Math 211 - Hw #2

B 5/2.2. $y' = xy + y$

$\frac{dy}{dx} = xy + y$. Separate variables (assuming $y \neq 0$):

$\frac{dy}{y} = (x+1)dx$. Integrate:

$\int \frac{dy}{y} = \int (x+1)dx \Rightarrow \ln|y| = \frac{x^2}{2} + x + c$. Exponentiate:

$|y(x)| = e^{\frac{x^2}{2}+x+c} = e^c \cdot e^{\frac{x^2}{2}+x}$

Hence $y(x) = \pm e^c \cdot e^{\frac{x^2}{2}+x}$ or $y(x) = D \cdot e^{\frac{x^2}{2}+x}$

where D is an arbitrary constant.

Remark: $D=0$ gives us $y(x)=0$ which is a solution, too.

B 15/2.2 $y' = \frac{\sin x}{y}$, $y(\pi/2) = 1$

Separate variables and integrate:

$\int y\,dy = \int \sin x\,dx$

$\frac{y^2}{2} = -\cos x + c$.

From $y(\pi/2) = 1$, one gets $\frac{1}{2} = 0 + c \Rightarrow \underline{c = \frac{1}{2}}$

The implicit solution is $\frac{y^2}{2} + \cos x = \frac{1}{2}$

Explicit solution: $y^2 = 1 - 2\cos x \Rightarrow y(x) = \sqrt{1 - 2\cos x}$

Interval of existence: $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, because $1 - 2\cos x > 0$.

B27/2.2. If $N(t)$ is the amount of ^{32}P at time t, then $N(t) = N_0 \cdot e^{-\lambda t}$ (N_0 - initial amount)

and $T_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$ (half-life)

In our problem: $N_0 = 1{,}000$ mg and $N(10) = 615$

Hence $1{,}000 \cdot e^{-10\lambda} = 615 \Rightarrow \lambda = \frac{\ln \frac{615}{1{,}000}}{-10} \approx .0486$

and $T_{\frac{1}{2}} = \frac{\ln 2}{\lambda} \approx 14.26$ hours

B35/2.2. If $T(t)$ is the temperature of the body at time t, from Newton's law of cooling, we get:

$$\frac{dT}{dt} = -k(T-A)$$

(A - surrounding medium temp.)

Separate variables and integrate:

$$\int \frac{dT}{T-A} = \int -k\,dt \Rightarrow \ln|T-A| = -kt + c$$

$$\Rightarrow |T-A| = e^{-kt+c} = e^{c} \cdot e^{-kt}$$

$$\Rightarrow T-A = \pm e^{c} \cdot e^{-kt} = D \cdot e^{-kt}$$

$$\Rightarrow T(t) = A + D \cdot e^{-kt}$$

(D - arbitrary constant)

from $T(0) = T_0$ - initial temperature, one gets

$$\underline{T(t) = A + (T_0 - A)e^{-kt}}$$

b) We can assume that at midnight $\underline{t = 0}$

So $T(0) = 31$, $T(1) = 29$. We look for a time t such that $T(t) = 37$.

From $T(0) = 31 \Rightarrow T(t) = 21 + 10 \cdot e^{-kt}$

Using $T(1) = 29 \Rightarrow 21 + 10 \cdot e^{-k} = 29 \Rightarrow k = \ln \frac{10}{8} \approx .223$

Now, asking for $T(t) = 37$, one gets

$$t = \frac{\log \frac{10}{16}}{k} \approx -2.1 \approx -2 \text{ hours and } 6 \text{ minutes}$$

This means that the time of death is $\underline{21:54}$

B 4/2.3. Assuming that $a = 100\,m/s^2$ is motor's acceleration, in the 60 seconds

$$v(t) = (a - 9.8)t = 90.2t$$

$$x(t) = \frac{1}{2}(a - 9.8)t^2 = 45.1t^2$$

hence $v(60) = 5{,}412\ m/s$, $x(60) = 162{,}360\ m$

For the second part of upward motion:

$$\frac{dv}{dt} = -g \qquad v(0) = 5{,}412\ m/s \quad x(0) = 162{,}360\ m$$

Hence $v(t) = -gt + 5{,}412$, $x(t) = -\frac{1}{2}gt^2 + 5412t + 162{,}360$

Maximum altitude is reached at a time t_1 when $v(t_1) = 0$

Hence $t_1 = 552.2\,s$ and $x_{max} \sim 1.657 \times 10^6$

The last part (downward) motion: $v(0) = 0$; $x(0) = x_{max}$

Rocket reaches the earth at a time t_2 when $x(t_2) = 0$

$x(t) = -\frac{1}{2}gt^2 + x_{max}$, hence $t_2 \approx 581.5\,s$

Total trip: $60\,s + 552.2\,s + 581.5\,s \approx 1{,}193.7\,s$

B 9/2.3. Newton's second law:

$$m\frac{dv}{dt} = -mg - rv$$

gives a solution $v(t) = C\cdot e^{-\frac{r}{m}t} - \frac{mg}{r}$

and $v_{term} = -\frac{mg}{r}$

r can be obtained from $-1 = -r\cdot(0.2) \Rightarrow \underline{r=5}$

and $v_{term} = -\frac{mg}{r} = -.196\ \frac{m}{s}$

B 11/2.3. Newton's second law:

$$m\frac{dv}{dt} = -mg \Rightarrow \frac{dv}{dt} = -g$$

Also, from the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dy}\cdot\frac{dy}{dt} = v\frac{dv}{dy}$$

where $y(t)$ is the position at time t.

Hence $v\frac{dv}{dy} = -g \Rightarrow \int v\,dv = \int -g\,dy$

We can use definite integrals:

$t=0$: $y_0 = 1.5$, v_0 - initial velocity

t_{max}: $y_{max} = 15$, $v = 0$

Hence $\int_{v_0}^{0} v\,dv = \int_{1.5}^{15} -g\,dy \Rightarrow$

$$\frac{v^2}{2}\Big|_{v_0}^{0} = -gy\Big|_{1.5}^{15} \Rightarrow -\frac{v_0^2}{2} = -g(15-1.5)$$

Therefore $v_0 = \sqrt{2\cdot 9.8\cdot 13.5} \approx 16.266\ \frac{m}{s}$

Assuming air resistance:

$$m\frac{dv}{dt} = -mg - rv \text{ or (similar to the first part)}$$

$$v\,dv = \left(-g - \frac{r}{m}v\right)dy$$

Separate variables and integrate:

$$\int_{v_0}^{0} \frac{v}{g+\frac{r}{m}v}\,dv = \int_{1.5}^{15} -1\,dy$$

Left-side equals: $\frac{m}{r}\int_{v_0}^{0}\left(1 - \frac{g}{g+\frac{r}{m}v}\right)dv =$

$$= \frac{m}{r}\left(v - \frac{mg}{r}\ln\left(g+\frac{r}{m}v\right)\right)\Big|_{v_0}^{0}$$

$$= \frac{m}{r}\left(-\frac{mg}{r}\ln g - v_0 + \frac{mg}{r}\ln\left(g+\frac{r}{m}v_0\right)\right)$$

$$= \frac{m^2g}{r^2}\ln\left(g+\frac{r}{m}v_0\right) - \frac{m}{r}v_0 - \frac{m^2g}{r^2}\ln g$$

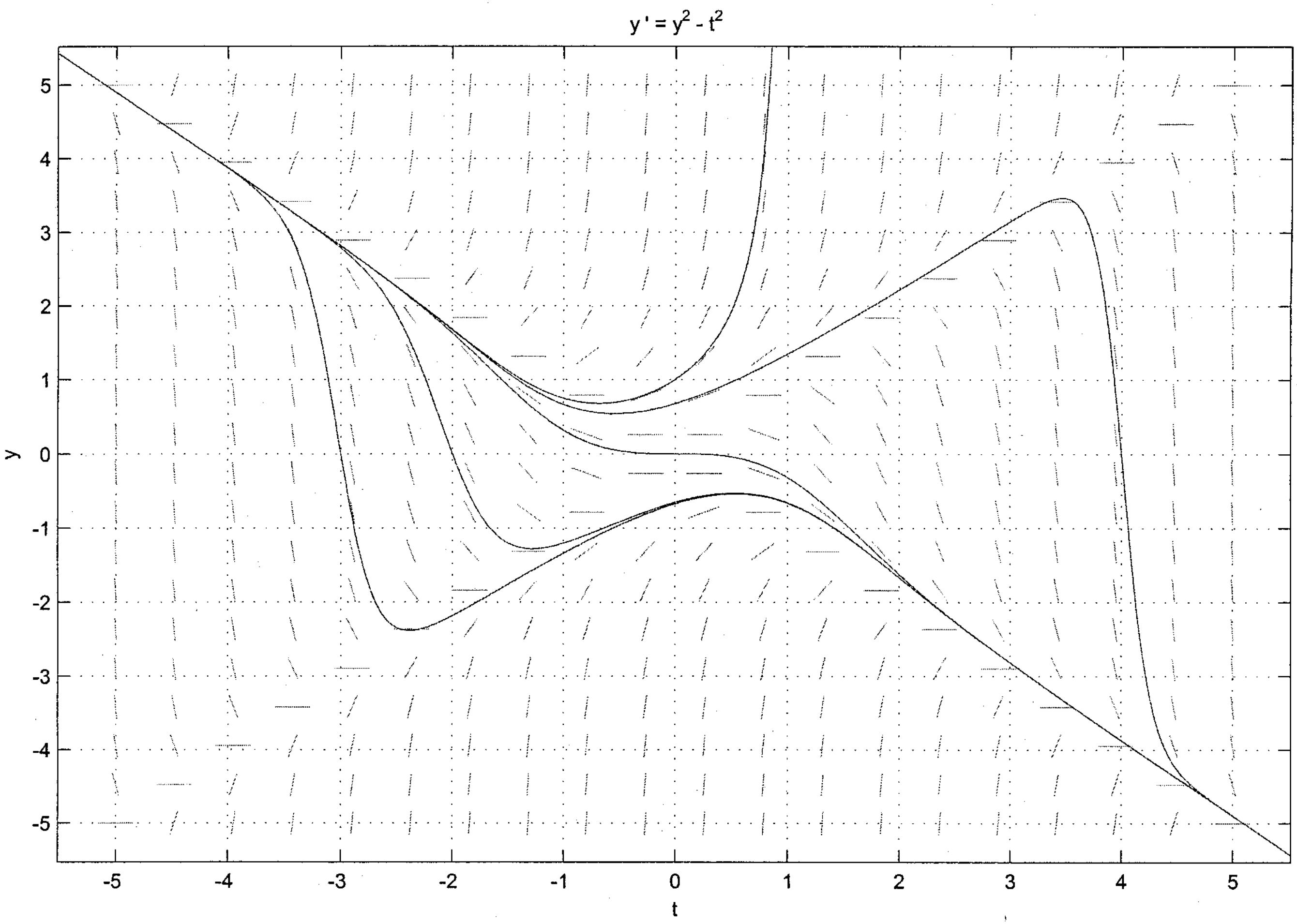
$$= \frac{m^2g}{r^2}\ln\left(1+\frac{r}{mg}v_0\right) - \frac{m}{r}v_0$$

Need to solve: $245\ln\left(1+\frac{1}{49}v_0\right) - 5v_0 = -13.5$

Using a calculator (or MATLAB): $v_0 \sim 18.114$ m/s

<u>Remark</u>: An alternative method is to rescale variables.

M 2/Ch 3



M 6/Ch 3