Math 211 - Hw #10

B 9/7.6.

$$\begin{vmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & -1 & -2 \end{vmatrix} \underset{\text{add } -4R_1 \text{ to } R_2}{\overset{\text{add } 3R1 \text{ to } R2}{=}} \begin{vmatrix} \boxed{1} & 0 & 4 \\ 0 & \boxed{3} & 10 \\ 0 & -1 & -18 \end{vmatrix} \underset{\text{to } R_3}{\overset{\text{add } \frac{1}{3}R2}{=}} \begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & 10 \\ 0 & 0 & -\frac{44}{3} \end{vmatrix}$$

$$= 1 \cdot 3 \cdot \left(-\frac{44}{3}\right) = \underline{\underline{-44}}$$

B 20/7.6.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{vmatrix} \underset{\text{by row 3}}{\overset{\text{expand}}{=}} (-1)^{3+1} \cdot (-1) \begin{vmatrix} -1 & 3 & 4 \\ 2 & -2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + $$

$$+ (-1)^{3+2} \cdot 2 \begin{vmatrix} 2 & 3 & 4 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 3 & 4 \\ 2 & -2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3 & 4 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{vmatrix} \qquad (*)$$

Take the first determinant and expand by row 2:

$$\begin{vmatrix} -1 & 3 & 4 \\ 2 & -2 & 0 \\ 3 & 1 & 2 \end{vmatrix} = (-1)^{2+1} \cdot 2 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + (-1)^{2+2} \cdot (-2) \begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= -2 \cdot (3 \cdot 2 - 4 \cdot 1) + (-2)(-1 \cdot 2 - 4 \cdot 3) = 24$$

Expand the second determinant by row 2:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-2) \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = -2 \cdot (2 \cdot 2 - 4 \cdot (-1)) = -16$$

Hence, from $(*)$ we have the answer: $-24 - 2 \cdot (-16) = \boxed{8}$

27. We row reduce in order to compute the determinant:

$$\begin{vmatrix} \boxed{1} & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} \boxed{1} & -2 & -4 \\ 0 & 5 & 10 \\ 0 & 6 & 12 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 6 & 12 \end{vmatrix}$$

$$= 5 \cdot \begin{vmatrix} \boxed{1} & -2 & -4 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{vmatrix} = 5 \cdot 1 \cdot 1 \cdot 0 = \boxed{0}$$

Because the determinant is 0, the matrix has a nontrivial subspace (notice that there are non-pivot columns).

From the row-reduction, we can obtain a basis, by solving:

$$\begin{aligned} x_1 - 2x_2 - 4x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned} \Rightarrow x_2 = -2x_3 \; ; \; x_1 = 0$$

x_3 - free

Null(A): $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$; hence a basis is $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

From the fact that the determinant is 0, we conclude that the column vectors are linearly dependent (this can be also noticed from the row-reduced matrix).

B 37/7.6. We use determinants. The given matrix has a nontrivial nullspace if and only if the determinant is 0.

$$\begin{vmatrix} 2-x & 0 & 0 \\ -1 & -x & 2 \\ 0 & -2 & 5-x \end{vmatrix} \underset{\text{by row 1}}{\overset{\text{expand}}{=}} (-1)^{1+1}(2-x) \begin{vmatrix} -x & 2 \\ -2 & 5-x \end{vmatrix}$$

$$= (2-x)\left(-x(5-x) - 2(-2)\right) =$$

$$= (2-x)(x^2 - 5x + 4) = (2-x)(x-1)(x-4).$$

$\det = 0 \Rightarrow \boxed{x = 1, 2, 4}$

B 50/7.6. (a) Because each row of $-2U$ is a multiple of a row of U (by -2) we get

$$\det(-2U) = (-2)\cdot(-2)\cdot(-2)\cdot(-2)\det U = 16(-3) = \boxed{-48}$$

(b) $$\det(U^3) = \det(U \times U \times U) = \det(U)\cdot\det(U)\cdot\det(U) = [\det(U)]^3 = \underline{\underline{-27}}$$

(c) $$\det(U^{-1}) = \frac{1}{\det U} = \underline{-\frac{1}{3}}$$

B 3/8.1. (3) Autonomous system of dimension 3.

B 11/8.1. $y'' + 2y' + 4y = 3\cos 2t \quad y(0) = 1, \; y'(0) = 0$

Let $x_1 = y$, $x_2 = y'$. Then

$$x_1' = y' = x_2$$

$$x_2' = y'' = -2y' - 4y + 3\cos 2t = -4x_1 - 2x_2 + 3\cos 2t$$

Matrix form: $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3\cos 2t \end{bmatrix} \text{ with } \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

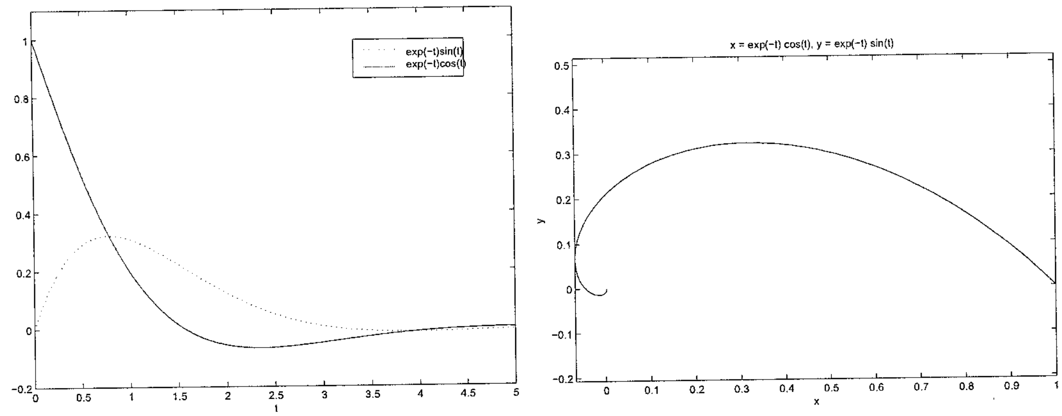
or $$\underline{\bar{x}' = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\bar{x} + \begin{bmatrix} 0 \\ 3\cos 2t \end{bmatrix} \text{ and } \bar{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

(This shows that the system is linear, inhomogeneous)

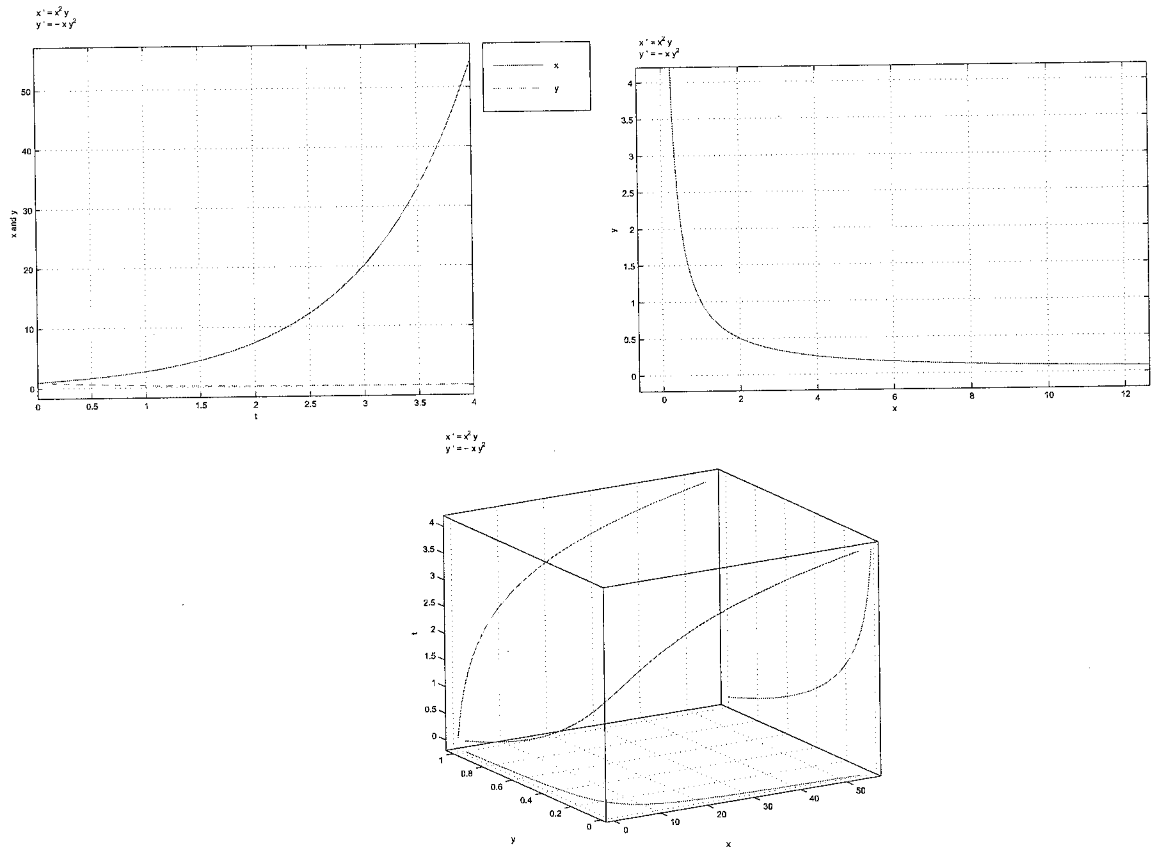
Equivalent form:

$$\underline{\underline{\bar{x}' = \begin{bmatrix} x_2 \\ -4x_1 - 2x_2 + 3\cos 2t \end{bmatrix}}} \qquad \bar{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

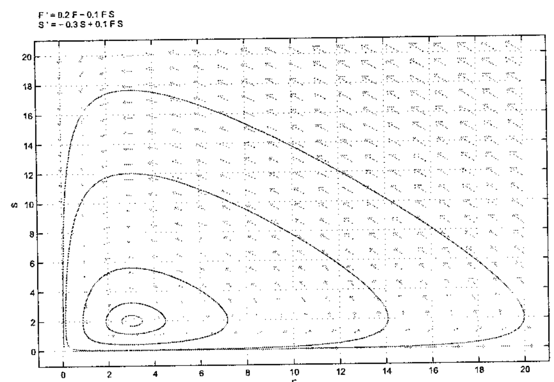
B5/8.2



B23/8.2



B 26/8.2 (a) The solutions seem to be closed in the phase space. This indicates a periodic oscillation of the two species. (b) The solution that starts at $(3,2)$ remains there for all times. This is because $(3,2)$ is an equilibrium point.



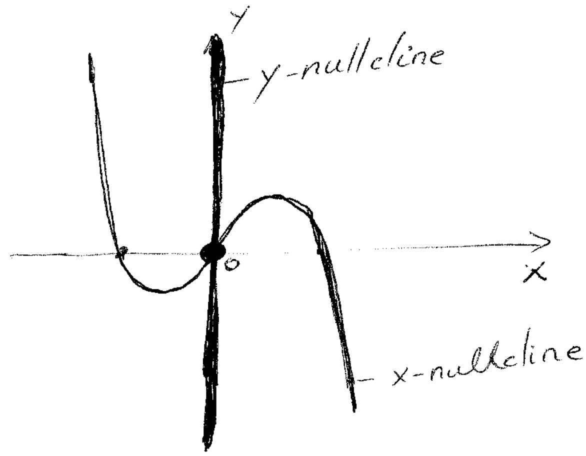
B 3/8.3. $x' = x - y - x^3$

$y' = x$

x-nullcline: $x - y - x^3 = 0$ or $\underline{y = x - x^3}$

y-nullcline: $\underline{x = 0}$

Equilebrium points: $\begin{cases} x - y - x^3 = 0 \\ x = 0 \end{cases} \Rightarrow \underline{x = 0,\ y = 0}$



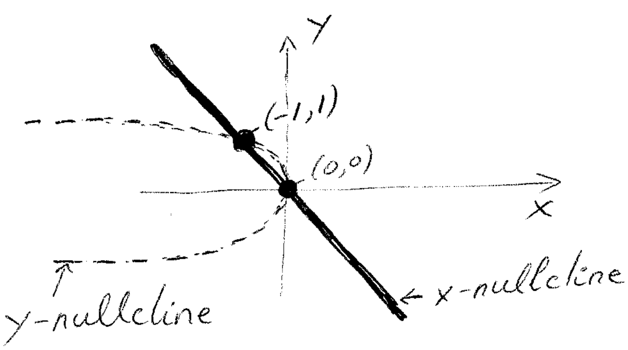
B 6/8.3. $x' = x + y^2$ x-nullcline: $x + y^2 = 0$ or $x = -y^2$

$y' = x + y$ y-nullcline: $x + y = 0$ or $y = -x$

Eg. points: $\begin{cases} x + y^2 = 0 \\ x + y = 0 \end{cases} \Rightarrow y = -x$, and $x + x^2 = 0$

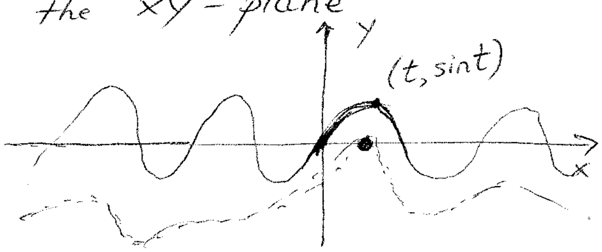
Hence $\underline{x = 0;\ y = 0}$ or

$\underline{x = -1,\ y = 1}$.



B 7/8.3. By direct verification $x(t) = t$, $y(t) = \sin t$ is a solution

In the xy-plane



The other solution starts at $(\frac{\pi}{2}, 0)$ below the solution $(t, \sin t)$. Since the system is autonomous, the phase plots cannot intersect hence $y(t) < \sin x(t)$.