Math 211 - Hw #3

B1/2.5

Let $x(t)$ - amount of sugar at time t

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= 3 \cdot 0.2 - 3 \cdot \frac{x(t)}{100} = 0.6 - 3 \cdot \frac{x(t)}{100}$$

Solving this linear equation: $x(t) = 20 + C \cdot e^{-3t/100}$

Since $x(0) = 0$, $C = -20$ and $\underline{x(t) = 20 - 20 \cdot e^{-3t/100}}$

(a) $x(20) = 20 - 20 \cdot e^{-60/100} \approx \underline{9.038 \text{ lb}}$

(b) $x(t) = 15 \Rightarrow t = \frac{100 \ln 4}{3} \approx \underline{46.2 \text{ min}}$

(c) $\lim_{t \to \infty} x(t) = \underline{20}$.

B5/2.5

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= 4 \text{ gal/min} \times 0.5 \text{ lb/gal} - 2 \text{ gal/min} \times \frac{x(t)}{\underbrace{20+2t}_{\text{Volume at time } t}}$$

$$= -\frac{x(t)}{t+10} + 2$$

Solving this linear equation (with integrating factor).

$$x(t) = \frac{t^2 + 20t + C}{t+10}.$$ Use $x(0) = 0$ to get $C = 0$.

So $$\underline{x(t) = \frac{t^2 + 20t}{t+10}}$$

Tank is full after 15 min, and $\boxed{x(15) = 21 \text{ lb.}}$

B 12/2.5

$x(t)$ - amount of salt in tank A
$y(t)$ - amount of salt in tank B

$$\begin{cases} \dfrac{dx}{dt} = 5\cdot 0 - 5\cdot \dfrac{x(t)}{100} & x(0)=20 \\[2ex] \dfrac{dy}{dt} = 5\cdot\dfrac{x(t)}{100} - 2.5\,\dfrac{y(t)}{200+2.5t} & y(0)=40 \end{cases}$$

Solving for $x(t)$: $x(t) = 20\cdot e^{-t/20}$

Solving for $y(t)$: $\dfrac{dy}{dt} = e^{-t/20} - \dfrac{y}{80+t}$ (linear equation)

Integrating factor: $u(t) = 80+t$

General solution: $y(t) = \dfrac{C}{80+t} - 20\cdot e^{-t/20} - \dfrac{400}{80+t}e^{-t/20}$

From $y(0)=40$, $C = 5200$.

Tank B contains 250 gal when $200+2.5t = 250$, i.e. $t=20$.

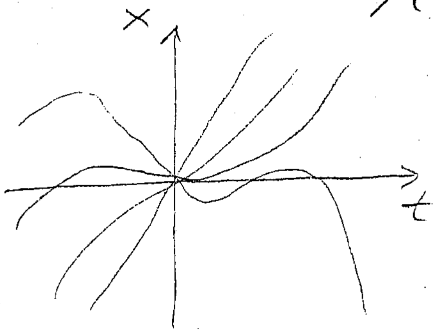
Hence $y(20) = \dfrac{5200}{100} - 20\cdot e^{-1} - 4\cdot e^{-1} \approx 43.17$ lb

B 7/2.7

$$ty' - y = t^2\cos t$$
$$y' - \frac{1}{t}y = t\cos t$$

Linear equation with general solution:

$$y(t) = t\sin t + Ct.$$



Notice that all solutions pass throug (0,0), b/c $y(0)$ is always 0.
Therefore, the IVP $y(0) = -3$ has no solutions
This does not contradict the existence thm b/c $1/t$ is not continuous at 0.

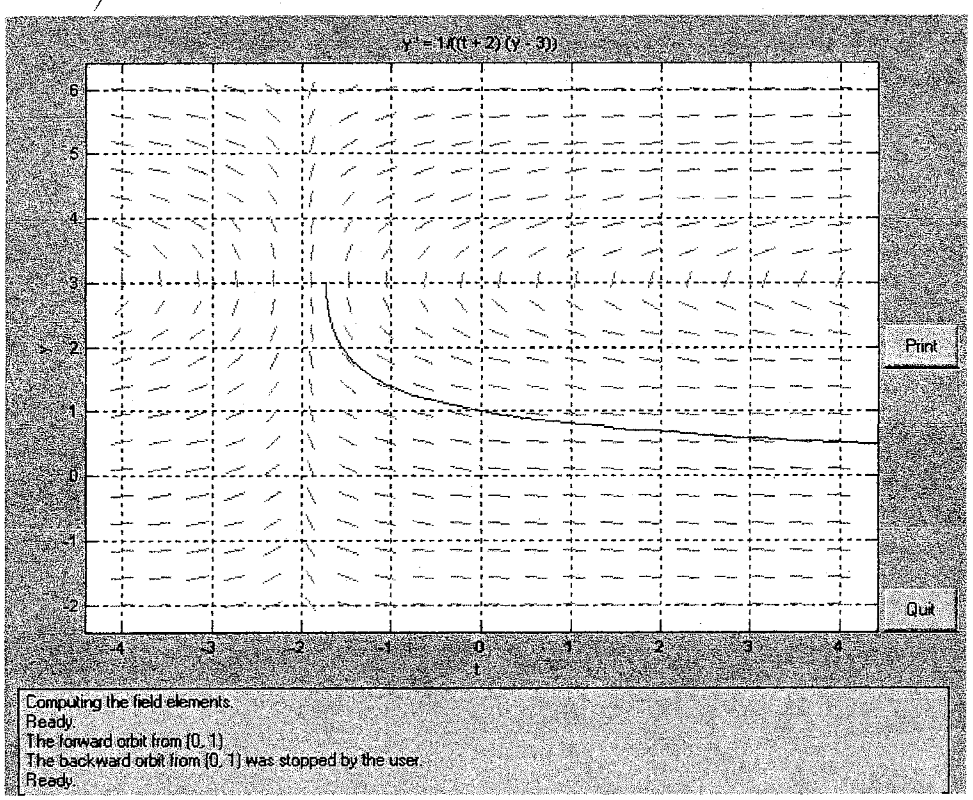
B 5/2.7 $\quad x' = \dfrac{t}{x+1}, \quad x(0)=0.$

Notice that the right-hand side $f(t,x) = \dfrac{t}{x+1}$ is continuous on a small rectangle around $(0,0)$, and $\dfrac{\partial f}{\partial x}(t,x) = -\dfrac{t}{(x+1)^2}$ is also continuous on such a rectangle.
A unique solution is guaranteed by Theorem 7.16.

B6/2.7 $\quad y' = \dfrac{1}{x}y + 2, \quad y(0)=1.$

The right-hand side $f(x,y) = \dfrac{1}{x}y+2$ is not even well-defined for $x=0$, so a unique solution is not guaranteed. Theorem 7.16 cannot be applied.

B 14/27 a)



It seems that the solver gets stuck when t is somewhere between $(-2,-1)$ and when y approaches 3. $\longrightarrow$

B 9/2.7. By direct verification, both $y(t)=0$ and $y(t)=t^3$ satisfy $y'=3y^{2/3}$, $y(0)=0$. This does not contradict the uniqueness theorem because the right-hand side of the equation $f(t,y)=3y^{2/3}$ has $\frac{\partial f}{\partial y}=2y^{-1/3}=\frac{2}{\sqrt[3]{y}}$ which is not continuous at $y=0$

B 14/2.7 b) $\frac{dy}{dt}=\frac{1}{(t+2)(y-3)}$ $y(0)=1$.

Continuation

Separating variables: $\int(y-3)\,dy=\int\frac{dt}{t+2}$

$$\frac{1}{2}y^2-3y=\ln|t+2|+c$$

Because $y(0)=1$, $c=-\frac{5}{2}-\ln 2$, and $t>-2$.

Therefore $\frac{1}{2}y^2-3y=\ln(t+2)-\frac{5}{2}-\ln 2$

$$y(t)=3\pm\sqrt{9+2\left(\ln(t+2)-\frac{5}{2}-\ln 2\right)}$$
$$=3\pm\sqrt{4+2\ln(t+2)-2\ln 2}$$

Because $y(0)=1 \Rightarrow y(t)=3-\sqrt{4+2\ln(t+2)-2\ln 2}$

The interval of existence:

$$4+2\ln(t+2)-2\ln 2>0$$

$\Rightarrow \ln(t+2)>\ln 2-2 \Rightarrow t+2>e^{\ln 2-2}=2\cdot e^{-2}$

Therefore $t>2\cdot e^{-2}-2\approx-1.73$, and the interval of existence is $(2\cdot e^{-2}-2;\ \infty)$.

The problem is that when $t\searrow 2\cdot e^{-2}-2$, $y\nearrow 3$ and $\frac{1}{(t+2)(y-3)}\to-\infty$ hence the solution cannot be extended beyond this point.

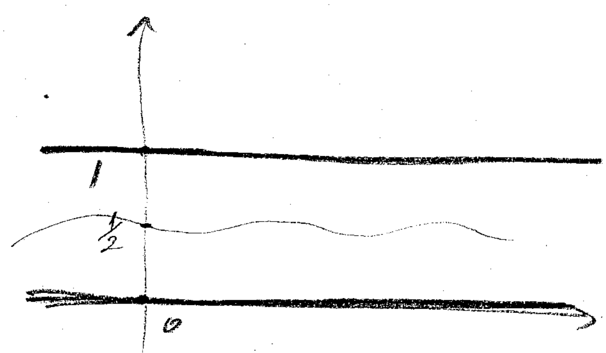
B30/2.7 $x' = \dfrac{x^3 - x}{1+t^2x^2}$, $x(0) = 1/2$.

Need to show $0 < x(t) < 1$.

Notice that the constant functions $x_1(t) = 0$ and $x_2(t) = 1$ are solutions of the given equation. Indeed, by plugging-in, $x_1' = 0$ and $\dfrac{x_1^3(t) - x_1(t)}{1+t^2x_1^2(t)} = \dfrac{0-0}{1+t^2 \cdot 0} = 0$

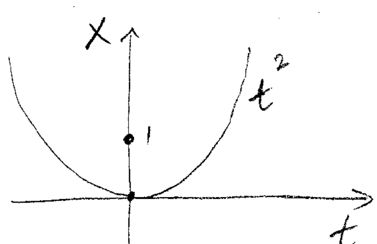
$x_2' = 0$ and $\dfrac{x_2^3(t) - x_2(t)}{1+t^2x_2^2(t)} = \dfrac{1-1}{1+t^2} = 0.$

Also the right-hand side $f(t,x) = \dfrac{x^3 - x}{1+t^2x^2}$ is continuous and $\dfrac{\partial f}{\partial x}$ is also continuous, which guarantees uniqueness of solutions. Geometrically this means that two solution curves cannot intersect.



Since $x(0) = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$, then $0 < x(t) < 1$ for all times.

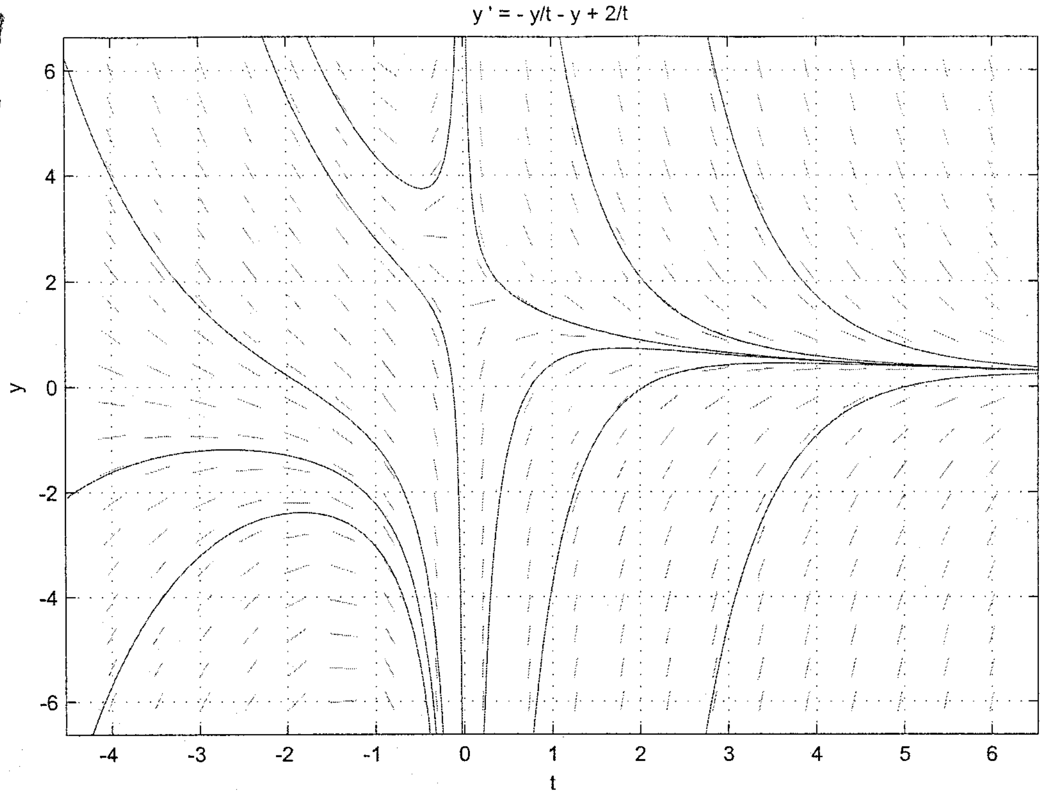
B31/2.7 Similarly, one checks that $x_1(t) = t^2$ is a solution of $x' = x - t^2 + 2t$. For a solution starting at $(0,1)$ we conclude that it has to stay inside the parabola, i.e. $\underline{\underline{x(t) > t^2}}$.



($f(t,x) = x - t^2 + 2t$ is continuous; $\frac{\partial f}{\partial x}$ - cont.)

M9/Ch 3

a)



b) It seems that, if $t_0 < 0$ then a solution is defined on $(-\infty, 0)$ and if $t_0 > 0$, the interval of existence is $(0, \infty)$. In the latter case, also $y(t)$ approaches 0 as $t \to \infty$.

c) The linear equation is

$$y' = \left(-\frac{1}{t} - 1\right) \cdot y + \frac{2}{t}$$

Integrating factor: $u(t) = e^{\ln|t| + t} = |t| \cdot e^t$

Solution: $$y(t) = \frac{\int \frac{2}{t} |t| \cdot e^t}{|t| \cdot e^t} = \boxed{\begin{cases} \dfrac{2e^t + C}{t \cdot e^t} & \text{if } t > 0 \\[2ex] \dfrac{-2e^t + C}{-t \cdot e^t} & \text{if } t < 0 \end{cases}}$$

Depending on t_0, solutions are defined either on $(-\infty, 0)$, or $(0, \infty)$.

d) If $t_0 > 0$, then $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \frac{2e^t + C}{t \cdot e^t} = 0.$