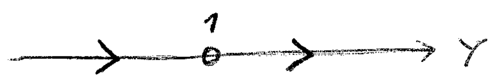
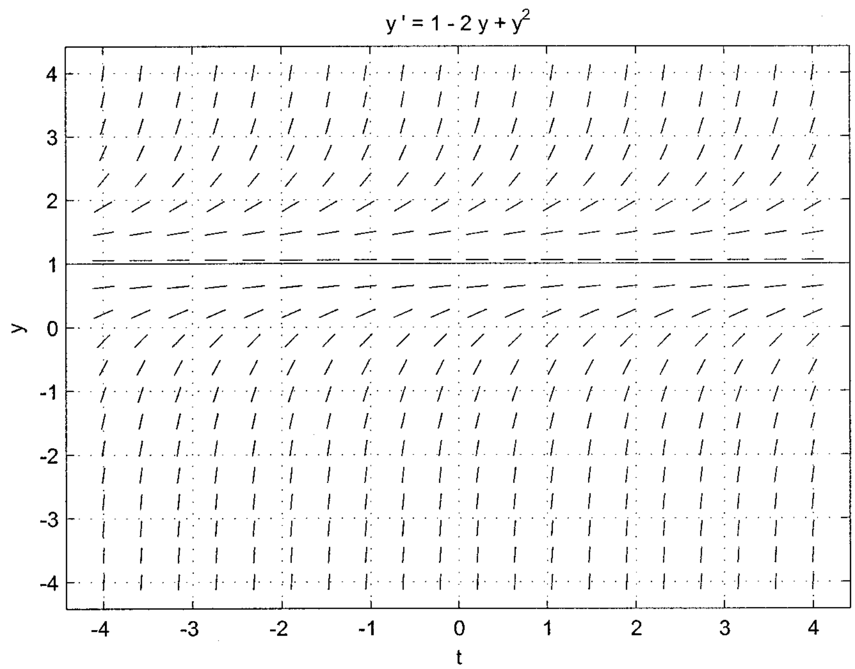
Math 211 - Hw #4

B 2/2.9. $Y' = 1 - 2Y + Y^2$

Equilibrium solutions: $1 - 2Y + Y^2 = 0 \Rightarrow (1-Y)^2 = 0 \Rightarrow \underline{Y = 1}$

Phase line: 

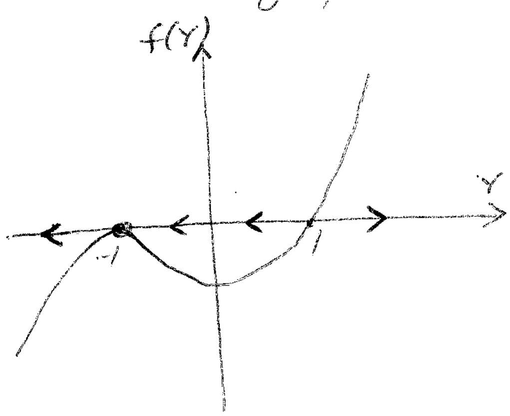
unstable equilibrium



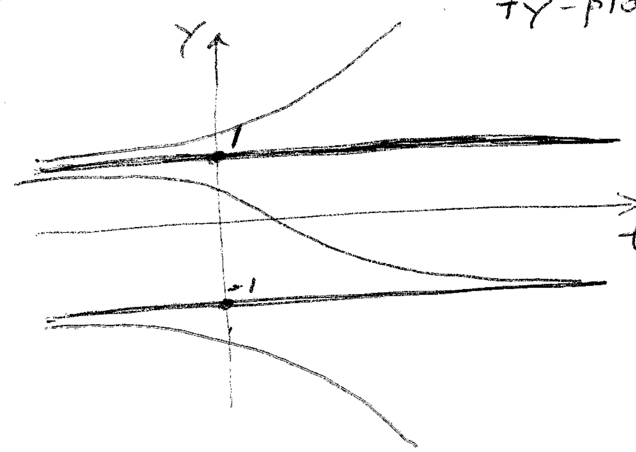
B 9/2.9. From the graph of f, there are 2 equilibrium points

$Y = -1$ (unstable)

$Y = 1$ (unstable)



tY-plane

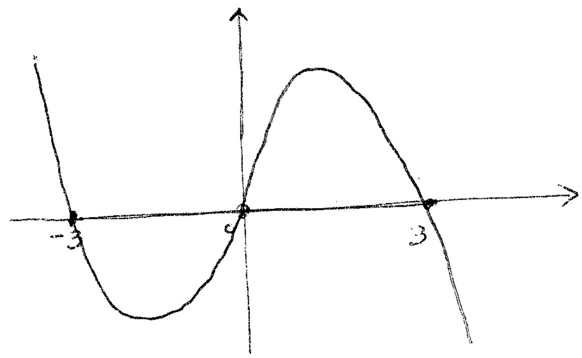


B 19/2.9 $\quad y' = 9y - y^3$

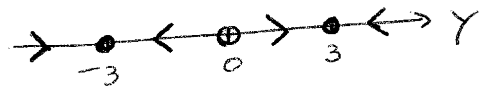
Equilibrium points: $9y - y^3 = 0$

$y(3-y)(3+y) = 0 \quad \begin{cases} y = 0 \\ y = 3 \\ y = -3 \end{cases}$

Graph of $f(y) = 9y - y^3$



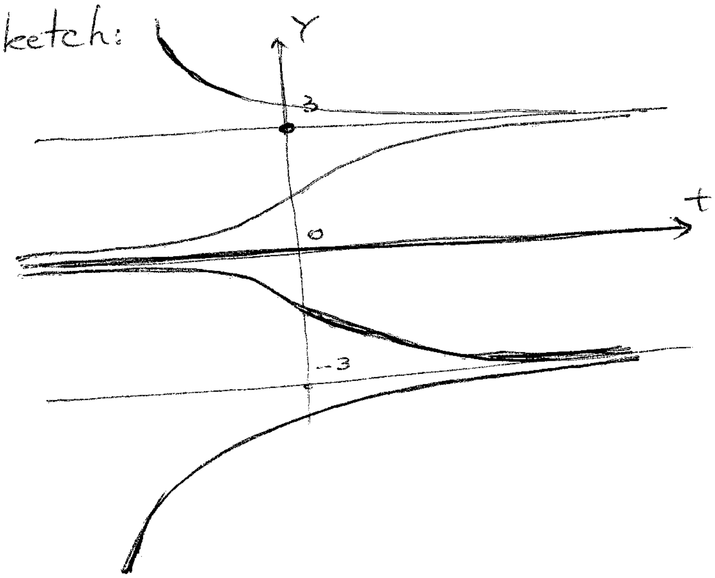
Phase line:



0 - unstable

-3, 3 - asymptotically stable

ty-plane sketch:



B 25/2.9. $\quad y' = (1+y)(5-y) \qquad y(0) = 2$

i) Separate variables: $\dfrac{dy}{(1+y)(5-y)} = dt$

Integrate $\displaystyle\int \frac{1}{(1+y)(5-y)} = \int dt$

$\displaystyle\int \frac{1}{(1+y)(5-y)} = \frac{1}{6}\int \left(\frac{1}{1+y} + \frac{1}{5-y}\right) dy$ (partial fractions)

$= \dfrac{1}{6}\left(\ln|1+y| - \ln|5-y|\right) = \dfrac{1}{6} \ln\left|\dfrac{1+y}{5-y}\right|$

Hence $\frac{1}{6} \ln \left| \frac{1+y}{5-y} \right| = t + c$

$\Rightarrow \frac{1+y}{5-y} = A \cdot e^{6t}$. From $y(0)=2$, we get $A=1$

and $\frac{1+y}{5-y} = e^{6t} \Rightarrow 1+y = 5 \cdot e^{6t} - y \cdot e^{6t}$. Solve for y:

$$y\left(1+e^{6t}\right) = 5 \cdot e^{6t} - 1$$

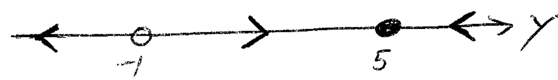
$$y(t) = \frac{5 \cdot e^{6t} - 1}{1+e^{6t}}.$$

ii) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{5 \cdot e^{6t} - 1}{1+e^{6t}} = \lim_{t\to\infty} \frac{\cancel{e^{6t}}\left(5 - e^{-6t}\right)}{\cancel{e^{6t}}\left(1 + e^{-6t}\right)} = 5.$

iii) Qualitative analysis:

Equilibrium points: $y=-1$, $y=5$

Phase line:



-1 - unstable

5 - asy. stable

If we start with $y(0)=2$, then the solution approaches 5 as $t\to\infty$

B 2/3.1. $\frac{dP}{dt} = rP$, hence $P(t) = P_0 \cdot e^{rt}$.

We know $P(1) = 1000$, $P(2) = 3000$. Therefore

$P_0 \cdot e^{r} = 1000$, $P_0 \cdot e^{2r} = 3000 \Rightarrow \frac{\cancel{P_0} \cdot e^{2r}}{\cancel{P_0} \cdot e^{r}} = 3$

$\Rightarrow e^{r} = 3$; $\underline{\underline{r = \ln 3}}$

$P_0 = \frac{1000}{e^{\ln 3}} = \frac{1000}{3} \approx \underline{\underline{333 \text{ cells}}}$

B 12/3.1. $\dfrac{dP}{dt} = rP\left(1-\dfrac{P}{K}\right)$ $K = 20{,}000$; $P(0) = 1{,}000$
$P(8) = 1{,}200$

The solution of the ODE is:

$$P(t) = \frac{KP_0 \cdot e^{rt}}{K-P_0+P_0 \cdot e^{rt}} = \frac{K\,P_0}{P_0+(K-P_0)\cdot e^{-rt}}$$

From $P(8) = 1{,}200$ we get: $1{,}200 = \dfrac{20{,}000 \cdot 1{,}000}{1{,}000 + 19{,}000 \cdot e^{-8r}}$

Hence $e^{-8r} = \dfrac{20 \cdot 10^6 - 12 \cdot 10^5}{19{,}000 \cdot 1{,}200} = \dfrac{188}{228}$

$$r = \frac{\ln \frac{188}{228}}{-8} \approx 0.0241$$

We want $P(t) = \dfrac{75}{100} \cdot K$, hence

$$\frac{\cancel{K} \cdot P_0}{P_0+(K-P_0)\cdot e^{-rt}} = \frac{3}{4}\cancel{K} \Rightarrow$$

$$\Rightarrow \quad 4P_0 = 3P_0 + 3(K-P_0)\cdot e^{-rt}$$

$$\Rightarrow \quad e^{-rt} = \frac{P_0}{3(K-P_0)} \Rightarrow t = \frac{\ln \frac{P_0}{3(K-P_0)}}{-r} \approx \underline{167.67h}$$

B 15/3.1. (a) Since 100 fish are removed each day, the logistic equation becomes:

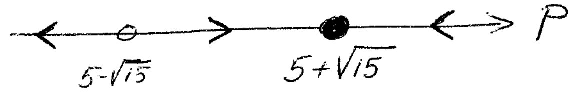
$$\frac{dP}{dt} = 0.1P\left(1-\frac{P}{10}\right) - 0.1 \qquad (P \text{ in thousands})$$

(b) Equilibrium points: $0.1P - 0.01P^2 - 0.1 = 0$

$$P^2 - 10P + 10 = 0$$

$$P = \frac{10 \pm \sqrt{60}}{2} = 5 \pm \sqrt{15}$$

Phase line :



$5-\sqrt{15}$ - unstable

$5+\sqrt{15}$ - asymptotically stable

(c) If $P_0 = 1$ (in thousands) then the population will decline to 0.

If $P_0 = 2$ (in thousands) then the population will tend toward $5+\sqrt{15} \approx 8.87$ (in thousands).

4/3.3. $\dfrac{dP}{dt} = rP + D$ $\qquad P_0 = 0 \qquad r = 6.25\%$

$D = ? \qquad P(18) = 50{,}000$

The solution of the ODE is:

$$P(t) = -\frac{D}{r} + \left(P_0 + \frac{D}{r}\right)\cdot e^{rt} = -\frac{D}{r} + \frac{D}{r}\cdot e^{rt}$$

$$= \frac{D}{r}\left(e^{rt} - 1\right)$$

From $P(18) = 50{,}000$:

$$\frac{D}{r}\left(e^{rt} - 1\right) = 50{,}000$$

$$D = \frac{50{,}000 \cdot r}{e^{18r} - 1} \approx \underline{\underline{1{,}502.25}} \text{ /year}$$

9/3.3. Let $S(t)$ be José's salary at time t (in thousands)

$$\frac{dS}{dt} = 0.01\, S \Rightarrow S(t) = S_0 \cdot e^{0.01t} = 28 \cdot e^{0.01t}$$

Let $P(t)$ be his retirement balance at time t (in thousands)

$$\frac{dP}{dt} = rP + \beta S(t) \quad \text{where } r = 6\% \quad P_0 = 25$$

$$= rP + \beta \cdot 28 \cdot e^{0.01t}$$

$\underline{\beta \text{ - unknown}}$

We solve this linear ODE:

integrating factor $u(t) = e^{-\int r\,dt} = e^{-rt}$

General solution: $P(t) = \dfrac{\int e^{-rt} \beta S(t)}{e^{-rt}} =$

$$= \frac{\int e^{-0.06t} \cdot \beta \cdot 28 \cdot e^{0.01t}\,dt}{e^{-0.06t}} = \frac{28\beta \cdot e^{-0.05t} \cdot \frac{-1}{0.05} + c}{e^{-0.06t}} =$$

$$= c \cdot e^{0.06t} - 560\beta \cdot e^{+0.01t}$$

From $P(0) = 2.5$, we get $C = 2.5 + 560\beta$, hence

$$P(t) = (2.5 + 560\beta)\, e^{0.06t} - 560\beta \cdot e^{0.01t}$$

We want $P(20) = 50$, therefore

$$(2.5 + 560\beta)\, e^{1.2} - 560\beta \cdot e^{0.2} = 50$$

$$\beta = \frac{50 - 2.5 \cdot e^{1.2}}{560\,(e^{1.2} - e^{0.2})} \approx 0.035 \quad \text{or } 3.5\%$$

15 a) Let $P(t)$ loan balance at time t (in years)

$$\frac{dP}{dt} = rP - A \qquad r = 8\% \qquad P(0) = 12{,}000$$

A - yearly payment

$$P(t) = \frac{A}{r} + \left(P_0 - \frac{A}{r}\right) e^{rt}$$

$$= \frac{A}{r}\left(1 - e^{rt}\right) + P_0 \cdot e^{rt}$$

$$P(5) = 0 \Rightarrow A = \frac{r \cdot P_0 \cdot e^{r5}}{e^{r5} - 1} \Rightarrow \text{monthly payment} = \frac{A}{12} \approx 242.66$$

Remark: one can choose the unit of time to be 1 month, hence similar computations will give the monthly payment.

b) Interest compounded monthly:

Let $P(n)$ - remaining balance after n-months.

$$P(n+1) = \left(1+\frac{r}{12}\right)P(n) - M \qquad M\text{- monthly payment}$$

Similar to exercise 13, this recursive relation gives:

$$P(n) = \left(P_0 - \frac{12M}{r}\right)\left(1+\frac{r}{12}\right)^n + \frac{12M}{r}$$

We want $P(60) = 0$, hence

$$\left(P_0 - \frac{12M}{0.08}\right)\left(1+\frac{0.08}{12}\right)^{60} + \frac{12M}{0.08} = 0$$

$$(P_0 - 150M)(1+0.0067)^{60} + 150M = 0$$

$$(P_0 - 150M)\cdot 1.4898 + 150M = 0$$

$$M \approx 243.93$$