Discrete linear equations/systems

I $\quad x_{n+1} = a x_n \longrightarrow x_n = a^n x_0$

$\quad x_{n+1} = a x_n + b \longrightarrow x_n = a^n x_0 + b \cdot \dfrac{a^n - 1}{a - 1}$.

II $\quad x_{n+2} = a_1 x_{n+1} + a_2 x_n$

Associate the quadratic equation $\lambda^2 - a_1 \lambda - a_2 = 0$

- Two real roots λ_1, λ_2:

$$\boxed{x_n = c_1 \cdot \lambda_1^n + c_2 \lambda_2^n}$$

- Two complex conjugate roots ($\lambda_{1,2} = a \pm bi$)

  Write $a+bi$ in polar form: $r(\cos\theta + i \sin\theta)$

  (where $r = \sqrt{a^2+b^2}$; $\theta = \arctan b/a$

$$\boxed{x_n = c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta)}$$

- Double real root $\lambda_1 = \lambda_2 = \lambda$

$$\boxed{x_n = c_1 \lambda^n + c_2 n \cdot \lambda^n}$$

III

$$x_{n+1} = a_{11} x_n + a_{12} y_n$$
$$y_{n+1} = a_{21} x_n + a_{22} y_n$$

or

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

First, solve the characteristic equation:

$$\lambda^2 - T\lambda + D = 0 \quad \text{where} \quad T = a_{11} + a_{22}$$
$$D = a_{11} a_{22} - a_{12} \cdot a_{21}$$

- Two real eigenvalues $\lambda_1 \neq \lambda_2$

  Determine the corresponding eigenvectors $\overline{V}_1, \overline{V}_2$.

$$\overline{V}_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a_{11}}{a_{12}} \end{pmatrix} ; \quad \overline{V}_2 = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a_{11}}{a_{12}} \end{pmatrix}$$

  Then

$$\boxed{\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 \cdot \lambda_1^n \, \overline{V}_1 + c_2 \cdot \lambda_2^n \, \overline{V}_2 .}$$

- Two complex-conjugate eigenvalues $\lambda_{1,2} = a \pm bi$

  Determine the eigenvector $\overline{V}_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a_{11}}{a_{12}} \end{pmatrix} = \bar{a} + \bar{b} i$

  Write $\lambda_1 = a + bi = r(\cos\theta + i \sin\theta)$

  Then

$$\boxed{\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 \cdot r^n (\bar{a} \cos n\theta - \bar{b} \sin n\theta) + {} + c_2 \cdot r^n (\bar{a} \sin n\theta + \bar{b} \cos n\theta)}$$