Math 211 - Hw #4

17/2.8. (a) $x' = -2x + \sin t$, so $f(t,x) = -2x + \sin t$

Notice that $\frac{\partial f}{\partial x} = -2$, so $\underline{M = 2}$.

Hence $\underline{|x(t) - y(t)| \le |x(0) - y(0)| \cdot e^{2|t|}}$.

(b) $x(t) = \dfrac{2\sin t - \cos t}{5}$

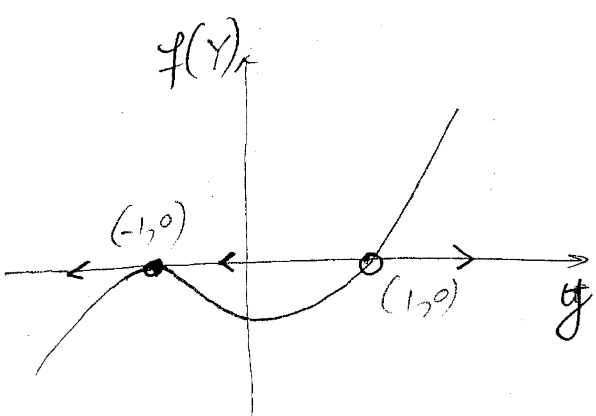
$y(t) = \dfrac{2\sin t - \cos t}{5} - \dfrac{e^{-2t}}{10}$

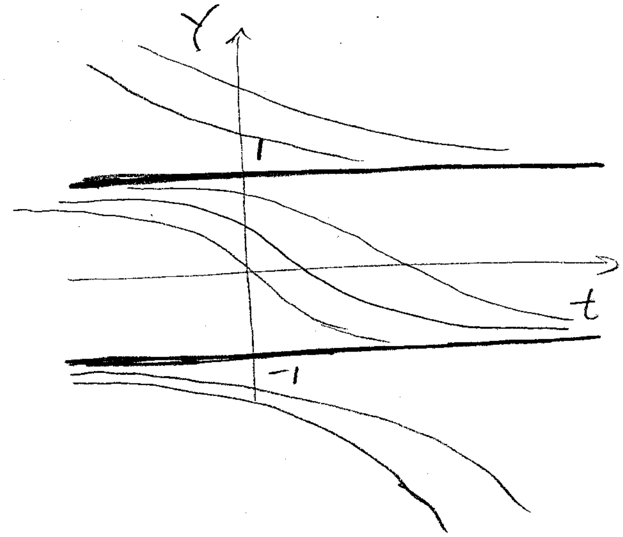
One checks that $|x(t) - y(t)| = \left|\dfrac{e^{-2t}}{10}\right| \le \left|-\frac{1}{5} + \frac{3}{10}\right| \cdot e^{2|t|}$

(because $\frac{1}{10}\left|e^{-2t}\right| \le \frac{1}{10} \cdot e^{2|t|}$)

(c) The inequality becomes equality when $t \le 0$.

9/2.9.



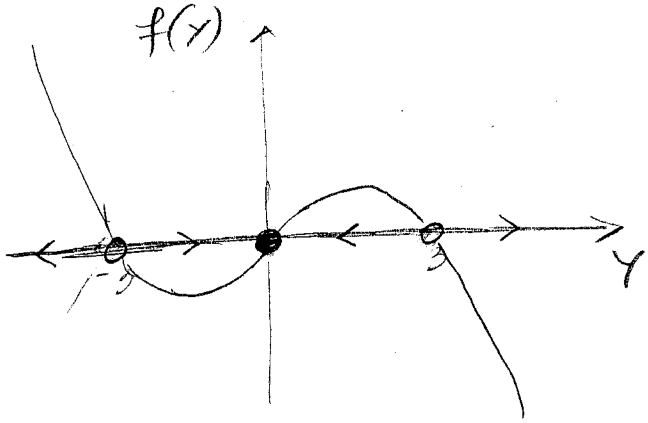


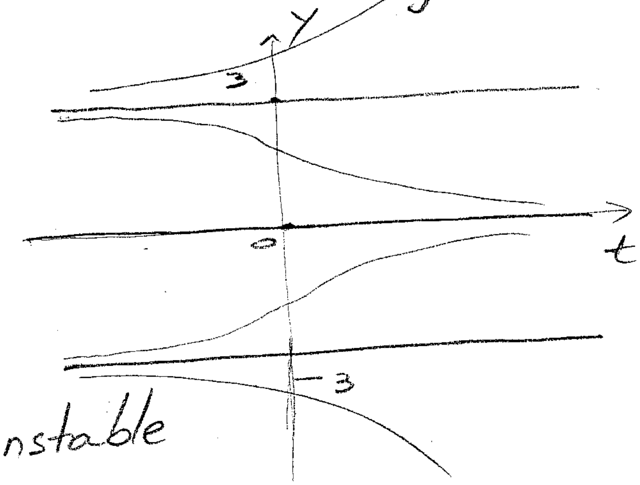
$(-1,0)$ - semistable

$(1,0)$ - unstable

19/2.9. Equilibrium points: $9y - y^3 = 0$

$y(9-y^2)=0$ so $y = 0, -3, 3$.



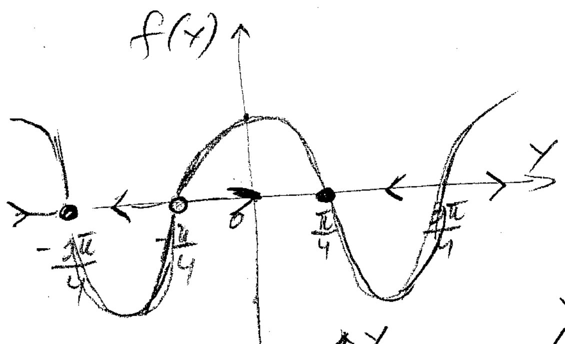


$y=0$ - stable; $y=-3$, $y=3$ unstable

22/2.9 $y' = \cos 2y$. Equilibrium points: $\cos 2y = 0$

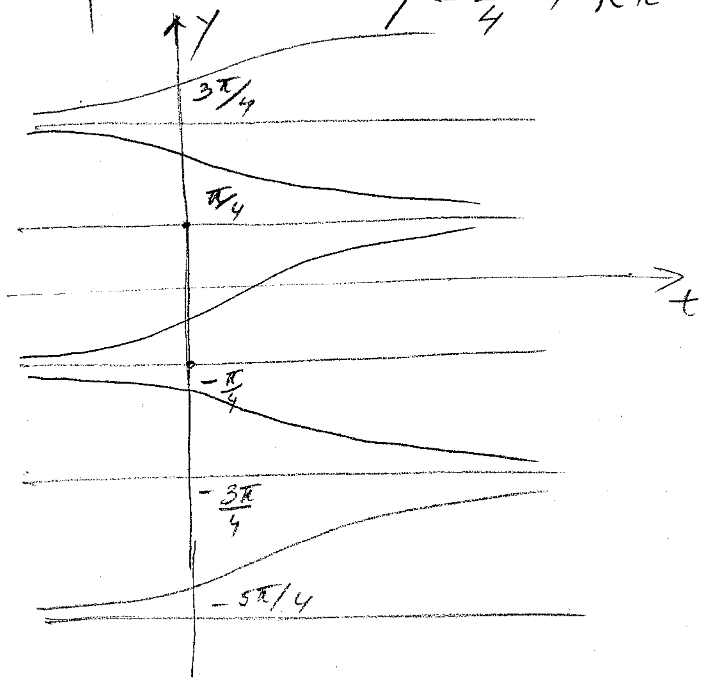
$$2y = \frac{\pi}{2} + k\pi$$

$$y = \frac{\pi}{4} + \frac{k\pi}{2}$$



$y = \frac{\pi}{4} + k\pi$ are stable points

$y = \frac{3\pi}{4} + k\pi$ are unstable points



25/2.9. $y' = (1+y)(5-y)$, $y(0)=2$

(i) Separate variables

$$\int \frac{dy}{(1+y)(5-y)} = \int 1\, dt$$

LHS: Use partial fractions: $\frac{1}{(1+y)(5-y)} = \frac{A}{1+y} + \frac{B}{5-y}$

to get $A = \frac{1}{6}$, $B = \frac{1}{6}$.

So $\frac{1}{6} \ln|1+y| - \frac{1}{6} \ln|5-y| = t + C$, or

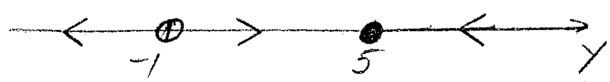
$\frac{1}{6} \ln\left|\frac{1+y}{5-y}\right| = t + C$. From $y(0)=2 \Rightarrow C=0$

Thus $\frac{1+y}{5-y} = \pm e^{6t}$. Since $y(0)=2$ we have

$$\frac{1+y}{5-y} = e^{6t} \Rightarrow \underline{y(t) = \frac{5e^{6t}-1}{e^{6t}+1} = \frac{5-e^{-6t}}{1+e^{-6t}}}$$

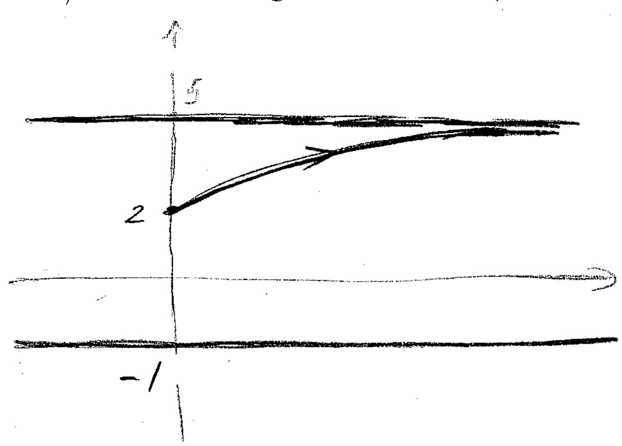
(ii) $\underline{\lim_{t\to\infty} y(t) = 5.}$

(iii) Equilibrium points: $y=-1$; $y=5$

Phase-line 

$y=-1$ unstable

$y=5$ stable



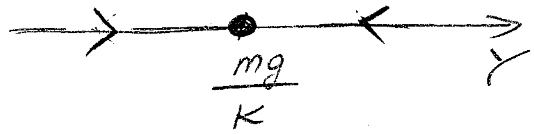
The solution starting at $y(0)=2$ converges towards $y=5$.

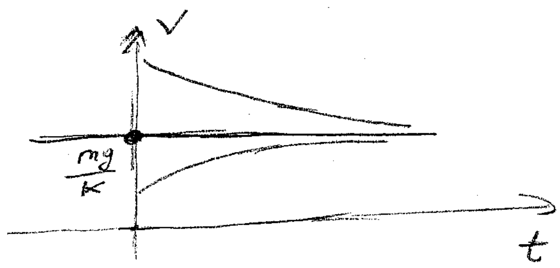
(Easier method)

30/2.9. $m\dfrac{dv}{dt} = mg - kv$; $\dfrac{dv}{dt} = g - \dfrac{k}{m}v$

Equilibrium point: $v = \dfrac{mg}{k}$

Phase-line





$y = \dfrac{mg}{k}$ - stable point
all solutions approach it
as $t \to \infty$.

3/3.1 $P(t) = P_0 \cdot e^{rt}$. We have $P(10) = 3P_0$, so

$P_0 \cdot e^{10r} = 3P_0 \Rightarrow r = \dfrac{\ln 3}{10}$

We also need t s.t. $P(t) = 2P_0$, so

$P_0 \cdot e^{rt} = 2P_0 \Rightarrow t = \dfrac{\ln 2}{r} \approx 6.3$ hours

13/3.1 $P' = rP\left(1 - \dfrac{P}{K}\right)$ where $K = 20{,}000$, $P_0 = 1{,}000$

Also $P(8) = 1{,}200$. Now

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)\cdot e^{-rt}}$$

Using $P(8) = 1{,}200$, one gets $r = 0.024$

Now, look for t such that $P(t) = 0.75K$ and get $t = 167.67$ hours.

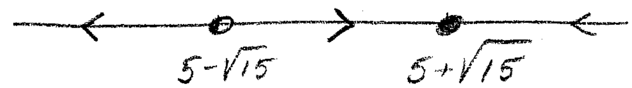
16/31 (a) $P' = 0.1\,P(1 - P/10) - 0.1$

(b) Equilibrium points:

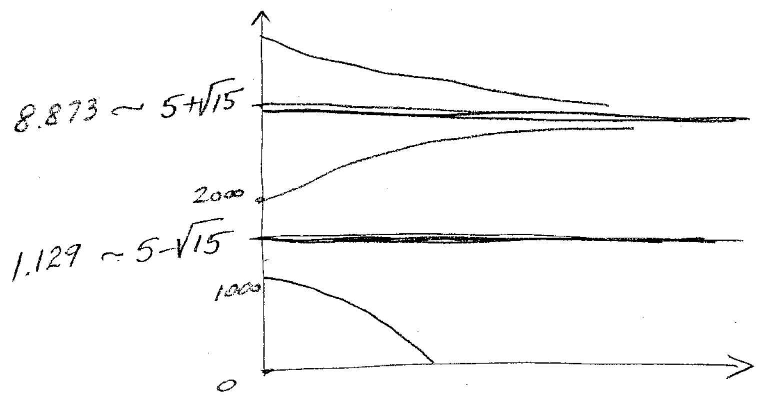
$$\frac{1}{10}P\left(1 - \frac{P}{10}\right) - \frac{1}{10} = 0 \quad \text{or} \quad -\frac{P^2}{10} + P - 1 = 0$$

So, $$P = \frac{-1 \pm \sqrt{1 - 2/5}}{-\frac{1}{5}} = 5 \pm 5\sqrt{3/5} = 5 \pm \sqrt{15}$$

Phase-line:



$P = 5 - \sqrt{15}$ unstable; $P = 5 + \sqrt{15}$ -stable



One can see that if $P_0 = 1{,}000$, then the population dies out. If $P_0 = 2{,}000$ the population approaches 8,873.