Hw # 9 - Math 211

B 11/7.4 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We solve $A\bar{x} = \bar{0}$ by row-reduction:

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \underset{\text{to } R2, R3}{\overset{\text{add } -R1}{\sim}} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \overset{R2 \leftrightarrow R3}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{-1} & -1 & 0 \\ 0 & 0 & \boxed{-1} & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -x_2 - x_3 &= 0 \\ -x_3 &= 0 \end{aligned} \quad \overset{\text{Backsolve}}{\Longrightarrow} \quad x_3 = 0 \;;\; x_2 = 0 \;;\; x_1 = 0.$$

Hence $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only solution. (This can be concluded from the fact that A has only nonzero entries along the diagonal, hence no free variables.)

A is therefore nonsingular.

B 12/7.4 $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Row-reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 0 & \boxed{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ -x_2 - x_3 &= 0 \end{aligned} \quad \Rightarrow x_2 = -x_3 \;;\; x_1 = -x_2 - 2x_3 = -x_3$$

x_3 - free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

A is singular.

B21/7.4. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ We apply row-reduction to $[A \mid I_3]$:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R2 \Leftrightarrow R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & \boxed{2} & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \overset{\frac{1}{2}R2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \boxed{1} & 0 & 1 & 0 \end{array}\right]$$ (pivots on the diagonal, A is nonsingular)

$$\overset{R2-\frac{1}{2}R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \overset{R1-2R1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & +1 & -1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

B23/7.4 The system has three variables and 2 equations, hence at least one column has no pivot, i.e. the system has either an infinite # of solutions (if it is consistent), or no solutions at all (if it is inconsistent).

B31/7.4. Reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 2 & -4 & 0 \\ 3 & -5 & 10 & a \\ 2 & -4 & 8 & b \end{array}\right] \overset{R1 \Leftrightarrow R3}{\sim} \left[\begin{array}{ccc|c} 2 & -4 & 8 & b \\ 3 & -5 & 10 & a \\ 0 & 2 & -4 & 0 \end{array}\right] \overset{\frac{1}{2}R1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 4 & b/2 \\ 3 & -5 & 10 & a \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$\underset{\sim}{R2-3R1}$ $\left[\begin{array}{ccc|c} \boxed{1} & -2 & 4 & b/2 \\ 0 & \boxed{1} & -2 & a-3b/2 \\ 0 & 2 & -4 & 0 \end{array}\right]$ $\underset{\sim}{R3-2R2}$ $\left[\begin{array}{ccc|c} 1 & -2 & 4 & b/2 \\ 0 & 1 & -2 & a-3b/2 \\ 0 & 0 & 0 & -2a+3b \end{array}\right]$

The system if consistent if and only if the bottom entry in the right column is 0. Hence, the condition

$$\underline{-2a+3b=0}$$

gives us a consistent system.

B 5/7.5. $A=\begin{bmatrix} 1 & 1 & 1 \\ -5 & -2 & -5 \\ 1 & 0 & 1 \end{bmatrix}$. We need to find null (A), i.e. the solution set of $A\bar{x}=\bar{0}$.

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ -5 & -2 & -5 \\ 1 & 0 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 + x_3 = 0$

$x_2 = 0$

x_3 - free

$$\bar{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{null(A)=\left\{t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ t\in\mathbb{R}\right\}}$$

B 14/7.5. $\bar{v}_1=\begin{bmatrix} -8 \\ 9 \\ -6 \end{bmatrix}$, $\bar{v}_2=\begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix}$, $\bar{v}_3=\begin{bmatrix} 8 \\ -18 \\ 40 \end{bmatrix}$

In order to study if the vectors are linearly independent or not, we study the null space of the associated matrix.

$$\left[\begin{array}{ccc|c} -8 & -2 & 8 & 0 \\ 9 & 0 & -18 & 0 \\ -6 & 7 & 40 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/4 & -1 & 0 \\ 9 & 0 & -18 & 0 \\ -6 & 7 & 40 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/4 & -1 & 0 \\ 0 & -9/4 & -9 & 0 \\ 0 & 34/4 & 34 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/4 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1/4 & -1 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The null space is nontrivial (there is a free variable) hence the vectors are <u>linearly dependent</u>.

To find a nontrivial linear combination, we solve the homogeneous system (to find the nullspace):

$$\begin{aligned} c_1 + \tfrac{1}{4} c_2 - c_3 &= 0 \\ c_2 + 4x_3 &= 0 \\ c_3 &- \text{free} \end{aligned} \qquad \Rightarrow \quad c_2 = -4c_3, \; c_1 = 2c_3$$

$$\text{null}(A) = \left\{ c_3 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, x_3 \in R \right\}$$

Pick $c_3 = 1 \Rightarrow c_2 = -4,\ c_1 = 2$ and then

$$\underline{2\bar{V}_1 - 4\bar{V}_2 + \bar{V}_3 = \bar{0}.}$$

B22/7.5. Basis for span $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ given by the vectors corresponding to pivot columns (exercise 14), i.e. $\bar{V}_1$ and $\bar{V}_2$.

$$\mathcal{B} = \left\{ \begin{bmatrix} -8 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} \right\}.$$ Dimension is $\underline{2}$.

B31/7.5. Row reduce $\left[\begin{array}{cccc|c} 0 & -2 & 0 & -2 & 0 \\ 2 & -12 & -4 & -14 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -2 & 11 & 4 & 13 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & -6 & -2 & -7 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Hence $\quad x_1 - 6x_2 - 2x_3 - 7x_4 = 0$

$\qquad x_2 + x_4 = 0 \quad \Rightarrow x_2 = -x_4$ and

$\qquad x_3, x_4$ - free $\qquad x_1 = 6x_2 + 2x_3 + 7x_4$

$\qquad\qquad = 2x_3 + x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

A basis for null(A): $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

B 39/1.5. $\quad$ Solve $A\bar{x} = \bar{b}$ by row-reduction:

$$\left[\begin{array}{cccc|c} 0 & -2 & 0 & -2 & 0 \\ 2 & -12 & -4 & -14 & 6 \\ 0 & 1 & 0 & 1 & 0 \\ -2 & 11 & 4 & 13 & -6 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -6 & -2 & -7 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Similar to previous exercise,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 + 2x_3 + x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(Notice that the solution set is obtained from null (A) by adding $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$).

Manual (Chapter 11)

Exercise 7

```
>> A=[-19,-128,81,38;8,61,-27,-16;2,4,12,-4;-8,-16,12,16];
>> b=[3;5;-21;-4];
>> M=[A,b]

M =

   -19  -128    81    38     3
     8    61   -27   -16     5
     2     4    12    -4   -21
    -8   -16    12    16    -4

>> rref(M)

ans =

     1     0     0    -2     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     0     1
```

Looking at the last row of the reduced echelon form, we conclude that the system is incon ↙
sistent.

Exercise 8

```
>> A=[2,0,2;1,0,1;-7,12,5];
>> b=[6;3;27];
>> M=[A,b]

M =

     2     0     2     6
     1     0     1     3
    -7    12     5    27

>> rref(M)

ans =

     1     0     1     3
     0     1     1     4
     0     0     0     0
```

Notice that the system is consistent and has a free variable (therefore infinitely many s ↙
olutions). If one tries to find the solution by using the
operation A\b, Matlab reports a warning. A\b can be used only if the system has a unique ↙
solution.

The general solution can be find by backsolving the equivalent system:

```
x1 +      x3=3
    x2 + x3=4   with x3 is a free variable. One obtains x2=4-x3 and x1=3-x3.
```

Exercise 13

One can use the row reduction procedure to find the nullspace and then a basis, or the MA ↙
TLAB command null(A,'r') to get a basis in a direct way

```
>> A=[2,-4,-9;0,0,2;0,0,1];
>> null(A,'r')

ans =

     2
     1
     0
```

A basis consists of one vector [2;1;0], hence the dimension of the nullspace is 1.

Exercise 21

One needs to study the nullspace of the matrix which has the vectors v1, v2, v3 as its co ↙
lumns. The 'null' command gives a basis of the nullspace.

```
>> v1=[1;1;1]; v2=[1;-1;1]; v3=[5;1;5];
>> V=[v1,v2,v3]
>> null(V,'r')

ans =

    -3
    -2
     1
```

Since the nullspace is nontrivial, the vectors v1,v2,v3 are linearly dependent. A nontriv ↙
ial linear combination would be

    -3v1 - 2v2 + v3 = 0

Exercise 22

```
>> v1=[1;0;1;0]; v2=[0;1;1;1]; v3=[5;-6;-1;-6];
>> V=[v1,v2,v3]
>> null(V,'r')

ans =

    -5
     6
     1
```

>> Since the nullspace is nontrivial, the vectors v1,v2,v3 are linearly dependent. A nont ↙
rivial linear combination would be

    -5v1 + 6v2 + v3 = 0