Hw # ~~7~~ 6 - Math 211

B 1/4.5. $\quad y'' + 3y' + 2y = 4\cdot e^{-3t}$

We look for a solution of the form $y_p(t) = A\cdot e^{-3t}$

Substitute into the equation:

$$9A\cdot e^{-3t} + 3\left(-3A\cdot e^{-3t}\right) + 2\cdot A\cdot e^{-3t} = 4\cdot e^{-3t}$$

This implies that $2A = 4 \Rightarrow A = 2$.

$\boxed{y_p(t) = 2\cdot e^{-3t}}$ - a particular solution

B 17/4.5. $\quad y'' + 3y' + 4y = t^3$

We look for $y_p(t) = At^3 + Bt^2 + Ct + D$

$$y_p'(t) = 3At^2 + 2Bt + C; \quad y_p''(t) = 6At + 2B$$

Hence

$$\begin{array}{r} 6At + 2B + \\ 9At^2 + 6Bt + 3C + \\ 4At^3 + 4Bt^2 + 4Ct + D = \\ \hline t^3 \end{array}$$

$\Rightarrow \quad 4A = 1; \quad 9A + 4B = 0; \quad 6A + 6B + 4C = 0, \quad 2B + 3C + D = 0$

$\Rightarrow \quad A = 1/4; \quad B = -9/16; \quad C = 15/32; \quad D = -9/128$

$$\therefore \quad y_p(t) = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}$$

B 21/4.5. $y'' - 2y' + 5y = 3\cos t$, $y(0) = 0$; $y'(0) = -2$

The homogeneous equation $y'' - 2y' + 5y = 0$ has the general solution $y(t) = e^{t}(C_1 \cos 2t + C_2 \sin 2t)$

(b/c $\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$)

Look for a particular solution of the inhomogeneous eqn:

$$y_p(t) = A\cos t + B\sin t$$

$$y_p'(t) = -A\sin t + B\cos t; \quad y_p''(t) = -A\cos t - B\sin t$$

Hence $-A\cos t - B\sin t + 2A\sin t - 2B\cos t + 5A\cos t + 5B\sin t$

$$= 3\cos t$$

$$\cos t(-A - 2B + 5A) + \sin t(-B + 2A + 5B) = 3\cos t$$

$$\Rightarrow \begin{matrix} 4A - 2B = 3 \\ 2A + 4B = 0 \end{matrix} \quad \Rightarrow \quad A = \frac{3}{5}; \quad B = -\frac{3}{10}$$

The general solution of the inhomogeneous eqn. is:

$$y(t) = e^{t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

Using $y(0) = 0$ and $y'(0) = -2 \Rightarrow C_1 = -\frac{3}{5}$; $C_2 = -\frac{11}{20}$.

$$\therefore\ y(t) = e^{t}\left(-\frac{3}{5}\cos 2t - \frac{11}{20}\sin 2t\right) + \frac{3}{5}\cos t - \frac{3}{10}\sin t$$

B 25/4.5 $\quad y'' - y' - 2y = 2\cdot e^{-t}$

Look for $\quad y_p(t) = At\cdot e^{-t}$

Then $\quad y_p'(t) = A\cdot e^{-t} - At\cdot e^{-t}$

$$y_p''(t) = -A\cdot e^{-t} - A\cdot e^{-t} + At\cdot e^{-t}$$
$$= At\cdot e^{-t} - 2A\cdot e^{-t}$$

Substitute into the eqn:

$$\cancel{At\,e^{-t}} - 2A\cdot e^{-t} - A\cdot e^{-t} + \cancel{At\cdot e^{-t}} - \cancel{2At\cdot e^{-t}} = 2e^{-t}$$

$$\Rightarrow \quad -3A = 2 \quad \Rightarrow \quad A = -\tfrac{2}{3}.$$

$$\therefore \quad \underline{y_p(t) = -\tfrac{2}{3}\,e^{-t}}$$

B 37/4.5. $\quad y'' + 4y' + 4y = e^{-2t} + \sin 2t$

A particular solution is obtained by adding the particular solutions of $\quad y'' + 4y' + 4y = e^{-2t}$

and $\quad y'' + 4y' + 4y = \sin 2t$

For $\quad y'' + 4y' + 4y = e^{-2t}$

Since e^{-2t}, $t\cdot e^{-2t}$ are already solutions of the homogeneous equation, we look for $Y_1(t) = A\cdot t^2\cdot e^{-2t}$.

Substituting into the eqn, one gets (after some calculations) that $2A = 1$, hence $A = \frac{1}{2}$

For $y'' + 4y' + 4y = \sin 2t$

Look for $y_2(t) = A\cos 2t + B\sin 2t$.

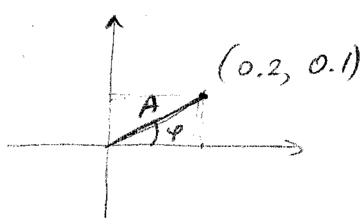
Substitute and calculate:

$$\cos 2t\,(-4A + 8B + 4A) + \sin 2t\,(-4B - 8A + 4B) = \sin 2t$$

$$\Rightarrow \quad \begin{array}{l} 8B = 0 \\ -8A = 1 \end{array} \quad \Rightarrow \quad B = 0, \quad A = -\tfrac{1}{8}$$

Hence $y_p(t) = \frac{1}{2}t^2 \cdot e^{-t} - \frac{1}{8}\cos 2t$ is a particular solution

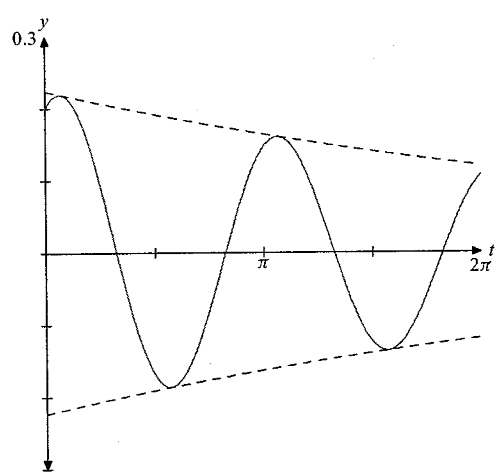
B 9/4.4. $\quad y(t) = e^{-0.1t}(0.2\cos 2t + 0.1\sin 2t)$



$$A = \sqrt{(0.2)^2 + (0.1)^2} = \sqrt{0.05}$$

$$\varphi = \arctan\frac{0.1}{0.2} = \arctan\frac{1}{2} \approx 0.4636$$

$$y(t) = e^{-0.1t} \cdot \sqrt{0.05}\cos(2t - 0.4636)$$



B 11/4.4. $\quad m = 0.2$ kg $\quad k = 5\,kg/s^2$; $\; y(0) = 0.5\,m$; $\; y'(0) = 0$

Notice that $\omega_0 = \sqrt{\frac{k}{m}} = 5$, and the equation of motion is:

$$y'' + 25y = 0.$$

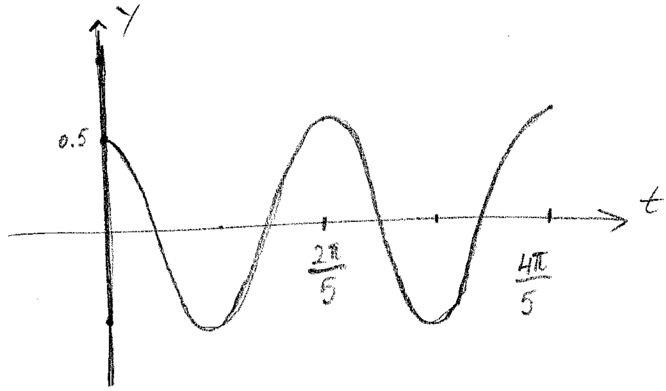
General solution $y(t) = c_1 \cos 5t + c_2 \sin 5t$

From $y(0) = 0.5$ and $y'(0) = 0 \Rightarrow c_1 = 0.5$; $c_2 = 0$.

Hence $\; y(t) = 0.5 \cos 5t \;\Rightarrow\; A = 0.5$ (amplitude)

$\varphi = 0$ (phase)

$\omega_0 = 5$ (frequency)



B 16/4.4 $\quad K = \frac{F}{y} = \frac{mg}{y} = \frac{1 \cdot 9.8}{4.9} = 2\ N/m$

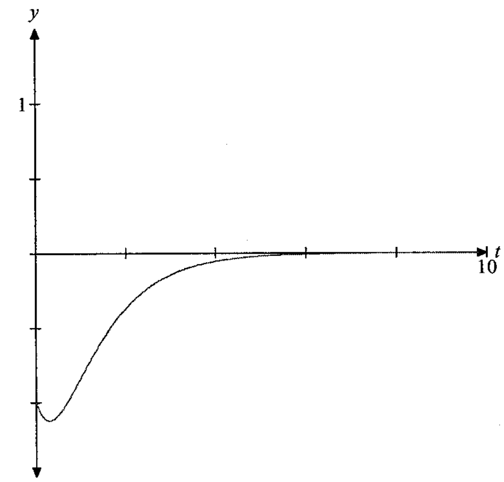
$m = 1$ kg, $K = 2\ N/m$, $\mu = 3$ kg/s hence the harmonic equation is $\; y'' + 3y' + 2y = 0$, with $y(0) = -1$, $y'(0) = -1$

The roots of $\lambda^2 + 3\lambda + 2 = 0$ are $\lambda_1 = -1$, $\lambda_2 = -2$. Hence

$$y(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{-2t}$$

Using $y(0) = -1$, $y'(0) = -1 \Rightarrow c_1 = -3$, $c_2 = 2$ and

$$y(t) = -3 \cdot e^{-t} + 2 \cdot e^{-2t}.$$



B 9/4.7. (a) $m = 1$ kg $K = 4$ kg/s² $f(t) = 4\cos\omega t$ N

The harmonic equation is:

$$y'' + 4y = 4\cos\omega t \qquad y(0)=0;\ y'(0)=0.$$

The solution is:

$$y(t) = \frac{4}{4-\omega^2}\left(\cos\omega t - \cos 2t\right)$$

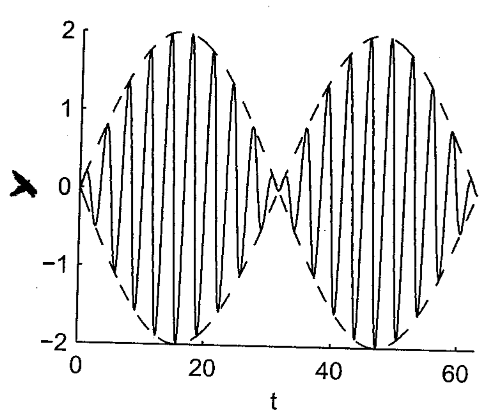
(b) Set $\bar\omega = (2+\omega)/2$; $\delta = \dfrac{2-\omega}{2}$ and the solution becomes

$$y(t) = \frac{2}{\bar\omega\,\delta}\sin\delta t\,\sin\bar\omega t.$$

Take a value close to $\omega_0 = 2$, for example $\omega = 1.8$.

Then $\bar\omega = 1.9$, and $\delta = 0.1$, so

$$y(t) = \frac{2}{0.19}\sin(0.1t)\sin(1.9t)$$



B 19/4.7 $x'' + 4x' + 5x = 3\sin t \quad x(0)=0, \; x'(0) = -3.$

The homogeneous eqn. $x''+4x'+5x$ has the general solution:

$$x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) \quad (\text{complex roots } \lambda = -2 \pm i)$$

Look for a particular solution $x_p(t) = A\cos t + B\sin t$

Substitute into the equation and calculate:

$$\cos t\,(4A+4B) + \sin t\,(-4A+4B) = 3\sin t$$

$$\Rightarrow \quad \begin{aligned} 4A+4B &= 0 \\ -4A+4B &= 3 \end{aligned} \quad \Rightarrow \quad A = -\frac{3}{8}, \; B = \frac{3}{8}$$

$$x_p(t) = -\frac{3}{8}\cos t + \frac{3}{8}\sin t$$

$$x(t) = e^{-2t}(c_1\cos t + c_2 \sin t) - \frac{3}{8}\cos t + \frac{3}{8}\sin t$$

From $x(0)=0$, $x'(0) = -3 \Rightarrow c_1 = \frac{3}{8}$, $c_2 = -\frac{21}{8}$.

$$x(t) = e^{-2t}\left(\frac{3}{8}\cos t - \frac{21}{8}\sin t\right) - \frac{3}{8}\cos t + \frac{3}{8}\sin t.$$