# Math 211 - Hw #3

**B 4/2.4** $y' + 2ty = 5t$

Integrating factor: $u(t) = e^{\int 2t\,dt} = e^{t^2}$

General solution: $u(t) = \dfrac{\int e^{t^2} \cdot 5t\,dt}{e^{t^2}} = \dfrac{\frac{5}{2}e^{t^2} + c}{e^{t^2}}$

$= \frac{5}{2} + \underline{C \cdot e^{-t^2}}$

**B 5/2.4** $x' - 2x/(t+1) = (t+1)^2$

$a(t) = \dfrac{2}{t+1}$; $f(t) = (t+1)^2$

Integrating factor: $u(t) = e^{\int -a(t)dt} = e^{-\int \frac{2}{t+1}dt}$

$= e^{-2\ln|t+1|} = (t+1)^{-2}$

General solution: $x(t) = \dfrac{\int u(t) f(t)}{u(t)} = \dfrac{\int 1\,dt}{(t+1)^{-2}} =$

$= \dfrac{t+c}{(t+1)^{-2}} = t(t+1)^2 + \underline{c(t+1)^2}.$

**B 16/2.4** $(1+t^2)y' + 4ty = (1+t^2)^{-2}$ $\quad y(1) = 0$

Divide by $(1+t^2)$: $y' + \dfrac{4t}{1+t^2}y = (1+t^2)^{-3}$

Integrating factor: $u(t) = e^{\int \frac{4t}{1+t^2}dt} = (1+t^2)^2$

General solution: $x(t) = \dfrac{\int (1+t^2)^{-1}dt}{(1+t^2)^2} = \dfrac{\arctan(t) + c}{(1+t^2)^2}$

From $y(1) = 0$, $C = -\frac{\pi}{4}$, hence $x(t) = \dfrac{\arctan t - \pi/4}{(1+t^2)^2}$.

B 21/2.4. $(1+t)x' + x = \cos t, \; x(-\pi/2) = 0.$

Divide by $(1+t)$: $\quad x' + \frac{1}{1+t} x = \frac{\cos t}{1+t}$

Integrating factor: $u(t) = e^{\int \frac{1}{1+t} dt} = |1+t|$

Because the initial moment is $-\frac{\pi}{2}$ and $\frac{1}{1+t}$ is not continuous at $t = -1$, we need to solve for $t < -1$.

Therefore $u(t) = -(1+t)$, and

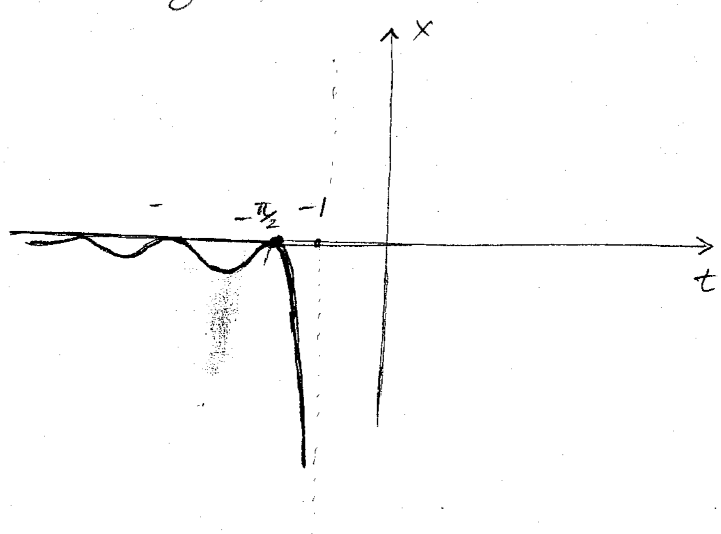
$$x(t) = \frac{\int -\cos t \, dt}{-(1+t)} = \frac{-\sin t + c}{-(1+t)}$$

From $x(-\pi/2) = 0$, we find $c = -1$ and

$$x(t) = \frac{-\sin t - 1}{-(1+t)} = \frac{\sin t + 1}{1+t}.$$

The maximal interval of existence is $(-\infty, -1)$.

Sketch of graph:



B 1/2.5 $\quad x(t)$ - amount of sugar at time $t$

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$
$$= 3 \cdot 0.2 - 3 \cdot \frac{x(t)}{100}$$
$$= 0.6 - \frac{3x}{100}$$

Solving this linear equation: $x(t) = 20 + C \cdot e^{-3t/100}$

Since $x(0) = 0$, $C = -20$ and $\underline{x(t) = 20 - 20 \cdot e^{-3t/100}}$

a) $x(20) = 20 - 20 \cdot e^{-60/100} \approx \underline{9.038 \text{ lb}}$

b) $x(t) = 15 \Rightarrow 20 - 20 \cdot e^{-3t/100} = 15 \Rightarrow t = \frac{100 \ln 4}{3} \underline{\approx 46.2}$

c) $\lim\limits_{t \to \infty} x(t) = \underline{20}$

B 6/2.5 $\quad x(t)$ - amount of salt at time $t$

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$
$$= 2 \cdot 0 - 3 \cdot \frac{x(t)}{100 - t}$$

(the volume decreases by 1 gal/min)

So $\frac{dx}{dt} = \frac{-3x}{100-t}$, $x(0) = 100 \cdot 0.05 = 5$ lb.

Separating variables: $x(t) = A \cdot (100-t)^3$

Since $x(0) = 5$, $A = 5 \times 10^{-6}$ and $x(t) = 5 \cdot 10^{-6} (100-t)^3$

If the volume is 50 gal $\Rightarrow t = 50$ min and $x(50) = 0.625$ lb

B 12/2.5

$x(t)$ - amount of salt in tank A
$y(t)$ - amount of salt in tank B

$$\begin{cases} \dfrac{dx}{dt} = 5 \cdot 0 - 5 \cdot \dfrac{x(t)}{100} & x(0) = 20 \\[2ex] \dfrac{dy}{dt} = 5 \cdot \dfrac{x(t)}{100} - 2.5 \dfrac{y(t)}{200 + 2.5t} & y(0) = 40 \end{cases}$$

Solving for $x(t)$: $x(t) = 20 \cdot e^{-t/20}$

Solving for $y(t)$: $\dfrac{dy}{dt} = e^{-t/20} - \dfrac{y}{80+t}$ (linear equation)

Integrating factor: $u(t) = 80 + t$

General solution: $y(t) = \dfrac{c}{80+t} - 20 \cdot e^{-t/20} - \dfrac{400}{80+t} e^{-t/20}$

From $y(0) = 40$, $c = 5200$.

Tank B contains 250 gal when $200 + 2.5t = 250$, i.e. $t = 20$.

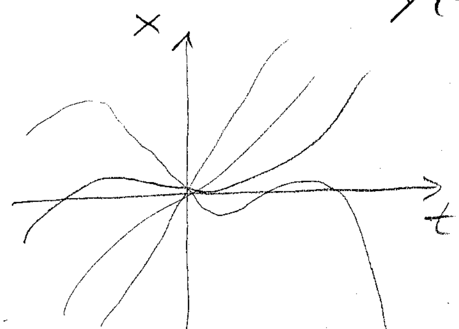
Hence $y(20) = \dfrac{5200}{100} - 20 \cdot e^{-1} - 4 \cdot e^{-1} \approx 43.17$ lb

B 7/2.7

$$t y' - y = t^2 \cos t$$
$$y' - \frac{1}{t} y = t \cos t$$

Linear equation with general solution:

$$y(t) = t \sin t + Ct.$$




Notice that all solutions pass throug $(0,0)$, b/c $y(0)$ is always 0.
Therefore, the IVP $y(0) = -3$ has no solutions
This does not contradict the existence thm b/c $1/t$ is not continuous at 0.

B 9/2.7. By direct verification, both $y(t)=0$ and $y(t)=t^3$ satisfy $y'=3y^{2/3}$, $y(0)=0$. This does not contradict the uniqueness theorem because the right-hand side of the equation $f(t,y)=3y^{2/3}$ has $\frac{\partial f}{\partial y}=2y^{-1/3}=\frac{2}{\sqrt[3]{y}}$ which is not continuous at $y=0$

B 14/2.7. $\frac{dy}{dt}=\frac{1}{(t+2)(y-3)}$ $\quad y(0)=1$.

Separating variables: $\int (y-3)\,dy=\int \frac{dt}{t+2}$

$$\tfrac{1}{2}y^2-3y=\ln|t+2|+c$$

Because $y(0)=1$, $c=-\frac{5}{2}-\ln 2$, and $t>-2$.

Therefore $\frac{1}{2}y^2-3y=\ln(t+2)-\frac{5}{2}-\ln 2$

$$y(t)=3\pm\sqrt{9+2\left(\ln(t+2)-\tfrac{5}{2}-\ln 2\right)}$$
$$=3\pm\sqrt{4+2\ln(t+2)-2\ln 2}$$

Because $y(0)=1 \Rightarrow y(t)=3-\sqrt{4+2\ln(t+2)-2\ln 2}$

The interval of existence:

$$4+2\ln(t+2)-2\ln 2>0$$

$\Rightarrow \quad \ln(t+2)>\ln 2-2 \Rightarrow t+2>e^{\ln 2-2}=2\cdot e^{-2}$

Therefore $t>2\cdot e^{-2}-2\approx -1.73$, and the interval of existence is $(2\cdot e^{-2}-2,\infty)$.

The problem is that when $t\searrow 2\cdot e^{-2}-2$, $y\nearrow 3$ and $\frac{1}{(t+2)(y-3)}\to-\infty$ hence the solution cannot be extended beyond this point.