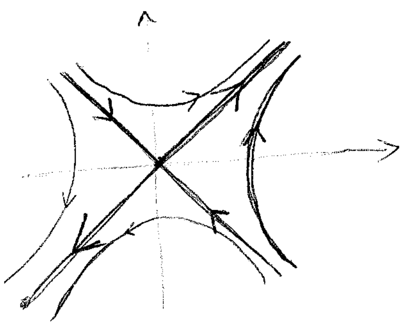
Math 211 - Hw #12

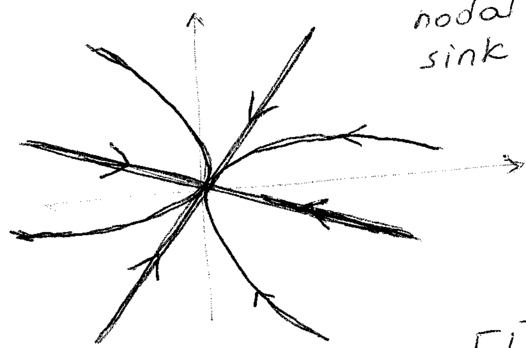
$\boxed{B\ 12/9.3}$ $y(t) = c_1 \cdot e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \cdot e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

saddle



$\boxed{B\ 13/9.3}$

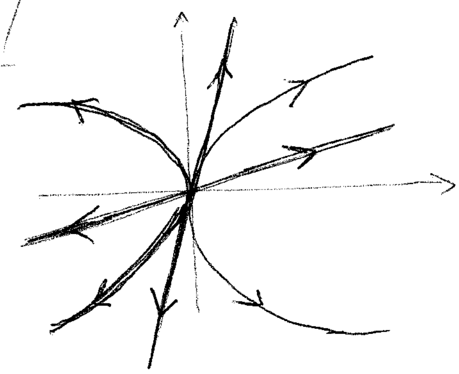
nodal sink



tangent direction: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{B\ 15/9.3}$

nodal source



tangent direction: $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

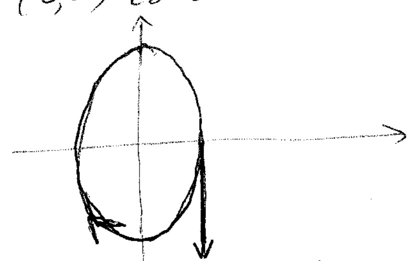
$\boxed{B\ 19/4.3}$

$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \bar{y}$

Eigenvalues: $\lambda^2 + 4 = 0$

$\Rightarrow \lambda = \pm 2i$

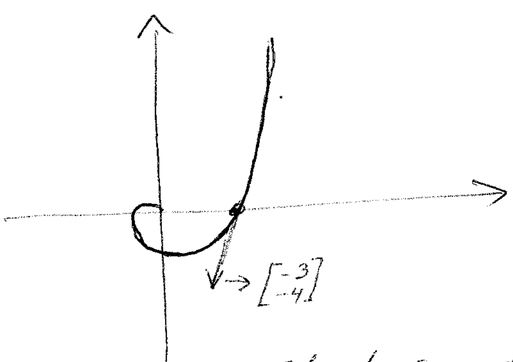
Hence $(0,0)$ is a center.



At $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the direction vector is $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$ (points downward), hence the direction on the ellipse is clockwise.

B23/9.3 Eigenvalues: $\lambda = -1 \pm 2i$

$(0,0)$ is a spiral sink



Clockwise spiral

B 45/9.3 $\quad A = \begin{bmatrix} 2 & 1 \\ -10 & -5 \end{bmatrix}$; $\quad Y' = \begin{bmatrix} 2 & 1 \\ -10 & -5 \end{bmatrix} Y$

Notice that $\det A = 0$, hence the system has an infinite number of equilibrium points obtained from

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -10 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$2x_1 + x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

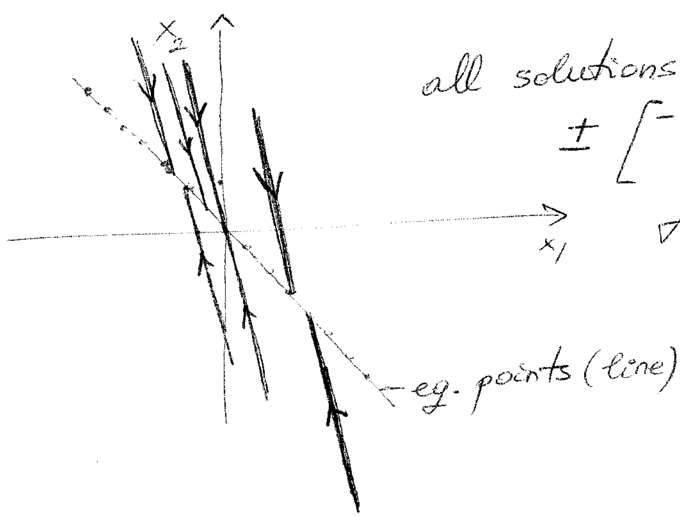
The general solution is obtained after computing the eigenvalues and the corresponding eigenvectors:

Eigenvalues: $\lambda^2 - T\lambda + D = 0$; $\lambda^2 + 3\lambda = 0 \Rightarrow \lambda = 0, -3$

For $\lambda = -3$: $(A + 3I)\bar{v} = \bar{0} \Rightarrow \left[\begin{array}{cc|c} 5 & 1 & 0 \\ -10 & -2 & 0 \end{array}\right] \Rightarrow \bar{v} = t \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix}$

For $\lambda = 0$: $A\bar{v} = \bar{0} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -10 & -5 & 0 \end{array}\right] \Rightarrow \bar{v} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Hence, the general solution is $\quad c_1 \cdot e^{-3t} \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$



all solutions are lines of direction $\pm \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix}$ translated along the line of eq. points

B 7/9.4 $A = \begin{bmatrix} 4 & -5 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & -5 & 4 \\ 0 & -1-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -1.$

Eigenvectors: for $\lambda = 4$: $(A - 4I)\bar{v} = 0$ $\left[\begin{array}{ccc|c} 0 & 5 & 4 & 0 \\ 0 & -5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$

$\Rightarrow v_3 = 0, v_2 = 0, v_1$ - free

$\bar{v} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = -1$: $\bar{v} = s \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 1$: $\bar{v} = s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

Hence, the general solution is:

$$\boxed{c_1 e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \cdot e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}.$$

13. We need $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow c_1 = 2, c_2 = -5$
$c_3 = 2$

Hence $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = 2e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 5 \cdot e^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

23. $A = \begin{bmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{bmatrix}$

Eigenvalues: $\begin{vmatrix} 6-\lambda & 0 & -4 \\ 8 & -2-\lambda & 0 \\ 8 & 0 & -2-\lambda \end{vmatrix} = 0$. First-row expansion $\Rightarrow$

$$\Rightarrow (-1)^{1+1}(6-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} + (-1)^{1+3}(-4)\begin{vmatrix} 8 & -2-\lambda \\ 8 & 0 \end{vmatrix} = 0$$

$$(6-\lambda)(-2-\lambda)^2 + (-4)(-8)(-2-\lambda) = 0$$

$$(-2-\lambda)(\lambda^2 - 4\lambda + 20) = 0 \Rightarrow \lambda_1 = -2$$

$$\lambda_{2,3} = 2 \pm 4i$$

Eigenvectors: for $\lambda_1 = -2$: $(A+2I)\bar{v} = \bar{0}$

$$\left[\begin{array}{ccc|c} 8 & 0 & -4 & 0 \\ 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 8 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$8v_1 - 4v_3 = 0$

$v_3 = 0$ $\Rightarrow$ $\bar{v} = v_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

v_2 - free

for $\lambda = 2+4i$: $(A-(2+4i)I)\bar{v} = \bar{0}$

$$\left[\begin{array}{ccc|c} 4-4i & 0 & -4 & 0 \\ 8 & -4-4i & 0 & 0 \\ 8 & 0 & -4-4i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1-i & 0 & -1 & 0 \\ 0 & -4-4i & 4+4i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1-i & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(1-i)v_1 - v_3 = 0$

$v_2 - v_3 = 0$

v_3 - free

$$\Rightarrow \bar{v} = v_3 \begin{bmatrix} \frac{1}{1-i} \\ 1 \\ 1 \end{bmatrix} = s \cdot \begin{bmatrix} 1+i \\ 2 \\ 2 \end{bmatrix}$$

So, a complex solution would be: $c_1 \cdot e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \cdot e^{t(2+4i)}\begin{bmatrix} 1+i \\ 2 \\ 2 \end{bmatrix} + c_3 \cdot e^{t(2-4i)}\begin{bmatrix} 1-i \\ 2 \\ 2 \end{bmatrix}$.

We look for real solutions, thus

$$e^{t(2+4i)}\begin{bmatrix}1+i\\2\\2\end{bmatrix} = e^{2t}(\cos 4t + i\sin 4t)\begin{bmatrix}1+i\\2\\2\end{bmatrix} =$$

$$= e^{2t}\begin{bmatrix}\cos 4t - \sin 4t + i(\cos 4t + \sin 4t)\\ 2(\cos 4t + i\sin 4t)\\ 2(\cos 4t + i\sin 4t)\end{bmatrix}.$$

$$= e^{2t}\begin{bmatrix}\cos 4t - \sin 4t\\ 2(\cos 4t)\\ 2\cos 4t\end{bmatrix} + ie^{2t}\begin{bmatrix}\cos 4t + \sin 4t\\ 2\sin 4t\\ 2\sin 4t\end{bmatrix}$$

Hence $\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix} = c_1\cdot e^{-2t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_2\cdot e^{2t}\begin{bmatrix}\cos 4t - \sin 4t\\ 2\cos 4t\\ 2\cos 4t\end{bmatrix} + c_3\cdot e^{2t}\begin{bmatrix}\cos 4t + \sin 4t\\ 2\sin 4t\\ 2\sin 4t\end{bmatrix}$

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34 $\quad A = \begin{bmatrix}2 & 0 & 0\\ -6 & 2 & 3\\ 6 & 0 & -1\end{bmatrix}$ $\quad$ Eigenvalues: $\begin{vmatrix}2-\lambda & 0 & 0\\ -6 & 2-\lambda & 3\\ 6 & 0 & -1-\lambda\end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)\begin{vmatrix}2-\lambda & 3\\ 0 & -1-\lambda\end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda)(-1-\lambda) = 0$$

$\Rightarrow \lambda = 2$ (repeated root)

$\lambda = -1$

Eigenvectors: for $\lambda = 2$: $(A-2I)\bar{V} = \bar{0}$

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ -6 & 0 & 3 & 0\\ 6 & 0 & -3 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}2 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\bar{V} = \begin{bmatrix}V_3/2\\ V_2\\ V_3\end{bmatrix} = V_2\begin{bmatrix}0\\1\\0\end{bmatrix} + V_3\begin{bmatrix}1/2\\0\\1\end{bmatrix}$$

Geometric multiplicity is 2 (same as algebraic multiplicity) for $\lambda = -1$: $(A + I)\bar{v} = \bar{0} \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ -6 & 3 & 3 & 0 \\ 6 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\bar{V} = \begin{bmatrix} 0 \\ -v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

We have obtained three linearly independent eigenvectors,

hence $$\bar{y}(t) = c_1 \cdot e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \cdot e^{2t} \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} + c_3 \cdot e^{-t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$