Math 211 - Hw #11

B 8.4/13 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$

B 8.4/31 By direct verification, one can check that $\bar{y}_1(t) = \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix}$ and $\bar{y}_2(t) = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}$ satisfy $\bar{y}' = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}\bar{y}$

Since $\bar{y}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\bar{y}_2(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and obviously $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are l.i. it follows that $\bar{y}_1(t)$ and $\bar{y}_2(t)$ are l.i. Thus, the general solution is $\bar{y}(t) = c_1\bar{y}_1(t) + c_2\bar{y}_2(t) = \underline{c_1\begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}}$

$\bar{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or $\begin{matrix} c_1 + c_2 = 3 \\ c_2 = 1 \end{matrix}$.

Hence $c_2 = 1$ and $c_1 = 2 \Rightarrow \underline{\bar{y}(t) = 2\begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} + e^{-4t} \\ e^{-4t} \end{bmatrix}}$

B 8.4/37 $\frac{dx}{dt}$ = rate in - rate out

$= 0 - \frac{x(t)}{200}\cdot 10 = -\frac{x(t)}{20}$

$\frac{dy}{dt}$ = rate in - rate out $= \frac{x(t)}{20} - \frac{y(t)}{200}\cdot 10 = \frac{x}{20} - \frac{y}{20}$

$\frac{dz}{dt}$ = rate in - rate out $= \frac{y(t)}{20} - \frac{z(t)}{200}\cdot 10 = \frac{y}{20} - \frac{z}{20}$

Hence $\begin{cases} x' = -x/20 \\ y' = \frac{x}{20} - \frac{y}{20} \\ z' = \frac{y}{20} - \frac{z}{20} \end{cases}$

B 5/9.2 $\quad \bar{y}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \bar{y}$

Eigenvalues: $\det(A - \lambda I) = 0 \quad (\text{or} \quad \lambda^2 - T\lambda + D = 0)$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3.$$

Eigenvectors for $\lambda_1 = 2$: $\quad (A - 2I)\bar{v} = \bar{0}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-v_1 + 2v_2 = 0 \quad \Rightarrow \quad \bar{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

v_2 - free

Eigenvectors for $\lambda_2 = 3$: $\quad (A - 3I)\bar{v} = \bar{0}$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\bar{v} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution:

$$\bar{y}(t) = c_1 \cdot e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

B 11/9.2 $\quad \bar{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Hence $c_1 = 1, \quad c_2 = 1$

$$\bar{y}(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} + e^{3t} \\ e^{2t} + e^{3t} \end{bmatrix}$$

B 27/9.2. $A=\begin{bmatrix}-1 & 3\\ -3 & -1\end{bmatrix}$

Eigenvalues: $\lambda^2+2\lambda+10=0$

$$\lambda_{1,2}=\frac{-2\pm\sqrt{-36}}{2}=-1\pm 3i$$

Eigenvectors for $\lambda=-1+3i$: $(A-\lambda I)\bar{v}=\bar{0}$.

$$\left[\begin{array}{cc|c}-3i & 3 & 0\\ -3 & -3i & 0\end{array}\right]\sim\left[\begin{array}{cc|c}+1 & i & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\Rightarrow v_1+iv_2=0 \Rightarrow v_1=-iv_2 \qquad \bar{v}=t\begin{bmatrix}-i\\ 1\end{bmatrix}$$

From the fundamental solution $e^{(-1+3i)t}\begin{bmatrix}-i\\ 1\end{bmatrix}$ we can obtain two real fundamental solutions:

$$e^{(-1+3i)t}\begin{bmatrix}-i\\ 1\end{bmatrix}=e^{-t}(\cos 3t+i\sin 3t)\begin{bmatrix}-i\\ 1\end{bmatrix}=$$

$$=e^{-t}\begin{bmatrix}-i\cos 3t+\sin 3t\\ \cos 3t+i\sin 3t\end{bmatrix}$$

$$=e^{-t}\underline{\begin{bmatrix}+\sin 3t\\ \cos 3t\end{bmatrix}}+i\cdot \underline{e^{-t}\begin{bmatrix}-\cos 3t\\ \sin 3t\end{bmatrix}}.$$

Fundamental set of (real) solutions: $\bar{y}_1(t)=e^{-t}\begin{bmatrix}+\sin 3t\\ \cos 3t\end{bmatrix}$

$$\bar{y}_2(t)=e^{-t}\begin{bmatrix}-\cos 3t\\ \sin 3t\end{bmatrix}$$

(This answer is not unique!)

B 33/9.2. The general solution of the previous problem:

$$\bar{y}(t)=c_1\cdot e^{-t}\begin{bmatrix}\sin 3t\\ \cos 3t\end{bmatrix}+c_2e^{-t}\begin{bmatrix}-\cos 3t\\ \sin 3t\end{bmatrix}$$

From $\bar{y}(0)=\begin{bmatrix}3\\ 2\end{bmatrix} \Rightarrow c_1\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+c_2\cdot\begin{bmatrix}-1\\ 0\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix} \Rightarrow c_2=3,\ c_1=2$

Hence $\bar{y}(t)=e^{-t}\begin{bmatrix}2\sin 3t+3\cos 3t\\ -3\sin 3t+2\cos 3t\end{bmatrix}$

B 39/9.2. $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

Eigenvalues: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2$ repeated eigenvalue

Eigenvectors for $\lambda = 2$: $(A - 2I)\bar{v}_1 = \bar{0}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_1^1 - v_2^1 = 0 \Rightarrow \bar{v}_1 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We got one fundamental solution: $e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \bar{y}_1(t)$

A second fundamental solution is: $e^{2t}(\bar{v}_2 + t\,\bar{v}_1)$

where $(A - 2I)\bar{v}_2 = \bar{v}_1$

Hence $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_2^1 - v_2^2 = 1$

$\Rightarrow \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Pick $\bar{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(t=0)$

Therefore $\bar{y}_2(t) = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

General solution: $\bar{y}(t) = c_1 \cdot e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \cdot e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(Not unique formula: you can pick arbitrary eigenvectors)

B 45/9.2. $\bar{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -1 \\ c_2 = 3 \end{array}$

Hence $\bar{y}(t) = -e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3e^{2t}\begin{bmatrix} t+1 \\ t \end{bmatrix}$

$$= e^{2t}\begin{bmatrix} 3t+2 \\ 3t-1 \end{bmatrix} = \begin{bmatrix} e^{2t}(3t+2) \\ e^{2t}(3t-1) \end{bmatrix}.$$