Hw # 8

3/8.5. $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

9/8.5 Check that $\bar{x}'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \bar{x}(t)$; $\bar{y}'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \bar{y}(t)$

and also $\left(c_1 \bar{x}'(t) + c_2 \bar{y}(t)\right)' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \left(c_1 \bar{x}(t) + c_2 \bar{y}(t)\right)$.

15/8.5. Need to check that $\bar{x}(0)$, $\bar{y}(0)$ are lin. indep

But $\bar{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\bar{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ - clearly lin indep

hence the solutions $\bar{x}(t), \bar{y}(t)$ are l.i.

$\bar{z}(t) = c_1 \bar{x}(t) + c_2 \bar{y}(t)$. Since $\bar{z}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

we get $\begin{matrix} c_1 + 0 = -2 \\ 0 + c_2 = 3 \end{matrix}$, hence $c_1 = -2$, $c_2 = 3$

and $\bar{z}(t) = -2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} + 3 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix}$.

27/8.5. tank A: rate in $= 5\ \ell/\text{min} \times 0\ \frac{kg}{\ell} + 4\ \ell/\text{min} \times \frac{x_B(t)}{3}\ \frac{kg}{\ell}$

$= \frac{4}{3} x_B(t)\ kg/\text{min}$

rate out $= 9\ \ell/\text{min} \times \frac{x_A(t)}{3}\ kg/\ell$

$= 3 x_A(t)\ kg/\ell$

Hence $\boxed{\frac{dx_A}{dt} = \frac{4}{3} x_B(t) - 3 x_A(t)}$ $\underline{x_A(0) = 1}$

tank B: rate in $= 9 \cdot \frac{x_A(t)}{3} = 3\, x_A(t)$ kg/min

rate out $= 4 \cdot \frac{x_B(t)}{3} + 5 \cdot \frac{x_B(t)}{3}$

$= 3\, x_B(t)$ kg/min

So $\boxed{\frac{dx_B}{dt} = 3\, x_A(t) - 3\, x_B(t)}$ $\quad \underline{x_B(0) = 0}$

5/9.2 $\quad A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$; eigenvalues: $\lambda^2 - 5\lambda + 6 = 0$

so $\lambda_1 = 2;\ \lambda_2 = 3$

Find eigenvectors.

For $\lambda_1 = 2$: $(A - 2I)\bar{v} = \bar{0}$; $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so $-v_1 + 2v_2 = 0$, v_2 - free $\Rightarrow \bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (one choice)

For $\lambda_2 = 3$: $(A - 3I)\bar{w} = \bar{0}$; $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so $-w_1 + w_2 = 0$, w_2 - free $\Rightarrow \bar{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (one choice)

The general solution $\bar{y}(t) = c_1 \cdot e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

11/9.2 $\quad$ Use $\bar{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to get $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Solve $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \end{array}\right] \Rightarrow c_2 = 1;\ c_1 = 1.$

So $\bar{y}(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

17/9.2 $A = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix}$; eigenvalues $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = \underline{1 \pm 2i}$$

Eigenvectors:

$\lambda_1 = 1+2i$: $(A - (1+2i)I)\bar{v} = 0$; $\left[\begin{array}{cc|c} -2-2i & -2 & 0 \\ 4 & 2-2i & 0 \end{array}\right] \sim$

$$\sim \left[\begin{array}{cc|c} 2 & 1-i & 0 \\ -2-2i & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = -\frac{1-i}{2} v_2$ $\qquad \bar{v} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$ (if $v_2 = 2$)

v_2 - free

We use the complex fundamental solution $e^{\lambda_1 t}\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$ to generate two real fundamental solutions.

$$e^{(1+2i)t}\begin{bmatrix} -1+i \\ 2 \end{bmatrix} = e^t(\cos 2t + i \sin 2t)\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

$$= e^t\begin{bmatrix} -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + i e^t \begin{bmatrix} -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$$

So the general (real) solution is:

$$\bar{Y}(t) = c_1 e^t \begin{bmatrix} -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 \cdot e^t \begin{bmatrix} -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$$

(By choosing a different eigenvector $\bar{v}$, one gets different real fundamental solutions; they are not unique, of course).

23/9.2 Use $\bar{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to get

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} -c_1 + c_2 = 0 \\ 2c_1 \quad = 1 \end{matrix}$$

So $c_1 = 1/2$ and $c_2 = +1/2$ Thus

$$\bar{y}(t) = \frac{1}{2} e^t \left(\begin{bmatrix} -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + \begin{bmatrix} -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix} \right)$$

$$= \frac{1}{2} e^t \begin{bmatrix} -2\sin 2t \\ 2\cos 2t + 2\sin 2t \end{bmatrix} = e^t \begin{bmatrix} -\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix}.$$

33/9.2 $A = \begin{bmatrix} -2 & 1 \\ -9 & 4 \end{bmatrix}$; eigenvalues : $\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = \lambda_2 = 1$

Eigenvector for $\lambda = 1$: $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ -9 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

So $\bar{V} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (for a choice of $V_2 = 3$)

For a second (generalized) eigenvector $\bar{w}$ we need to solve:

$(A - \lambda I)\, \bar{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so

$\left[\begin{array}{cc|c} -3 & 1 & 1 \\ -9 & 3 & 3 \end{array}\right] \Rightarrow \bar{w} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$ (for a choice of $w_2 = 0$)

So the general solution is:

$$c_1 \cdot e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\, e^t \left(\begin{bmatrix} -1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

41/9.2. $A=\begin{bmatrix}2&4\\-1&6\end{bmatrix}$; $\lambda^2-8\lambda+16=0$ so $\lambda_1=\lambda_2=4$.

Eigenvectors:

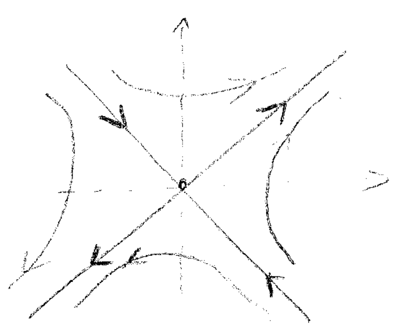
For $\lambda=4$: $(A-4I)\bar{v}=0$, so $\left[\begin{array}{cc|c}-2&4&0\\-1&2&0\end{array}\right]$

with one solution $\bar{v}=\begin{bmatrix}2\\1\end{bmatrix}$.

For a second generalized eigenvector we solve $(A-4I)\bar{w}=\bar{v}$ to get $\bar{w}=\begin{bmatrix}-1\\0\end{bmatrix}$; the general solution is

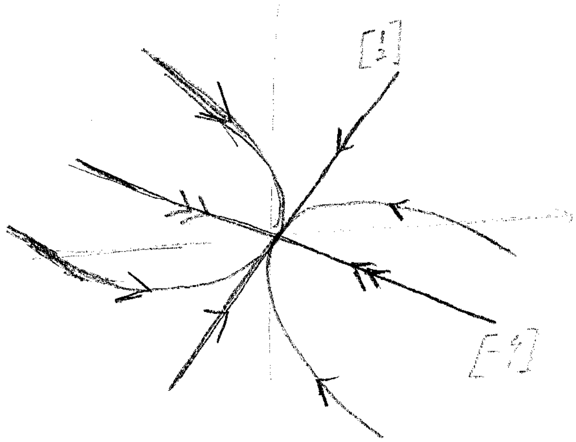
$$c_1\cdot e^{4t}\begin{bmatrix}2\\1\end{bmatrix}+c_2\cdot e^{4t}\left(\begin{bmatrix}-1\\0\end{bmatrix}+t\cdot\begin{bmatrix}2\\1\end{bmatrix}\right).$$

12/9.3. $\bar{y}(t)=c_1\cdot e^{t}\begin{bmatrix}1\\1\end{bmatrix}+c_2\cdot e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$



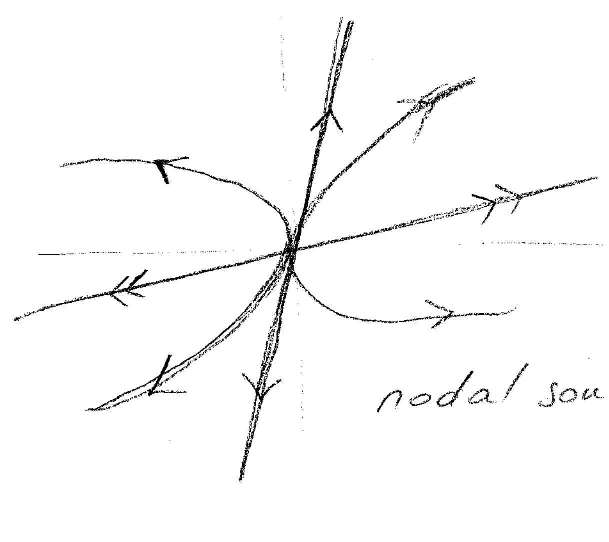
saddle

13/9.3. $\bar{y}(t)=c_1\cdot e^{-3t}\begin{bmatrix}-4\\1\end{bmatrix}+c_2\cdot e^{-t}\begin{bmatrix}1\\2\end{bmatrix}$



nodal sink

15/9.3.



nodal source

19/9.3. $\bar{Y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \bar{Y}$ eigenvalues: $\lambda^2 + 4 = 0$

$\lambda_{1,2} = \pm 2i$

So $(0,0)$ is a center.

If one wants to know more precisely the elliptical shape of each orbit:

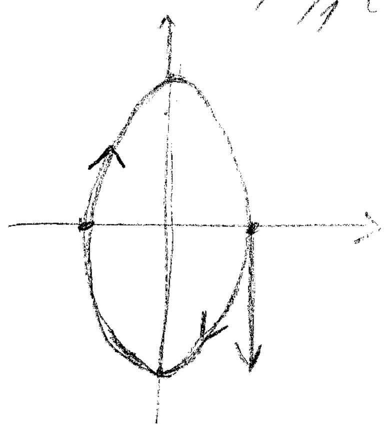
$$y_1' = y_2 \quad | 4y_1$$

$$y_2' = -4y_1 \quad | y_2$$

$$\Rightarrow \quad 4y_1' y_1 + y_2' y_2 = 0 \quad \text{or}$$

$$2\left(y_1^2(t)\right)' + \frac{1}{2}\left(y_2^2(t)\right)' = 0$$

So $4y_1^2(t) + y_2^2(t) = \text{constant}$



The tangent vector at $(1,0)$ is $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$; it points down so we have a clockwise direction

23/9.3. Eigenvalues: $\lambda^2 + 2\lambda + 5 = 0$

$\lambda_{1,2} = -1 \pm 2i$

So we have a spiral sink. The vector field at $(1,0)$ is $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ points down so the motion around $(0,0)$ is clockwise.