Math 211 - Hw #6

3/7.3. We row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & 2 \\ -1 & -1 & 2 & 4 \\ 1 & 3 & 6 & 7 \end{array}\right] \underset{\text{add } -R1 \text{ to } R2}{\overset{\text{add } R1 \text{ to } R2}{\sim}} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & 2 \\ 0 & \boxed{1} & 4 & 6 \\ 0 & 1 & 4 & 5 \end{array}\right].$$

$$\overset{R3-R2}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & 2 \\ 0 & \boxed{1} & 4 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Since there is a pivot in the last column, the system is <u>inconsistent.</u>

7/7.3.

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 4 \\ 1 & 2 & -2 & 6 \\ 1 & 4 & -6 & 14 \end{array}\right] \overset{R2 \leftrightarrow R1}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 2 & -2 & 6 \\ 0 & 1 & -2 & 4 \\ 1 & 4 & -6 & 14 \end{array}\right] \underset{\text{to } R2}{\overset{\text{add } -R1}{\sim}}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -2 & 6 \\ 0 & \boxed{1} & -2 & 4 \\ 0 & 2 & -4 & 8 \end{array}\right] \underset{\text{to } R3}{\overset{\text{add } -2R2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equivalent system is:

$$\left.\begin{array}{r} x_1 + 2x_2 - 2x_3 = 6 \\ x_2 - 2x_3 = 4 \end{array}\right\} \xrightarrow{\text{Backsolve}} \begin{array}{l} x_2 = 4 + 2x_3 \\ x_1 = 6 - 2x_2 + 2x_3 \\ \quad = -2 - 2x_3 \end{array}$$

x_3 - free

So the solution is $$\boxed{\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-2x_3 \\ 4+2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \ (\text{if } t = x_3)}$$

8/7.3.

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 6 \\ 1 & 2 & -1 & 2 \\ 2 & 1 & -2 & 4 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -4 & 6 \\ 2 & 1 & -2 & 4 \end{array}\right] \underset{\text{to } R_2}{\overset{\text{add } -2R_1}{\sim}}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & -4 & 6 \\ 0 & -3 & 0 & 0 \end{array}\right] \underset{\text{to } R_3}{\overset{\text{add } 3R_2}{\sim}} \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & -4 & 6 \\ 0 & 0 & \boxed{-12} & 18 \end{array}\right]$$

So

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 - 4x_3 &= 6 \\ -12x_3 &= 18 \end{aligned} \quad \xrightarrow{\text{backsolve}} \quad \begin{aligned} x_3 &= -\frac{3}{2} \\ x_2 &= 6 + 4x_3 = 0 \\ x_1 &= 2 - 2x_2 + x_3 = \frac{1}{2} \end{aligned}$$

There is a unique solution $\left(\frac{1}{2}, 0, -\frac{3}{2}\right)$.

21/7.3. The corresponding system is:

$$\begin{aligned} x_1 - x_2 + x_3 \quad - x_5 &= 1 \\ x_4 + x_5 &= 0 \end{aligned}$$

Thus, x_2, x_3, x_5 are free variables and

$x_4 = -x_5$, $x_1 = 1 + x_2 - x_3 + x_5$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + x_2 - x_3 + x_5 \\ x_2 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

32/7.3. rref gives

$$\sim \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -25/4 & 29/4 \\ 0 & \boxed{1} & 0 & -11/2 & 11/2 \\ 0 & 0 & \boxed{1} & 1 & 1 \end{array}\right]$$

This gives

$$x_1 - \frac{25}{4} x_4 = \frac{29}{4}$$
$$x_2 - \frac{11}{2} x_4 = \frac{11}{2}$$
$$x_3 + x_4 = 1 \qquad x_4 \text{ - free}$$

So $x_3 = 1 - x_4$; $x_2 = \frac{11}{2} + \frac{11}{2} x_4$, $x_1 = \frac{29}{4} + \frac{25}{4} x_4$

Solution is:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 29/4 + 25/4\, x_4 \\ 11/2 + 11/2\, x_4 \\ 1 - x_4 \\ x_4 \end{bmatrix} = \underline{\begin{bmatrix} 29/4 \\ 11/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 25/4 \\ 11/2 \\ -1 \\ 1 \end{bmatrix}}$$

7/7.4. Notice that A is already row-reduced, so:

$$x_1 - x_3 + 3x_5 = 0$$
$$x_2 + 2x_3 - 5x_5 = 0$$
$$x_4 + 2x_5 = 0 \qquad \text{and } x_3, x_5 \text{ are free.}$$

Hence $x_4 = -2x_5$; $x_2 = -2x_3 + 5x_5$; $x_1 = x_3 - 3x_5$.

and
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_5 \\ -2x_3 + 5x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = \underline{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 5 \\ 0 \\ -2 \\ 1 \end{bmatrix}}.$$

15/7.4. (a) There are 3 equations and 6 variables, so there must be at least $6-3=$ <u>3 free</u> variables.

(b) Row-reduction gives
$$\begin{bmatrix} \boxed{1} & 0 & -2 & 0 & 1 & 2 \\ 0 & \boxed{1} & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so actually there are <u>4 free</u> variables <u>(x_3, x_4, x_5, x_6)</u>.

3/7.6. Row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 3 & -3 & b_1 \\ -1 & -1 & 1 & b_2 \\ 3 & 5 & -1 & b_3 \end{array}\right] \underset{\sim}{\tfrac{1}{3}R1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & b_1/3 \\ -1 & -1 & 1 & b_2 \\ 3 & 5 & -1 & b_3 \end{array}\right]$$

add R_1 to R_2

add $-3R_1$ to R_2

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1/3 \\ 0 & 0 & 0 & b_2 + b_1/3 \\ 0 & 2 & 2 & b_3 - b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & b_1/3 \\ 0 & \boxed{2} & 2 & b_3 - b_1 \\ 0 & 0 & 0 & b_1/3 + b_2 \end{array}\right]$$

The system has solutions only if $b_1/3 + b_2 = 0$ (or $b_1 + 3b_2 = 0$)
The equation $b_1/3 + b_2 = 0$ represents a plane in $\mathbb{R}^3$.
The coefficient matrix is singular.

7/7.6 Row-reduce the matrix, and get:

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ -2 & 3 & 3 \\ -2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{3} & 1 \\ 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

Each diagonal element is nonzero; thus the matrix is <u>nonsingular</u>

23/7.6.

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \end{array}\right] \underset{R1-R3}{\overset{R2-R3}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(notice that A is invertible)

$$\overset{R_1-R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\underbrace{\hphantom{xxxx}}_{I_3}$ $\underbrace{\hphantom{xxxx}}_{A^{-1}}$

Thus $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.