Positive Lyapunov exponent for ergodic Schrödinger Operators

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Let $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable sequences.
Let the potential $V : \mathbb{Z} \to \mathbb{R}$ be a bounded sequence.
Define the Schrödinger operator H by

$$\begin{aligned} H &: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ (Hu)(n) &= \underbrace{u(n+1) + u(n-1)}_{(\Delta u)(n)} + V(n)u(n). \end{aligned} \tag{1}$$

H is a bounded and self-adjoint operator.
If u solves $Hu = Eu$ for some energy E, then

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(E,n) \cdot \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \tag{2}$$

with

$$A(E,n) = \prod_{j=n}^{1} \begin{pmatrix} V(j) - E & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

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$$A(E,n) = \prod_{j=n}^{1} \begin{pmatrix} V(j) - E & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The Lyapunov exponent $L(E)$ describes the exponential growth of the norm of $A(E,n)$:

$$L(E) = \limsup_{n\to\infty} \frac{1}{n} \log \|A(E,n)\|. \tag{5}$$

For fixed energy E: $0 \le L(E) \le const.$
By the Combes–Thomas estimate, one also has

$$L(E) \gtrsim \operatorname{dist}(E, \sigma(H)). \tag{6}$$

We will be interested in positive Lyapunov exponent $L(E) > 0$, on the spectrum $\sigma(H)$. In the ergodic case

$$\sigma_{\mathrm{ac}}(H) = \overline{\{E : \quad L(E) = 0\}}^{\mathrm{ess}}. \tag{7}$$

Thus one can think of positive Lyapunov exponent as a criterion for localization.

Instead of looking at general potentials $V(n)$, we will restrict our attention to ergodic potentials.
So let (Ω, μ) be a probability space. $T : \Omega \to \Omega$ an invertible ergodic transformation. Ergodic means that if g satisfies

$$g \circ T = g \tag{8}$$

then g is almost surely constant. Ergodic implies that for $A \subseteq \Omega$

$$\frac{1}{n}\#\{1 \leq k \leq n : \quad T^k\omega \in A\} \to \mu(A), \quad a.e.\ \omega. \tag{9}$$

Let $f : \Omega \to \mathbb{R}$ a bounded function. We define for $\omega \in \Omega$ and the coupling $\lambda > 0$:

$$V_{\omega,\lambda}(n) = \lambda f(T^n\omega). \tag{10}$$

From the definition of $L_\omega(E, \lambda)$, it follows that

$$L_{T\omega}(E, \lambda) = L_\omega(E, \lambda). \tag{11}$$

Thus by ergodicity, the value is almost surely ω independent, and we denote it by $L(E, \lambda)$.

A 'silly' example: two periodic Schrödinger operator

Take $\Omega = \mathbb{Z}_2 = \{0, 1\}$ with uniform measure.

$$Tx = x + 1 \pmod 2, \tag{12}$$

and $f(x) = 2(x - \frac{1}{2})$. Then the potential is just

$$\dots, \lambda, -\lambda, \lambda, -\lambda, \dots \tag{13}$$

and we have for the spectrum

$$\sigma(H_\lambda) = \sigma_{\mathrm{ac}}(H_\lambda) = [-\sqrt{4+\lambda^2}, -\lambda] \cup [\lambda, \sqrt{4+\lambda^2}]. \tag{14}$$

And the Lyapunov exponent vanishes on it.

Known Results: 1/2.

- Anderson Model: $V(n)$ are independent identically distributed random variables.
  By Furstenberg, Kotani: $L(E, \lambda) > 0$ for all E and all $\lambda > 0$.
  By Pastur–Figotin:

$$0 < \lambda \ll 1 : \quad L(E, \lambda) \gtrsim \lambda^2. \tag{15}$$

- Rotations on the torus: (α irrational)

$$\Omega = \mathbb{R}/\mathbb{Z} \cong [0, 1], \quad Tx = x + \alpha \pmod 1. \tag{16}$$

  $V(n) = \lambda f(n\alpha)$ quasi-periodic, f analytic,

$$\lambda \gg 1 : \quad L(E, \lambda) \gtrsim \log \lambda \tag{17}$$

  for all E. Herman, Sorets–Spencer, Bourgain–Goldstein, Goldstein–Schlag, Bourgain.
  The Lyapunov exponent vanishes on the spectrum for small λ. Eliasson, Bourgain–Jitomirskaya.

Known Results: 2/2.

- Skew-Shift (α irrational)

$$T : [0,1]^2 \to [0,1]^2, \quad T(x,y) = (x+\alpha, x+y) \pmod 1.$$

$V(n) = \lambda f(n^2\alpha)$, f analytic. Bourgain–Goldstein–Schlag:

$$\lambda \gg 1 : \quad L(E,\lambda) \gtrsim \log(\lambda). \tag{18}$$

Conjecture: $L(E,\lambda) > 0$ for all $\lambda > 0$.
Partial results by Bourgain for $0 < \lambda \ll 1$.

- Doubling

$$T : [0,1] \to [0,1], \quad Tx = 2x \pmod 1. \tag{19}$$

$V(n) = \lambda f(2^n x)$. Chulaevsky–Spencer:

$$0 < \lambda \ll 1 : \quad L(\lambda, E) \gtrsim \lambda^2. \tag{20}$$

Also conjectured for all $\lambda > 0$.
Positivity is known by Damanik–Killip, but proof is non-quantitative (they use Kotani).

Main result: Pseudoformulation

Theorem

Assume an initial condition at an initial scale n then $L(E) > 0$ for most energies E.

I will give a more specific statement later.

Proof by metaproof.

Multiscale analysis:

Use energy elimination to pass from scale n to scale $100^j n$ on step j. □

Now: What kind of initial condition do we want?

Later: Exact statement and some ideas of the proof.

Choosing an initial condition: First try

Recall

$$L(E) = \limsup_{n\to\infty} \frac{1}{n} \log \|A(\omega, n, E)\| \tag{21}$$

for almost every ω.

First guess of an initial condition:

$$\mu(\{\omega : \quad \frac{1}{n} \log \|A(\omega, n, E)\| < \gamma\}) \leq \frac{1}{4} \tag{22}$$

for $\gamma > 0$ and n suitable.
This condition is necessary to obtain $L(E) \gtrsim \gamma$.
However, it might not be sufficient.

Let $\gamma > 0$ and n large, assume

$$\mu(\{\omega : \quad \frac{1}{n}\log\|A(\omega,n,E)\| < \gamma\}) \leq \frac{1}{4}. \qquad (23)$$

We know

$$A(\omega,2n,E) = A(T^n\omega,n,E)A(\omega,n,E). \qquad (24)$$

Want to use this to go from scale n to scale $2n$. However,

- We would lose probability.
- The matrix norm is not multiplicative, but only submultiplicative.

Solution 1: By Bourgain–Goldstein–Schlag:
Use Large Deviation Estimates and
the Avalanche Principle.
Problem: Large Deviation Estimates are not proved for doubling at large coupling.
Comment: Single energy method.

Solution 2: Multiscale Analysis

- Reformulate the initial condition as a condition for finite volume Green's function. For $x, y \in \Lambda \subseteq \mathbb{Z}$ define

$$G_{\omega,\Lambda}(E, x, y) = \langle \delta_x, (H_{\omega,\Lambda} - E)^{-1} \delta_y \rangle \tag{25}$$

where $H_{\omega,\Lambda}$ is the restriction of H_ω to $\ell^2(\Lambda)$.

$$|G_{\omega,[0,2n]}(E, 0, n)| \leq \mathrm{e}^{-\gamma n} \leftrightsquigarrow \frac{1}{n} \log \|A(\omega, n, E)\| \geq \gamma. \tag{26}$$

- Use the resolvent equation $x \in [a, b] \subseteq \Lambda = [0, N]$

$$\begin{aligned} G_{\omega,\Lambda}(E, 0, x) = &- G_{\omega,\Lambda}(E, 0, a-1) G_{\omega,[a,b]}(E, a, x) \\ &- G_{\omega,\Lambda}(E, 0, b+1) G_{\omega,[a,b]}(E, b, x) \end{aligned}$$

to pass from scale to scale.
- Use a Wegner estimate to avoid losing probability.

Adapted by Bourgain to prove positive Lyapunov exponent without any a priori Wegner estimate.
This approach has been heavily used for Random Schrödinger operators to prove Anderson Localization.

The main result

Theorem (arXiv:0905.1791)

Assume the initial condition for $x \in \{1, 2n-1\}$

$$\mu\left(\left\{\omega : \sup_{E \in [E_0, E_1]} |G_{\omega,[1,2n-1]}(E,n,x)| > \mathrm{e}^{-\gamma n}\right\}\right) \leq \frac{1}{8}, \qquad (27)$$

for $\gamma \cdot n \gg 1$ *(dependent on* $E_1 - E_0$*). Then there exists a set* $\mathcal{E}_b$ *such that*

$$|\mathcal{E}_b| \lesssim \mathrm{e}^{-c\gamma n} \qquad (28)$$

and for $E \in [E_0, E_1] \backslash \mathcal{E}_b$

$$L(E) \geq \frac{\gamma}{2}. \qquad (29)$$

We will now discuss how to prove an initial condition for ergodic Schrödinger operators at large coupling.
Then sketch some ideas for the proof.

Proof of the intial condition

Assume for some $\alpha > 0$ and all $E \in \mathbb{R}$

$$\mu(\{\omega : |f(\omega) - E| \leq \varepsilon\}) \lesssim \varepsilon^{\alpha}. \tag{30}$$

For example analytic $f : [0, 1] \to \mathbb{R}$ and μ the Lebesgue measure satisfy this assumption. Łojasiewicz.
This implies for any $N \geq 1$ that for $\lambda \gg 1$ ($V_{\omega,\lambda}(n) = \lambda f(T^n \omega)$)

$$\mu(\{\omega : \mathrm{dist}(E, \{V_{\omega,\lambda}(n)\}_{n=1}^{N}) < \lambda^{\alpha/2}\}) \leq \frac{1}{16}. \tag{31}$$

By $\|\Delta\| \leq 2$ and $H_\omega = \Delta + V_{\omega,\lambda}$, we thus obtain

$$\mu(\{\omega : \mathrm{dist}(E, \sigma(H_{\omega,\lambda,[1,N]})) < \frac{1}{2}\lambda^{\alpha/2}\}) \leq \frac{1}{16}. \tag{32}$$

Use the Combes–Thomas estimate to convert this into a Green's function estimate and to obtain the initial condition.

The doubling map

In particular:

Corollary

Let $f : [0, 1] \to \mathbb{R}$ *be analytic.*
Let λ *be large enough.*
Introduce the potential (almost every x*)*

$$V(n) = \lambda f(2^n x \pmod 1). \tag{33}$$

Then

$$|\{E : \quad L(E, \lambda) \leq \frac{1}{5} \log(\lambda)\}| \lesssim \mathrm{e}^{-c\lambda^\alpha} \tag{34}$$

for some $c > 0, \alpha > 0$.

Hence $L(E) \geq \frac{1}{5} \log \lambda$ for most energies E.
Note $[-2, 2] + f(0) \subseteq \sigma(\Delta + V)$.
Note the result holds also for example for any toral automorphism $[0, 1]^2 \to [0, 1]^2$.

Proof: 1 / 5

Recall for $x, y \in \Lambda \subseteq \mathbb{Z}$

$$G_{\omega,\Lambda}(E, x, y) = \langle \delta_x, (H_{\omega,\Lambda} - E)^{-1} \delta_y \rangle \tag{35}$$

We assume the initial condition

$$\mu\left(\left\{\omega : \sup_{E \in [E_0, E_1]} \sup_{x \in \{1, 2n-1\}} |G_{\omega,[1,2n-1]}(E, x, n)| > \mathrm{e}^{-\gamma \cdot n} \right\}\right) \leq \frac{1}{4}.$$

I will illustrate, how to do multiscale analysis without a Wegner estimate. To simplify things, I will assume the above holds for all ω.

I will go from scale n to scale 100 n.

Proof: 2 / 5

Assume for all ω, $\gamma \cdot n \gg 1$ and $x \in \{1, 2n-1\}$

$$\sup_{E \in [E_0, E_1]} |G_{\omega,[1,2n-1]}(E, x, n)| \leq \frac{1}{2} \mathrm{e}^{-\gamma \cdot n}. \qquad (36)$$

Then for $E \in [E_0, E_1]$

$$\begin{aligned} G_{\omega,[1,200n-1]}(E, 1, 100n) & \\ &= -G_{\omega,[1,200n-1]}(E, 1, 99n) \\ &\quad \cdot G_{\omega,[99n+1,101n-1]}(E, 99n+1, Ln) + \ldots \\ &= -G_{\omega,[1,200n-1]}(E, 1, 99n) \\ &\quad \cdot \underbrace{G_{T^{100n}\omega,[(1,2n-1]}(E, 1, n)}_{|\ldots| \leq \frac{1}{2}\mathrm{e}^{-\gamma \cdot n}} + \ldots \end{aligned}$$

Now iterate!

Proof: 3 / 5

Assume for all ω, $\gamma \cdot n \gg 1$ and $x \in \{1, 2n-1\}$

$$\sup_{E \in [E_0, E_1]} |G_{\omega,[1,2n-1]}(E, x, n)| \leq \frac{1}{2} \mathrm{e}^{-\gamma \cdot n}. \tag{37}$$

Iterating the above, we obtain

$$|G_{\omega,[1,200n-1]}(E, 1, 100n)| \leq \mathrm{e}^{-100\gamma \cdot n} \cdot \sup_{x,y \in [1,200n-1]} |G_{\omega,[1,200n-1]}(E, x, y)|.$$

Since $H_{\omega,[1,200n-1]}$ is self-adjoint

$$\sup_{x,y \in [1,200n-1]} |G_{\omega,[1,200n-1]}(E, x, y)| \leq \frac{1}{\mathrm{dist}(E, \sigma(H_{\omega,[1,200n-1]}))}. \tag{38}$$

Now $\sigma(H_{\omega,[1,200n-1]})$ consists of only $\approx 200n$ elements.

Proof: 4 / 5

We have

$$|G_{\omega,[1,200n-1]}(E,1,100n)| \leq \frac{\mathrm{e}^{-100\gamma\cdot n}}{\mathrm{dist}(E,\sigma(H_{\omega,[1,200n-1]}))}. \qquad (39)$$

with $\sigma(H_{\omega,[1,200N-1]})$ consisting of less then $200n$ elements.
To obtain

$$|G_{\omega,[1,200n-1]}(E,1,100n)| \leq \mathrm{e}^{-99\gamma\cdot n}, \qquad (40)$$

we only need

$$\mathrm{dist}(\sigma(H_{\omega,[1,200n-1]}),E) \geq \mathrm{e}^{-\gamma\cdot n}. \qquad (41)$$

Hence, we only need to eliminate a set of E of measure at most

$$400n \cdot \mathrm{e}^{-\gamma\cdot n} \qquad (42)$$

for each fixed ω.

Proof: 5 / 5

Let

$$\mathcal{B} = \{(E,\omega) \in [E_0, E_1] \times \Omega : \quad |G_{\omega,[1,200n-1]}(E, 1, 100n)| > \mathrm{e}^{-99\gamma \cdot n}\}.$$

We have

$$(Leb \times \mu)(\mathcal{B}) \leq 400n \cdot \mathrm{e}^{-\gamma \cdot n}. \tag{43}$$

Let $\mathcal{E}_b$ be the set such that

$$\mu\left(\left\{\omega : \quad |G_{\omega,[1,200n-1]}(E, 1, 100n)| > \mathrm{e}^{-99\gamma \cdot n}\right\}\right) > \frac{1}{4}$$

holds.By Markov's inequality we can thus conclude that

$$|\mathcal{E}_b| \leq 1600n \cdot \mathrm{e}^{-\gamma \cdot n}. \tag{44}$$

We see here why we need $E_1 - E_0$ to be big compared to $\mathrm{e}^{-\gamma \cdot n}$.
New γ is $\frac{99}{100}\gamma$.

Further aspects of the proof

We have seen now how to go from all ω and all E to most ω and most E.

We will need to do better:

- Need to ensure that $\mathcal{E}_b$ consists of finitely many intervals, so the above procedure can be iterated.
  Bourgain: Semialgebraic sets requires analytic functions.
  Here: Counting argument.
- Need to deal with not for all ω.
  This uses ergodicity and an improved iteration of the resolvent equation.

Further extensions

- The methods allow me to say something about $V(n) = \lambda \cos(n^k)$ for all small λ, letting k be large depending on λ.
  Here I use randomness to obtain an initial condition, following the methods Bourgain–Schlag used for the doubling map at small coupling.
- Assuming a (weak) Wegner estimate, one can remove the exceptional set of energies.
  However, this is an extra assumption.
  See Gan-K (`arXiv:0906.3300`).
- Open Problem: Prove an initial condition for $V(n) = \lambda \cos(n^2)$ for small $\lambda > 0$.