THE STRUCTURE OF EIGENVALUES FOR THE SKEW-SHIFT OPERATOR

HELGE KRÜGER

In this informal note, I wish to discuss the Schrödinger operator $H = \Delta + V$ with potential

$$V(n) = 1.8\cos(2\pi\sqrt{2}n^K), \tag{1}$$

where $K \in \{1, 2\}$. $K = 1$ corresponds to the subcritical Almost–Mathieu operator for which it is known that the spectrum is a Cantor set and purely absolutely continuous.

$K = 2$ corresponds to the skew-shift operator, for which it is believed that the spectrum is an interval and the Lyapunov exponent is strictly positive.

So let me brevely say what the following two pictures show. Consider the truncation of H to

$$\ell^2(\{1, 2, 3 \ldots, 50\}). \tag{2}$$

Then the operator can be written as a matrix

$$H = \begin{pmatrix} V(1) & 1 & & \\ 1 & V(2) & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & V(50) \end{pmatrix} \tag{3}$$

and we know from elementary linear algebra that H has 50 distinct eigenvalues. Denote these by E_j and by u_j the corresponding eigenfunctions, which we normalize by

$$\|u_j\| = \left(\sum_{n=1}^{50} |u_j(n)|^2 \right)^{\frac{1}{2}} = 1. \tag{4}$$

In the following two figures, a black dot is drawn at $(\frac{n}{10}, E_j)$ if $|u_j(n)| \geq 0.01$.

It is not hard to believe from these figures that in the skew-shift case, we have interval spectrum and localized eigenfunctions, whereas in the Almost–Mathieu case there is Cantor spectrum with extended states.

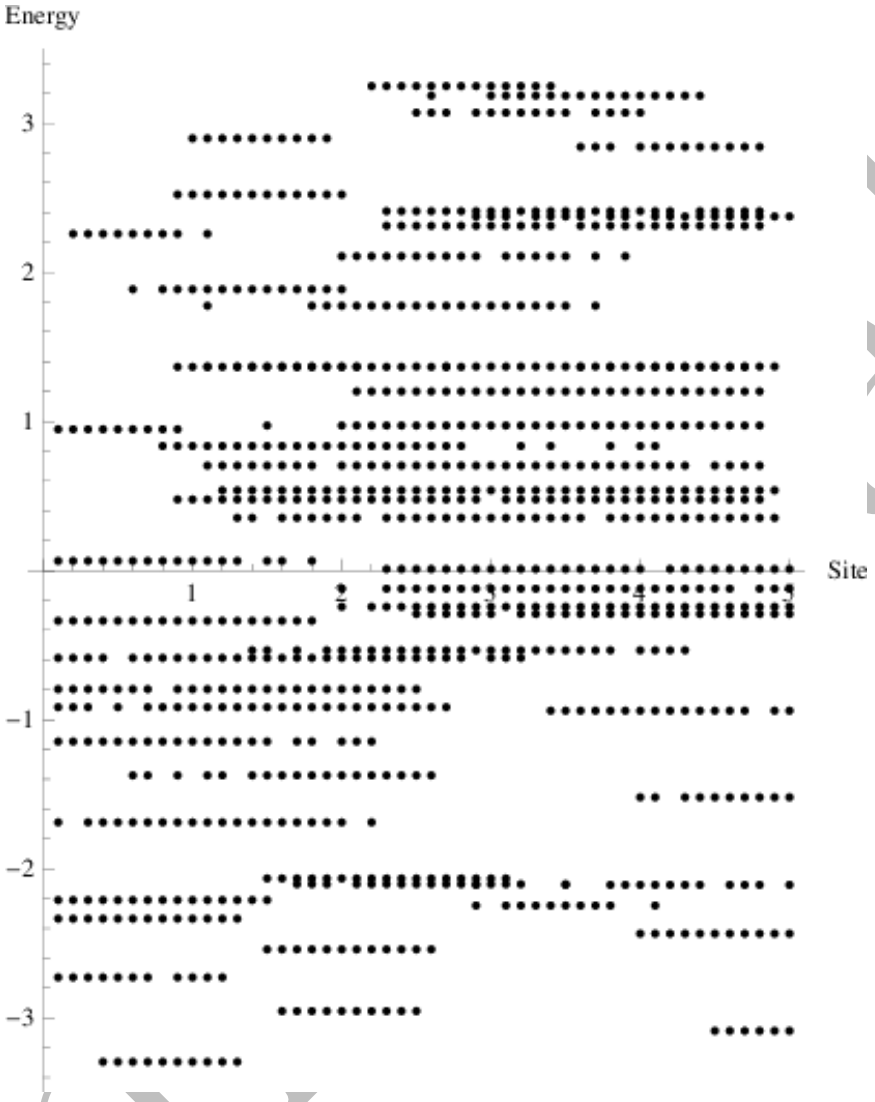

FIGURE 1. Skew-shift operator

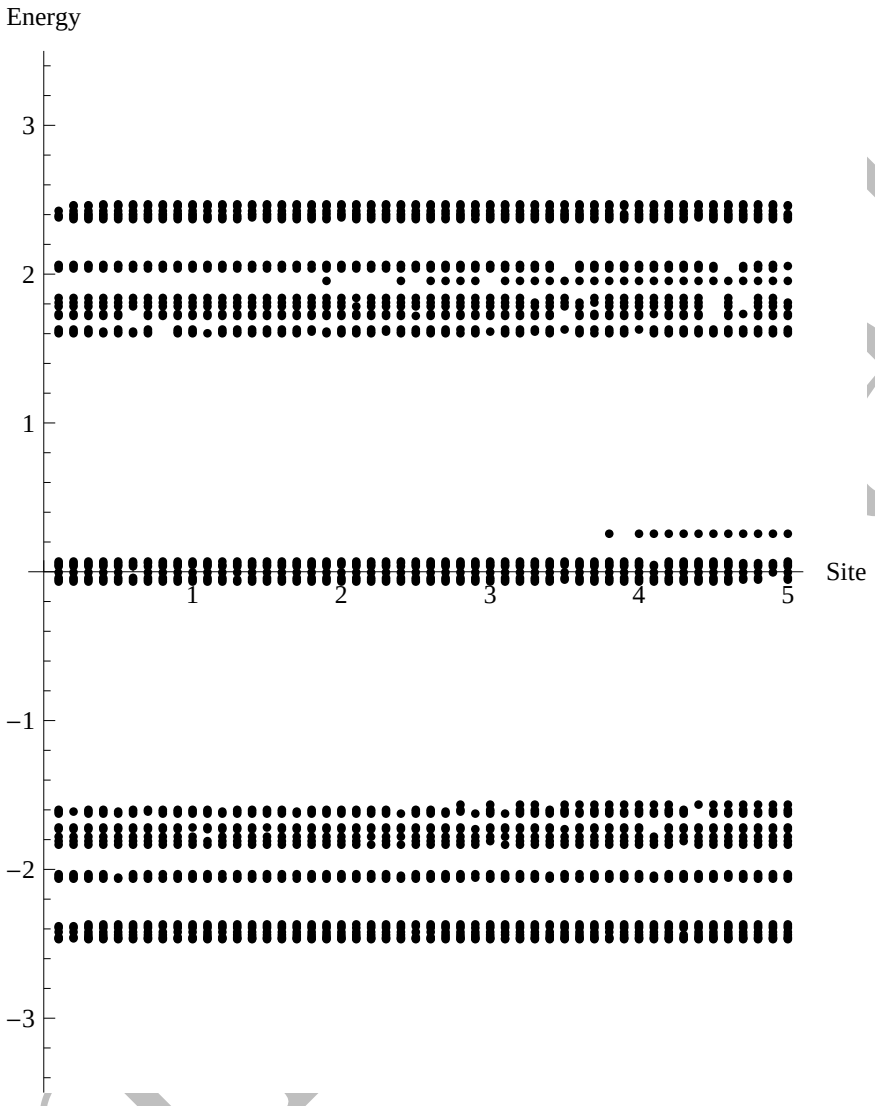

FIGURE 2. Almost–Mathieu operator