Linear differential equations

General form: $x'(t) = a(t)\,x(t) + b(t)$

Method of solving:

Step 1. Find an integrating factor: $u(t) = e^{-\int a(t)\,dt}$

Step 2. Find the general solution:

$$x(t) = \frac{\int u(t)\,b(t)\,dt}{u(t)}.$$

Example: $x' = \frac{1}{t+1}x + t$; $x(0) = 3$ (assume $t > -1$)

1. $u(t) = e^{-\int \frac{1}{t+1}dt} = e^{-\ln(t+1)} = \frac{1}{t+1}$

2. $$x(t) = \frac{\int \frac{1}{t+1}\cdot t\,dt}{\frac{1}{t+1}} = (t+1)\int \frac{t}{t+1}\,dt$$

$$= (t+1)\int\left(1 - \frac{1}{t+1}\right)dt = (t+1)\left(t - \ln(t+1) + c\right)$$

3. Use $x(0)=3$ to find c:

$$(0+1)(0 - \ln 1 + c) = 3 \Rightarrow c = 3$$

So $x(t) = (t+1)\left(t - \ln(t+1) + 3\right)$.

Applications:

1) time dependent population models (easy to come-up with examples)

2) Newton's law of cooling: the rate of change of an object's temperature (T) is proportional to the difference between its temperature and the ambient temperature (A). So we have:

$$\frac{dT}{dt} = -k(T-A) \quad ; \quad k\text{ - proportionality constant}$$

Solve this (as a linear eq. for example) to get

$$T(t) = A + (T_0 - A)e^{-kt} \qquad \text{where } T(0) = T_0.$$

Example (Polking's ODE book) A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temp. of the body is 29°C. Assume that the surrounding air temp. remains constant at 21°C. Find the time of victim's death.

Answer: $T(t) = A + (T_0 - A)\cdot e^{-kt}$, where $T(t)$ is the temp of the body at time t.

Assume that at midnight $t = 0$. So $T_0 = 31$; $T(1) = 29$.

From $T(1) = 29$: $21 + (31-21)e^{-k} = 29$; $k = \ln\frac{10}{8} \simeq .223$

We look for time t s.t. $T(t) = 37$.

$$21 + 10\cdot e^{-kt} = 37 \Rightarrow t = \frac{\ln(10/16)}{k} \simeq -2.1 \text{ hours}$$

The time of death is 21:54.