \frac{\lambda^2 \sigma_2}{8} \right)^2 d\nu^{\otimes N} = \frac{1}{N} \frac{\lambda^4 \sigma_4}{64 \sin(\kappa)^4}.$$

The claim then follows by Chebychev's inequality. \square

We will need the following result, which is Azuma's Inequality (Theorem 7.2.1. in Alon and Spencer [1]).

Theorem 12.6. *Let $X_1, X_2, \dots, X_N : [-1, 1]^N \rightarrow \mathbb{R}$ be functions satisfying the following three conditions:*

- (i) X_n only depends on V_1, \dots, V_n .
- (ii) $|X_n| \leq 1$.
- (iii) $\int_{[-1, 1]} X_n(V_1, \dots, V_{n-1}, V_n) d\nu(V_n) = 0$ for any $V_1, \dots, V_{n-1} \in [-1, 1]$.

Then

$$(12.14) \quad \nu^{\otimes N}(\{\underline{V} \in [-1, 1]^N : \left| \sum_{n=1}^N X_n(\underline{V}) \right| \geq \lambda \sqrt{N}\}) \leq e^{-\frac{1}{2} \lambda^2}.$$

We note that properties (i) - (iii) imply that X_1, \dots, X_N form a martingale.

Lemma 12.7. *We have that*

$$(12.15) \quad \nu^{\otimes N}(\{\underline{V} \in [-1, 1]^N : |\mathcal{F}_2| \geq \frac{1}{48} \gamma_1\}) \leq e^{-\frac{1}{4800} \gamma_1^2 N}.$$

Proof. In view of the definition of \mathcal{F}_2 , we introduce

$$X_n = \frac{\lambda}{4} \frac{V(n)}{\sin(\kappa)} (\zeta_\theta(n)\mu - \overline{\zeta_\theta(n)\mu}),$$

so that $\mathcal{F}_2 = \frac{1}{N} \sum_{n=1}^N X_n$. By (12.2), we have that $\zeta_\theta(n)\mu - \overline{\zeta_\theta(n)\mu}$ only depends on $V(1), \dots, V(n-1)$. Hence, we see that $\int X_n d\nu(V_n) = 0$, since $\int x d\nu = 0$. The other conditions of Theorem 12.6 are straightforward to check, and the result follows. \square

Lemma 12.8. *We have that*

$$(12.16) \quad \nu^{\otimes N}(\{\underline{V} \in [-1, 1]^N : |\mathcal{F}_j| \geq \frac{1}{48} \gamma_1\}) \leq e^{-\frac{1}{80000} \gamma_1^2 N}.$$

for $j = 3, 4$

Proof. Introduce

$$F_3 = -\frac{\lambda^2}{8N} \sum_{n=1}^N \left(\frac{V(n)}{\sin(\kappa)} \right)^2 \zeta(n)\mu$$

so that $\mathcal{F}_3 = F_3 + \overline{F_3}$. Now, decompose

$$F_3 = -\frac{\lambda^2}{8N \sin(\kappa)^2} \sum_{n=1}^N (V(n)^2 - \sigma_2^2) \zeta(n)\mu - \frac{\lambda^2 \sigma_2^2 \mu}{8N \sin(\kappa)^2} \sum_{n=1}^N \zeta(n).$$

We first observe that by (12.5), we have that

$$\left| \frac{\lambda^2 \sigma_2^2 \mu}{8N \sin(\kappa)^2} \sum_{n=1}^N \zeta(n) \right| \leq \frac{1}{172} \frac{\lambda^2 \sigma_2}{8 \sin(\kappa)^2} = \frac{\gamma_1}{172}.$$

Introduce $X_n = \frac{\lambda^2}{8} (V_n^2 - \sigma_2^2) \zeta(n)$, such that

$$|\mathcal{F}_3 - \frac{1}{N} \sum_{n=1}^N (X_n + \overline{X_n})| \leq \frac{\gamma_1}{96}.$$

Next, we observe that X_n obeys the condition of Theorem 12.6, and we can conclude that

$$\nu^{\otimes N}(\{\underline{V} \in [-1, 1]^N : \left| \frac{1}{N} \sum_{n=1}^N X_n \right| \geq \frac{\gamma_1}{172}\}) \leq e^{-\frac{1}{2}(\frac{\gamma_1}{172})^2 N}.$$

This finishes the proof of the first statement. A similar estimate works for \mathcal{F}_4 . \square

By the last sequence of lemma, we have shown Proposition 11.2.

13. A VARIANT OF THE MULTISCALE STEP

In this section, we will discuss a variant of the argument of Section 6. The main idea is instead of eliminating energies E as done in Lemma 6.6, we will assume a Wegner type estimate. In particular, this means that the results of this section will be very close in spirit to the ones used for random Schrödinger operators.

Theorem 13.1. *Assume that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$ -critical, $M \geq 3$ and (6.8) (that is $\frac{\sigma L}{M} \geq 2$). Furthermore assume that*

$$(13.1) \quad \#\{0 \leq l \leq L : \{V(n)\}_{n=0}^{N-1} \text{ is } ([k_l, k_l + \frac{16N(M+1)}{\sigma L}], \mathcal{E}, 2e^{-\sigma\delta}) \text{ resonant}\} \\ \leq \frac{\sigma}{4}(1 - 2\sigma) \frac{L}{M+1}.$$

Then $\{V(n)\}_{n=0}^{N-1}$ is also $(\tilde{\delta}, \tilde{\sigma}, \tilde{L}, \mathcal{E})$ -critical, with the quantities defined as in Theorem 6.2.

The proof of this theorem parallels the proof of Theorem 6.2. We define \tilde{k}_j as in (6.13), (6.14), whose properties stay the same. In particular \tilde{L} satisfies

$$(13.2) \quad (1 - 2\sigma) \frac{L}{M+1} \leq \tilde{L} \leq \frac{L}{M+1},$$

by the same argument as was used to show (6.15).

Instead of using Lemma 6.6 to find the set \mathcal{L} of good indices for \tilde{k}_l , we will proceed differently. Denote by $l \notin \tilde{\mathcal{L}}_0$ the set defined in (6.16), and the estimate (6.17) on its size still holds. We now let

$$\tilde{\mathcal{L}}_1 = \{0 \leq l \leq L : \{V(n)\}_{n=0}^{N-1} \text{ is } ([k_l, k_l + \frac{16N(M+1)}{\sigma L}], \mathcal{E}, 2e^{-\sigma\delta}) \text{ resonant}\},$$

with (13.1) now saying $\#\tilde{\mathcal{L}}_1 \leq \frac{\tilde{\sigma}}{2}\tilde{L}$ after a short computation. Hence, we introduce

$$\mathcal{L} = \tilde{\mathcal{L}}_0 \cup \tilde{\mathcal{L}}_1,$$

which satisfies $\#\mathcal{L} \leq \tilde{\sigma}\tilde{L}$. Now, we are ready for.

Proof of Theorem 13.1. One then sees that Lemma 6.7 still applies and the proof is finished in a similar fashion as the one of Theorem 6.2. \square

14. ADAPTATION OF THE MULTISCALE ARGUMENT

In this section $\sigma_j, \delta_j, L_j, M_j$ denote the same constants as in Section 7. We introduce

$$(14.1) \quad \varepsilon_j = 3e^{-\sigma_j \delta_j}.$$

We have the following lemma. We note that the choice of intervals, comes from (13.1).

Lemma 14.1. *Introduce the interval*

$$(14.2) \quad \mathcal{E} = [E - 2e^{-\sigma_j \delta_j}, E + 2e^{-\varepsilon_j \delta_j}]$$

Then we have that

$$(14.3) \quad \mathcal{E} + 2[-e^{-\sigma_j \delta_j}, 2e^{-\sigma_j \delta_j}] \subseteq [E - \varepsilon_j, E + \varepsilon_j],$$

for $0 \leq j \leq j_0 = j_0(J)$ and $\lim_{J \rightarrow \infty} j_0(J) = \infty$.

Proof. This follows from the fact that the sequence $\sigma_j \delta_j \gtrsim 10^{j^2}$. \square

We need the following lemma, one a numerical constant arising in Theorem 13.1.

Lemma 14.2. *Let K_j be the length required by Theorem 13.1, for $(\delta_j, \sigma_j, L_j, \mathcal{E})$, then*

$$(14.4) \quad K_j \leq \hat{K} \left(10^{(j+1)(j+2)}\right)^3, \quad \hat{K} = e^{4\sigma} e^{\frac{1}{99}} \frac{N}{L_0}.$$

Proof. First, observe that the K_j 's is given by $K_j = \frac{16N(M_j+1)}{\sigma_l L_j}$. By (7.8), we obtain that

$$\frac{N}{L_j} \leq e^{4\sigma} e^{\frac{1}{99}} 10^{j(j+1)} \frac{N}{L_0}.$$

By (7.4), we have that $M_j = 10^{(j+1)(j+2)}$, and since $j \leq j^2$, the result follows. \square

We furthermore collect the following lemma, which is similar to Lemma 9.1

Lemma 14.3. *Assume (3.41). There exists $\omega \in \Omega$, such that the following properties hold*

(i) *We have that*

$$(14.5) \quad L(E) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A_\omega(E, n)\|$$

for all E .

(ii) *There is $N_0 \geq 1$ such that for $N \geq N_0$, we have that $\{V_\omega(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, \lceil \frac{N}{K} - 1 \rceil, \mathcal{E})$ -critical*

(iii) *For $j \geq 1$, there is $N_j \geq 1$ such that for $N \geq N_j$, we have that*

$$(14.6) \quad \#\{0 \leq l \leq \frac{N}{K_0} : \{V_\omega\}_{n=0}^{N-1} \text{ is } ([lK_0, lK_0 + K_j], \{E\}, \varepsilon_j) \text{ resonant}\} \\ \leq \frac{2N}{K_0} C \cdot \frac{K_j^\beta}{|\log(\varepsilon_j)|^\rho}.$$

Proof. By total ergodicity, in particular Lemma 4.2, we may find a set $\Omega_0 \subseteq \Omega$ such that

$$\mu(\Omega_0) \geq 1 - \frac{1}{4}$$

and for any $\omega \in \Omega_0$, we have that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, \lceil \frac{N}{K} - 1 \rceil, \mathcal{E})$ -critical for $N \geq N_0$ (some N_0). Similarly, we may find by Lemma 4.2 for each $j \geq 1$ a set Ω_j such that

$$\mu(\Omega_j) \geq 1 - \frac{1}{4} \frac{1}{2^j}$$

and (14.6) holds for $N \geq N_j$. If we let

$$\Omega_\infty = \bigcup_{j=0}^{\infty} \Omega_j,$$

then we have that $\mu(\Omega_\infty) \geq \frac{1}{2}$. We will now fix $\omega \in \Omega_\infty \cap \Omega_{CS}$, where Ω_{CS} is as in Theorem 5.3. This finishes the proof. \square

In particular, we see that, we may choose $N/L = K(1 + o(1))$ in (14.4). We will now study the right hand side of (14.6).

Lemma 14.4. *Assume (3.41), (3.43), and (3.44). Then (14.6) implies (13.1) with $\delta = \delta_j$, $\sigma = \sigma_j$ and \mathcal{E} as in (14.2).*

Proof. The right hand side of (13.1) satisfies

$$\frac{\sigma_j}{4}(1 - 2\sigma_j) \frac{L_j}{M_j + 1} \geq \sigma e^{-4\sigma} e^{-\frac{1}{99}} L_0 10^{-3(j+1)(j+2)}$$

since $1 - 2\sigma_j \geq \frac{1}{2}$, $j + 4 \leq -2(j+1)(j+4)$, (7.4), and (7.8).

By (7.6), we have that $\sigma_j \delta_j \geq \sigma \delta 10^{j^2}$, and thus

$$|\log(\varepsilon_j)|^\rho \geq \left(\frac{\sigma \delta}{2}\right)^\rho 10^{\rho(j+1)(j+2)}.$$

Combining this with (14.4), we obtain the following estimate for the right hand side of (14.6)

$$\frac{2N}{K_0} C \cdot \frac{K_j^\beta}{|\log(\varepsilon_j)|^\rho} \leq 4C \cdot L_0 \cdot \frac{e^{4\beta\sigma} e^{\frac{\beta}{99}} (2K_0)^\beta 2^\rho}{(\sigma \delta)^\rho} \cdot 10^{-(\rho-3\beta)(j+1)(j+2)}.$$

Now (3.44) and (3.43) imply the claim. \square

Proposition 14.5. *Assume (3.43) and (3.44). Then, for every $j \geq 1$ and E , there exists an $N_0 \geq 1$, such that $\{V_\omega\}_{n=0}^{N-1}$ is $(\delta_j, \sigma_j, L_j, [E - \varepsilon_j, E + \varepsilon_j])$ -critical.*

Proof. By the last lemma, we can satisfy the conditions of Theorem 13.1 for all $i \leq j$, hence the claim follows. \square

Now, we are ready for.

Proof of Theorem 3.12. Applying the last proposition for sufficiently large j , we see that we can satisfy (7.13), and by sufficiently large N , that we satisfy (7.11). Furthermore (7.12) is automatically satisfied by our choice of ε_j . Hence, we can apply Theorem 7.4, to be in the same situation as discussed in Section 9. Applying the method of that section, we can conclude that there exists a set $\mathcal{E}_0 \subseteq \mathcal{E}$ of full measure, such that for every $E \in \mathcal{E}_0$, we have that

$$L(E) \geq e^{-8\sigma} e^{-\frac{1}{99}} \gamma.$$

We then even obtain the lower bound for every $E \in \mathcal{E}$ by subharmonicity of $L(E)$. This finishes the proof. \square

15. THE INTEGRATED DENSITY OF STATES

In this section, we quickly review some things about the integrated density of states. Let (Ω, μ) be a probability space, $T : \Omega \rightarrow \Omega$ an ergodic transformation, and $f : \Omega \rightarrow \mathbb{R}$ a bounded real valued function. We use the usual definition

$$(15.1) \quad V_\omega(n) = f(T^n \omega)$$

for $n \in \mathbb{Z}$ and $H(\omega)$ for the associated Schrödinger operator. For $\Lambda \subseteq \mathbb{Z}$, we let $H_\Lambda(\omega)$ be the restriction of $H(\omega)$ to $\ell^2(\Lambda)$. For some length scale $M \geq 1$, we introduce

$$(15.2) \quad k_M(E) = \frac{1}{M} \int_{\Omega} \text{tr}(P_{(-\infty, E)}(H_{[0, M-1]}(\omega))) d\mu(\omega).$$

We have the following lemma

Lemma 15.1. *Assume*

$$k_M(E + \frac{\varepsilon}{2}) - k_M(E - \frac{\varepsilon}{2}) \leq \frac{CM^\beta}{|\log(\varepsilon)|^\rho}$$

then

$$(15.3) \quad \mu(\{\omega : \exists \Lambda \subseteq [0, M-1] : \text{dist}(E, \sigma(H_{\omega, \Lambda})) \leq \frac{1}{2}\varepsilon\}) \leq \frac{CM^{2+\beta}}{|\log(\varepsilon)|^\rho}.$$

Proof. For fixed interval $\Lambda \subseteq [0, M-1]$, and ω , we have $\text{dist}(E, \sigma(H_{\omega, \Lambda})) \leq \frac{1}{2}\varepsilon$ implies that

$$\text{tr}(P_{(-\infty, E+\frac{1}{2}\varepsilon)}(H_{\omega, \Lambda}(\omega))) - \text{tr}(P_{(-\infty, E-\frac{1}{2}\varepsilon)}(H_{\omega, \Lambda}(\omega))) \geq 1.$$

For $\Lambda = [a, b]$, we have $H_{\omega, \Lambda} = H_{T^{-a}\omega, [0, b-a-1]}$. So we see by (15.2) that with $n = \#\Lambda$

$$\mu(\{\omega : \text{dist}(E, \sigma(H_{\omega, \Lambda})) \leq \frac{1}{2}\varepsilon\}) \leq k_n(E + \frac{1}{2}\varepsilon) - k_n(E - \frac{1}{2}\varepsilon) \leq \frac{Cn^\beta}{|\log(\varepsilon)|^\rho}.$$

The claim follows by that there are less than M subintervals of $[0, M-1]$ with n elements. \square

We furthermore remark the following lemma, whose prove is an exercise in elementary calculus.

Lemma 15.2. *Let $\alpha, \rho > 0$ and*

$$(15.4) \quad C(\alpha, \rho) = e^{-\rho} \left(\frac{\rho}{\alpha}\right)^\rho,$$

then for $0 < \varepsilon < \frac{1}{2}$

$$(15.5) \quad \varepsilon^\alpha \leq \frac{C(\alpha, \rho)}{|\log(\varepsilon)|^\rho}.$$

16. THE INTEGRATED DENSITY OF STATES FOR THE SKEW-SHIFT MODEL

In this section, we will prove Proposition 3.13. It turns out more convenient to prove the following theorem.

Theorem 16.1. *Let $\varepsilon > 0$ and $N \geq 1$ an integer. Then*

$$(16.1) \quad k_{\lambda, N}(E + \varepsilon) - k_{\lambda, N}(E) \leq 7 \cdot \max(1, \frac{1}{\lambda}) \cdot N^2 \varepsilon.$$

Before proving this theorem, let us first derive Proposition 3.13.

Proof of Proposition 3.13. This follows by Lemma 15.1 and 15.2. \square

In order to prove Theorem 16.1, we will need some preparations. For $\delta > 0$ and $N \geq 1$, introduce the set $\Omega(\delta, N)$ by

$$(16.2) \quad \Omega(\delta, N) = \{\underline{\omega} \in \Omega : (T^n \underline{\omega})_K \in [\delta, 1 - \delta], \quad 1 \leq n \leq N\}.$$

We have the following bound on the size of $\Omega(\delta, N)$.

Lemma 16.2. *We have*

$$(16.3) \quad |\Omega(\delta, N)| \geq 1 - 2N\delta.$$

Proof. Let

$$\Omega_b = \{\underline{\omega} \in \Omega : \omega_K \notin [\delta, 1 - \delta]\}.$$

We have that $|\Omega_b| = 2\delta$. Observe that

$$\Omega(\delta, N) = \Omega \setminus \bigcup_{n=1}^N T^{-n} \Omega_b.$$

The claim now follows by T being measure preserving. \square

We will need a bit of notation for $\underline{\omega} \in \Omega$, we will denote by $\underline{\omega}' \in \mathbb{T}^{K-1}$ the first $K-1$ components of $\underline{\omega}$, so

$$\underline{\omega} = (\underline{\omega}', \omega_K).$$

We will show the following bound.

Lemma 16.3. *Given $\rho : \mathbb{R} \rightarrow [0, 1]$ an increasing and differentiable function. The following bound holds*

$$(16.4) \quad \int_{\Omega(2\varepsilon, N)} \frac{\partial}{\partial \omega_K} \text{tr}(\rho(H_{\lambda, \underline{\omega}, [1, N]} - t)) d\underline{\omega} \leq N + 1.$$

Proof. We fix some $\underline{\omega}' \in \mathbb{T}^{K-1}$. We will let $\underline{\omega} = (\underline{\omega}', \vartheta)$, then $\frac{\partial}{\partial \omega_K}$ becomes $\frac{\partial}{\partial \vartheta}$. We have that the set

$$A = \{\vartheta : (\underline{\omega}', \vartheta) \in \Omega(2\varepsilon, N)\}$$

is some subset of $[0, 1]$ consisting of at most $N+1$ many intervals. So we may write

$$A = [\vartheta_0, \vartheta_1] \cup [\vartheta_2, \vartheta_3] \cup \dots \cup [\vartheta_{2N}, \vartheta_{2N+1}],$$

For $0 \leq p \leq N$, we have that for $H_{\lambda, (\underline{\omega}', \vartheta), [1, N]}$ and $H_{\lambda, (\underline{\omega}', \tilde{\vartheta}), [1, N]}$ differ by a rank one perturbation for $\vartheta, \tilde{\vartheta} \in [\vartheta_{2p}, \vartheta_{2p+1}]$. It is thus a standard fact, that

$$\begin{aligned} & \int_{[\vartheta_{2p}, \vartheta_{2p+1}]} \frac{\partial}{\partial \vartheta} \text{tr}(\rho(H_{\lambda, (\underline{\omega}', \vartheta), [1, N]} - t)) d\vartheta \\ & \leq \text{tr}(\rho(H_{\lambda, (\underline{\omega}', \vartheta_{2p+1}), [1, N]} - t)) - \text{tr}(\rho(H_{\lambda, (\underline{\omega}', \vartheta_{2p}), [1, N]} - t)) \leq 1 \end{aligned}$$

By summing up, and integrating over $\underline{\omega}' \in \mathbb{T}^{K-1}$ the claimed bound follows. \square

Now, we come to

Proof of Theorem 16.1. Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $\rho(x) = 1$ for $x \leq 0$ and $\rho(x) = 0$ for $x \geq \varepsilon$. We then observe that

$$\begin{aligned} & \text{tr}(P_{(-\infty, E+\varepsilon)} H_{\lambda, \underline{\omega}, [1, N]}) - \text{tr}(P_{(-\infty, E)} H_{\lambda, \underline{\omega}, [1, N]}) \\ & \leq \text{tr}(\rho(H_{\lambda, \underline{\omega}, [1, N]} - E - \varepsilon)) - \text{tr}(\rho(H_{\lambda, \underline{\omega}, [1, N]} - E + \varepsilon)) \\ & = \frac{1}{\lambda} \int_{E-\varepsilon}^{E+\varepsilon} \frac{\partial}{\partial t} \text{tr}(\rho(H_{\lambda, \underline{\omega}, [1, N]} - t)) dt. \end{aligned}$$

Since these functions are analytic, we can replace inside the set $\Omega(2\varepsilon, N)$ the t derivate by a ω_K derivate. Hence, we obtain that

$$\begin{aligned} & k_{\lambda, M}(E + \varepsilon) - k_{\lambda, M}(E) \\ & \leq \max\left(1, \frac{1}{\lambda}\right) \int_{\Omega(2\varepsilon, N)} \int_{E-\varepsilon}^{E+\varepsilon} \frac{\partial}{\partial \omega_K} \operatorname{tr}(\rho(H_{\lambda, \omega, [1, N]} - t)) dt d\omega \\ & \quad + |\Omega \setminus \Omega(2\varepsilon, N)| \cdot N, \end{aligned}$$

where we used the worst case estimate for $\omega \notin \Omega(2\varepsilon, N)$. The claim now follows by (16.4). \square

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