Math 211 - Hw #2

B27/2.2. If $N(t)$ is the amount of ^{131}I at time t,

then $N(t) = N_0 \cdot e^{-\lambda t}$ (N_0 - initial amount)

and $T_{1/2} = \dfrac{\ln 2}{\lambda}$.

We find $\lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{\ln 2}{8.04} = 0.0862$

We find $N(20) = 500 \cdot e^{-0.0862 \cdot 20} = 89.153$ mg

B32/2.2 (6) If $Q(t)$ denotes the amount of ^{14}C at time t; then:

$$Q(t) = Q_0 e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5568}$$

Look for t such that $Q(t) = 0.617\, Q_0$.

This means $Q_0 \cdot e^{-\lambda t} = 0.617\, Q_0$ or

$$t = \frac{\ln(0.617)}{-\lambda} = -\frac{\ln 0.617 \cdot 5568}{\ln 2} = \underline{\underline{3,879 \text{ years}}}$$

B33/2.2 The temperature $T(t)$ of the body at time t is $T(t) = A + (T_0 - A)\cdot e^{-kt}$

T_0 - initial temp.
A - medium temp.

Assume that at midnight $t=0$

So $T(0) = 31$, $T(1) = 29$.

From $T(1) = 29$: $21 + (31-21)\cdot e^{-k} = 29 \Rightarrow k = \ln\frac{10}{8} \approx .223$

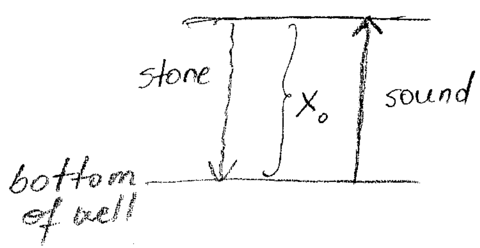
Now we look for time t such that $T(t) = 37$:

$$21 + 10\cdot e^{-kt} = 37 \Rightarrow t = \frac{\ln(10/16)}{k} \approx -2.1 \text{ hours}$$

$\approx$ 2 hours and 6 minutes

The time of death is 21:54

B3/2.4.



Position of stone: $x(t) = -\frac{1}{2}gt^2 + x_0$

When it hits the bottom of well: $x(t) = 0$ so $t_1 = \sqrt{\frac{2x_0}{g}}$

Time sounds travels the distance x_0: $t_2 = \frac{x_0}{V} = \frac{x_0}{340}$

We have $t_1 + t_2 = 8s$ so $\sqrt{\frac{2x_0}{9.8}} + \frac{x_0}{340} = 8$.

We get a quadratic equation: $\left(8 - \frac{x_0}{340}\right)^2 = \frac{2x_0}{9.8}$

with a valid solution $\boxed{x_0 = 257.1\,m}$

And $\boxed{t_1 = 7.243\,s}$

$\underrightarrow{B\,4/2.3}$ If one assumes that the net acceleration is $100\ m/s^2$, then in the first 60 sec the distance traveled is

$$\frac{1}{2}at^2 = \frac{1}{2}\,60 \cdot 100^2 = 180{,}000\ m$$

and the velocity after 60 sec is $at = 6{,}000\ m/s$.

From then on: $x''(t) = -g$ $\quad (x(0) = 180{,}000$

$\quad v(0) = x'(0) = 6{,}000)$

so $\underline{x(t) = -\frac{1}{2}gt^2 + 6{,}000t + 180{,}000\ ;\ v(t) = -gt + 6{,}000}$

Now, maximum altitude is reached when $v(t) = 0$,

so $t = \dfrac{6{,}000}{g} = 612.24\ s$, and $\boxed{x_{max} = 2.0167 \times 10^6\ m}$

The time when rocket hits the ground is obtained by setting $x(t) = 0$, so $t = 1{,}253.78\ s$

We just need to add the first 60 s, to get the total time $\boxed{= 1{,}313.78\,s}$

<u>Remark</u>. If one assumes that the net acceleration is $100 - g = 90.2\ m/s^2$ then

$x_{max} = 1.657 \times 10^6\ m$ and

$t_{end} \approx 1{,}193.7\,s$

B 9/2.3 The resistance force is $-rv$. If $v = 0.2$ m/s gives $-1N$, then $r = 5$. The terminal velocity is

$$V_{term} = -\frac{mg}{r} = -0.196 \text{ m/s}$$

B14/2.3. [redacted] (a) Follows from $a = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$.

(b)

$$v\,dv = -\frac{GM}{(R+y)^2}\,dy$$

$$\int_{v_0}^{v} v\,dv = -\int_0^y \frac{GM}{(R+s)^2}\,ds$$

$$\frac{1}{2}(v^2 - v_0^2) = -GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

$$v^2 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

(c) If y is the maximum height, the corresponding velocity is $v = 0$, so from (3.16)

$$0 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right).$$

Solving for y we get the result.

(d) If $v_0 < \sqrt{2GM/R}$, then $v_0^2 < 2GM/R$, and $2GM/R - v_0^2 > 0$. Hence by (c) the object has a finite maximum height and does not escape. However, when $v_0 = \sqrt{2GM/R}$, $2GM/R - v_0^2 = 0$, and there is no maximum height.

B 5/2.4. $x' - 2x/(t+1) = (t+1)^2$. Rewrite as: $x' = \frac{2x}{t+1} + (t+1)^2$.

So $a(t) = \frac{2}{t+1}$; $f(t) = (t+1)^2$

Integrating factor: $u(t) = e^{\int -a(t)dt} = e^{-\int \frac{2}{t+1} dt}$

$= e^{-2\ln|t+1|} = (t+1)^{-2}$

General solution: $x(t) = \frac{\int u(t) f(t)}{u(t)} = \frac{\int 1\, dt}{(t+1)^{-2}} =$

$= \frac{t+c}{(t+1)^{-2}} = t(t+1)^2 + c(t+1)^2$.

B 6/2.4. $tx' = 4x + t^4$

Rewrite: $x' = \frac{4}{t} x + t^3$

Integrating factor: $u(t) = e^{-\int \frac{4}{t} dt} = e^{-4\ln|t|} = \frac{1}{t^4}$

General solution: $x(t) = \frac{\int t^3 \cdot t^{-4} dt}{t^{-4}} = \frac{\ln|t| + C}{t^{-4}} =$

$= t^4 \ln|t| + Ct^4$.

Remark. Depending on t_0 (<0 or >0) the interval of existence is either $(-\infty, 0)$ or $(0, \infty)$.

B 21/2.4. $(1+t)x' + x = \cos t, \ x(-\pi/2) = 0.$

Divide by $(1+t)$: $\quad x' + \frac{1}{1+t}x = \frac{\cos t}{1+t}$

Integrating factor: $u(t) = e^{\int \frac{1}{1+t}dt} = |1+t|$

Because the initial moment is $-\frac{\pi}{2}$ and $\frac{1}{1+t}$ is not continuous at $t=-1$, we need to solve for $t<-1$.

Therefore $u(t) = -(1+t)$, and

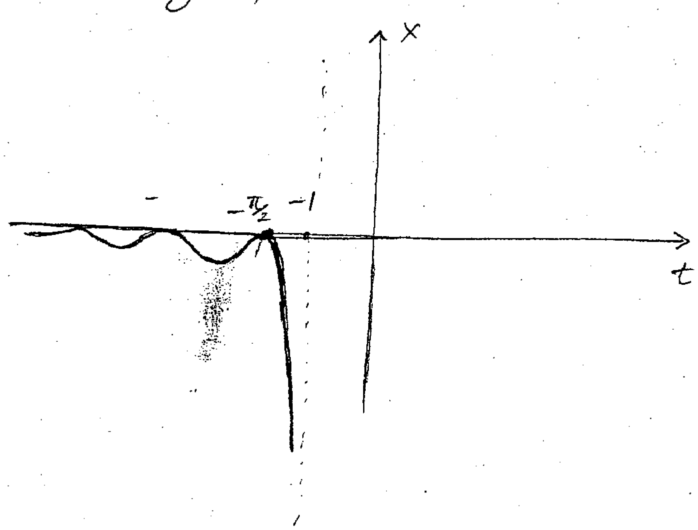
$$x(t) = \frac{\int -\cos t \, dt}{-(1+t)} = \frac{-\sin t + c}{-(1+t)}$$

From $x(-\frac{\pi}{2}) = 0$, we find $C=-1$ and

$$x(t) = \frac{-\sin t - 1}{-(1+t)} = \frac{\sin t + 1}{1+t}.$$

The maximal interval of existence is $(-\infty, -1)$.

Sketch of graph:



B 39/2.4. $y' + 2xy = 2x^3$, $y(0) = -1$

Using the variation of parameters technique, look first for a solution of $y' = -2xy$ which is

$$y_h(x) = e^{\int -2x\,dx} = e^{-x^2}$$

Now we look for solutions of $y' = -2xy + 2x^3$ of type $y(x) = v(x)\cdot e^{-x^2}$. Differentiate and plug-in to get:

$$v'(x)\cdot e^{-x^2} + \cancel{v(x)(-2x)\cdot e^{-x^2}} = \cancel{-2x\cdot v(x)\cdot e^{-x^2}} + 2x^3$$

Thus $v'(x) = 2x^3\cdot e^{x^2}$, so

$$v(x) = \int 2x^3 e^{x^2}\,dx \overset{x^2=u}{=\!=} \int u\cdot e^u\,du$$

$$\overset{\text{parts}}{=} u\cdot e^u - \int e^u\,du = e^u(u-1) + C$$

$$= e^{x^2}(x^2-1) + C$$

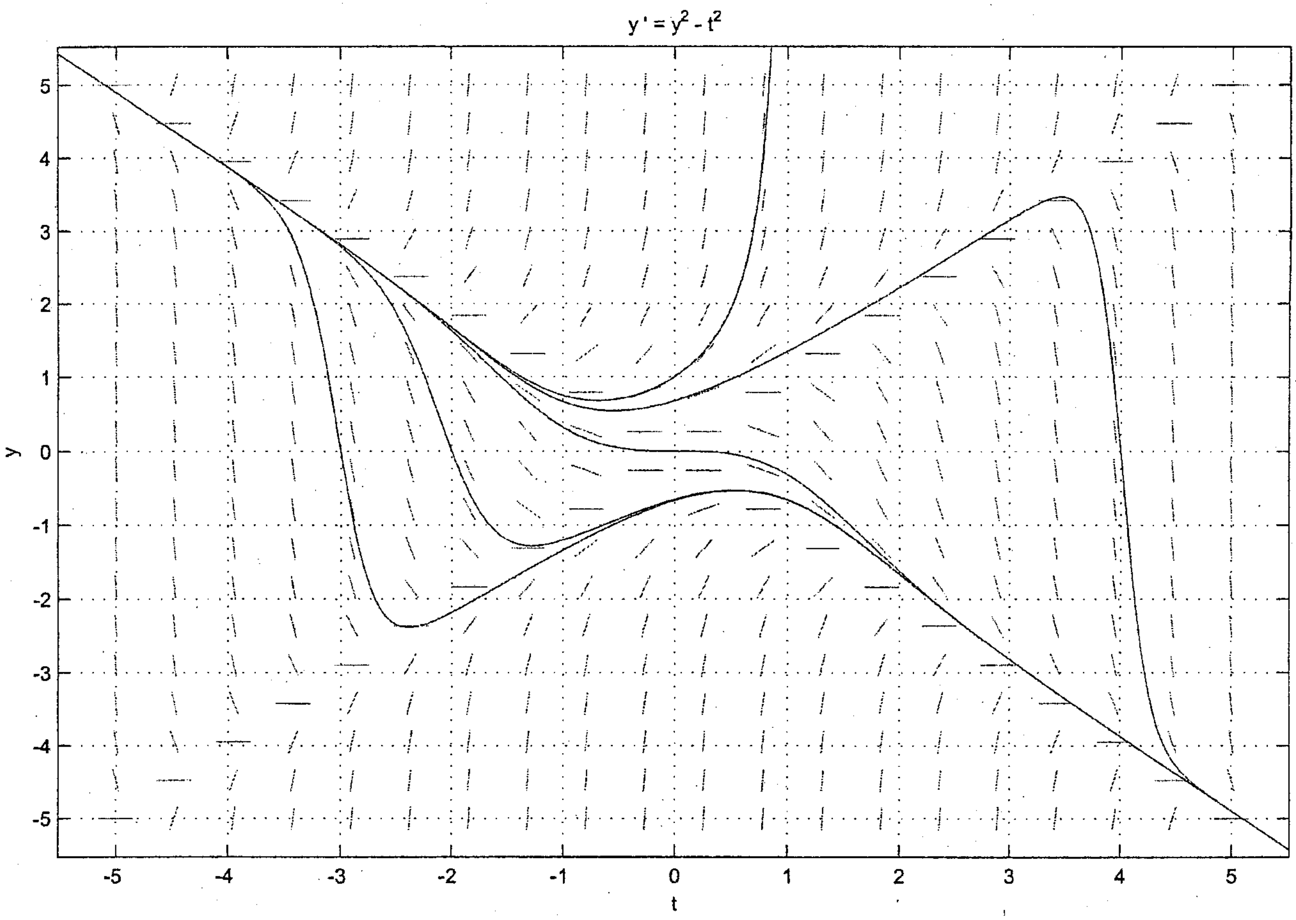
So $y(x) = \left(e^{x^2}(x^2-1) + C\right)\cdot e^{-x^2}$

$$= (x^2-1) + C\cdot e^{-x^2}$$

From $y(0) = -1$: $-1 + C\cdot 1 = -1 \Rightarrow \underline{C=0}$

So $\boxed{y(x) = x^2 - 1}$

M 2/Ch 3



M 6/Ch 3