An integral lift of contact homology

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Classical mechanics

The **phase space** $\mathbb{R}^{2n}$ of a system consists of the position and momentum of a particle.

Lagrange: The equations of motion minimize action
$\rightsquigarrow$ n second order differential equations.

Hamilton-Jacobi: The n Euler-Lagrange equations
$\rightsquigarrow$ a Hamiltonian system of $2n$ equations.

Motion is governed by conservation of *energy*, a Hamiltonian H.

- Flow lines of $X_H = -J_0 \nabla H$ are solutions.
- Phase space is (secretly) a symplectic manifold.
- Certain time dependent H give rise to **contact manifolds**.
- Flow lines of the **Reeb vector field** are solutions.

Contact geometry shows up in...

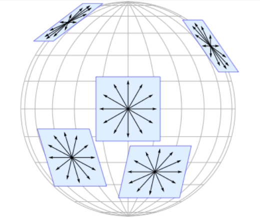
Restricted three body problems, Low energy space travel
Geodesic flow, Thermodynamics, Wave propogation, ...

Hyperplane fields

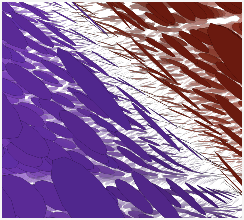
A **hyperplane field** ξ on M^n is the kernel of a 1-form α.
It is a smooth choice of an $\mathbb{R}^{n-1}$ subspace in T_pM at each point p.

Definition

ξ is **integrable** if locally there is a submanifold S with $T_pS = \xi_p$.



Nice and integrable.



Not so much.

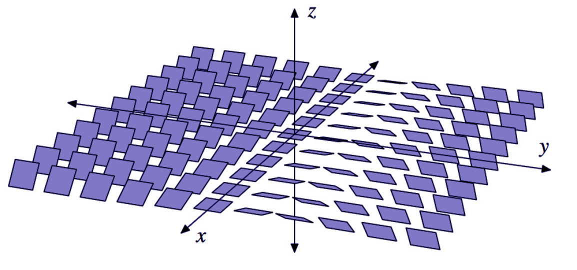
Definition

A **contact structure** is a maximally nonintegrable hyperplane field.

Contact forms

The kernel of a 1-form α on M^{2n+1} is a contact structure whenever

- $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow$ $d\alpha|_\xi$ is nondegenerate.



$\alpha = dz - ydx$ and $\xi = \ker \alpha$

$\xi = \text{Span}\left\{ \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\}$

$d\alpha = -dy \wedge dx = dx \wedge dy$

$$\Rightarrow \quad \alpha \wedge d\alpha \quad = \quad dx \wedge dy \wedge dz$$

Localized equality

Theorem (Darboux's theorem)

Let α be a contact form on the M^{2n+1} and $p \in M$. Then there are coordinates $(x_1, y_1, ..., x_n, y_n, z)$ on $U_p \subset M$ such that

$$\alpha|_{U_p} = dz - \sum_{i=1}^{n} y_i dx_i.$$

Thus locally all contact structures (and contact forms) look the same!
$\rightsquigarrow$ no local invariants like curvature for us to study.

Compact deformations do not produce new contact structures.

Theorem (Gray's stability theorem)

Let $\xi_{t,t\in[0,1]}$ agree off some compact subset of M. Then there is a family of diffeomorphisms $\phi_t : M \to M$ such that $d\phi_t(\xi_t) = \xi_0$.

Reeb vector fields and flow

Definition

The Reeb vector field R_α on (M, α) is uniquely determined by

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : M \to M$ is defined by $\dot{\varphi}_t(x) = R_\alpha(\varphi_t(x))$.

A closed **Reeb orbit** (modulo reparametrization) satisfies

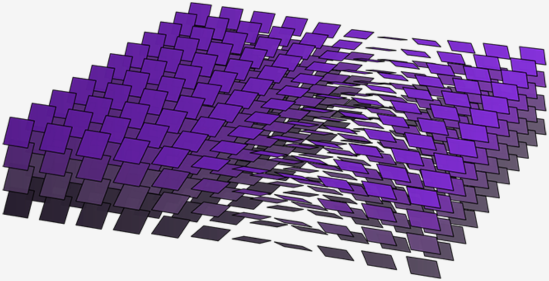
$$\gamma : \mathbb{R}/T\mathbb{Z} \to M, \quad \dot{\gamma}(t) = R_\alpha(\gamma(t)), \tag{0.1}$$

and is **embedded** whenever (0.1) is injective.

The linearized flow along γ defines a symplectic linear map of $(\xi, d\alpha)$. If 1 is not an eigenvalue then γ is **nondegenerate.**

The Reeb vector field on $(\mathbb{R}^3, \ker\alpha)$.

R_α satisfies $\alpha(R_\alpha) = 1$, $d\alpha(R_\alpha, \cdot) = 0$.
R_α is never parallel to ξ and not necessarily normal to ξ.



Let $\alpha = dz - ydx$, $\;d\alpha = dx \wedge dy$

$R_\alpha = \frac{\partial}{\partial z}, \quad \varphi_t(x, y, z) = (x, y, z + t)$

Reeb orbits on S^3

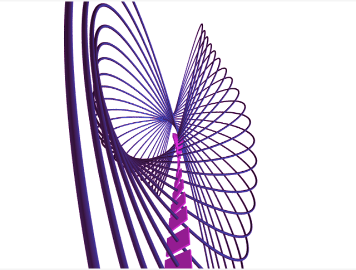
$$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \tfrac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv).$$

The orbits of the Reeb vector field form the Hopf fibration!

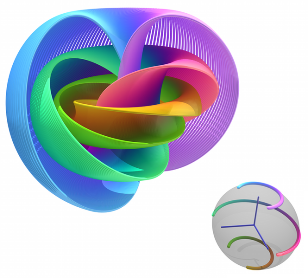
Why?

$$R_\alpha = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.



Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$

A video

https://www.youtube.com/watch?v=AKotMPGFJYk

The irrational ellipsoid

The ellipsoid is $E(a,b) := f^{-1}(1)$, $f := \frac{|u|^2}{a} + \frac{|v|^2}{b}$; $a, b \in \mathbb{R}_{>0}$.
The standard contact form is

$$\alpha_E = \tfrac{i}{2}\left((u d\bar{u} - \bar{u} du) + (v d\bar{v} - \bar{v} dv)\right).$$

The Reeb vector field

$$R_E = \frac{1}{a}\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}\right) + \frac{1}{b}\left(v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right),$$

rotates the u-plane at angular speed $\frac{1}{a}$ and the v-plane at speed $\frac{1}{b}$.

If a/b is irrational, there are only two nondegenerate embedded Reeb orbits living in the $u = 0$ and $v = 0$ planes.

By Gray's stability (S^3, α) and (E, α_E) are contact diffeomorphic.

Existence of periodic orbits

The Weinstein Conjecture (1978)

Let M be a closed oriented odd-dimensional manifold with a contact form α. Then the associated Reeb vector field R_α has a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder ('80's fun)
- Hofer (S^3)
- Taubes (dimension 3)

Tools > 1985: **Floer Theory** and **Gromov's** pseudoholomorphic curves.

A new era of contact topology

Helmut Hofer on turning 60:

> *Why did I come into symplectic and contact geometry? I had the flu, and the only thing to read was a copy of Rabinowitz's paper where he proves the existence of periodic orbits on star-shaped energy surfaces. It turned out to contain a fundamental new idea, which was to study a different action functional for loops in the phase space rather than for Lagrangians in the configuration space. Which actually if we look back, led to the variational approach in symplectic and contact topology, which is reincarnated in infinite dimensions in Floer theory and has appeared in every other subsequent approach. The flu turned out to be really good.*

Bounds on the number of embedded orbits

Theorem (Hofer-Wysocki-Zehnder 1998)

The dynamically convex S^3 admits 2 or ∞ embedded Reeb orbits.

Theorem (Hutchings-Taubes 2008)

Suppose (M^3, α) is nondegenerate. Then there exist at least 2 embedded Reeb orbits and if there are exactly two then both are elliptic and M is diffeomorphic to S^3 or a lens space.

Theorem (Cristofaro Gardiner - Hutchings - Pomerleano 2017)

Suppose (M^3, α) is nondegenerate and $c_1(\xi) \in H^2(M; \mathbb{Z})$ is torsion. Then there exist 2 or ∞ embedded Reeb orbits.

Morse theory

Let $f : M \to \mathbb{R}$ be a smooth "nondegenerate" function.
Let g be a "reasonable" metric.
Then the pair (f, g) is Morse-Smale.

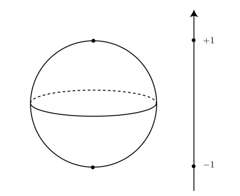
Ingredients:
$C_* = \mathbb{Z}\langle \mathrm{Crit}(f) \rangle$.
$* = \#\{\text{negative eigenvalues Hess}(f)\}$
∂ counts flow lines of $-\nabla f$ betwen critical points

Theorem

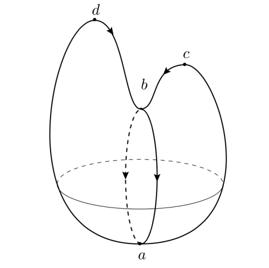
Morse $H_*(M, (f, g))$ = Singular $H_*(M)$!

Necessities:
Transversality (so the implicit function theorem holds)
Compactness (so $\partial^2 = 0$ and invariance holds)

More thoughts on spheres



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 & * = 0, 2, \\ 0 & \text{else.} \end{cases} \qquad \partial = 0$$



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & * = 2, \\ \mathbb{Z}_2 & * = 1, \\ \mathbb{Z}_2 & * = 0, \end{cases} \qquad \begin{aligned} \partial c = \partial d = b, \\ \partial b = 2a = 0. \end{aligned}$$

Theorem (Reeb)

Suppose there exists a Morse function on M that has only two critical points. Then M is homeomorphic to a sphere.

A new hope for a chain complex

Assume: (M, ξ) is a closed contact manifold and let α be a nondegenerate contact form such that $\xi = \ker \alpha$.

Floerify Morse theory on

$$\begin{array}{rrcl} \mathcal{A}: & C^\infty(S^1, M) & \to & \mathbb{R}, \\ & \gamma & \mapsto & \displaystyle\int_\gamma \alpha. \end{array}$$

Proposition

$\gamma \in \mathrm{Crit}(\mathcal{A}) \Leftrightarrow \gamma$ *is a closed Reeb orbit.*

- Grading on orbits given by Conley-Zehnder index,
- $C_*^{\mathbb{Q}}(M, \alpha) = \mathbb{Q}\langle\{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}\rangle$

The letter J is for pseudoholomorphic

$(\xi, d\alpha)$ symplectic vector bundle $\rightsquigarrow \overline{J}$ almost complex structure

Define J on $T(\mathbb{R} \times M) = \mathbb{R} \oplus \mathbb{R}\langle R_\alpha \rangle \oplus \xi$

$$\begin{aligned} J|_\xi &= \overline{J} \\ J\tfrac{\partial}{\partial\tau} &= R_\alpha \end{aligned}$$

Gradient flow lines are a no go; instead count **pseudoholomorphic cylinders** $u \in \mathcal{M}_J(\gamma_+;\gamma_-)$.

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$$

$$\bar{\partial}_{j,J}\; u := du + J \circ du \circ j \equiv 0$$

$$\lim_{s\to\pm\infty} \pi_{\mathbb{R}}\; u(s,t) = \pm\infty$$

$$\lim_{s\to\pm\infty} \pi_M\; u(s,t) = \gamma_\pm$$

up to reparametrization.

Note: J is S^1-INDEPENDENT

Cylindrical contact homology

- $\partial^{\mathbb{Q}} : C_*^{\mathbb{Q}} \to C_{*-1}^{\mathbb{Q}}$ is a weighted count of cylinders.
- Hope this is finite count.
- Hope the resulting homology is independent of our choices.

Conjecture (Eliashberg-Givental-Hofer '00)

Assume a minimal amount of things. Then $(C_*^{\mathbb{Q}}(M, \alpha, J), \partial^{\mathbb{Q}}))$ *is a chain complex and* $CH_*^{\mathbb{Q}} = H(C_*^{\mathbb{Q}}, \partial^{\mathbb{Q}})$ *is an invariant of* $\xi = \ker \alpha$.

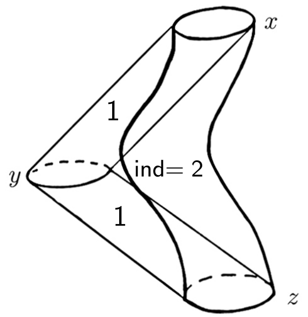
Theorem (Hutchings-N. 2014-present)

If (M^3, ξ) *admits a nondegenerate dynamically convex contact form* α *then* $\partial^{\mathbb{Q}}$ *is well-defined and* $\left(\partial^{\mathbb{Q}}\right)^2 = 0$. $CH_*^{\mathbb{Q}}$ *is invariant.*

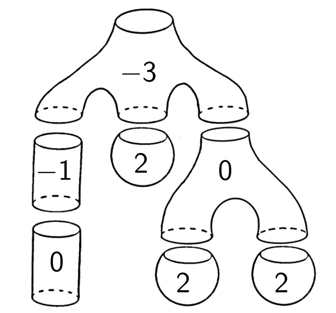
Approaches producing alternate contact invariants via Kuranishi structures are due to Bao-Honda ('15) and J. Pardon ('15).

The pseudoholomorphic menace

- Transversality for multiply covered curves is hard.
- Is $\mathcal{M}_J(\gamma_+;\gamma_-)$ more than a set?
- $\mathcal{M}_J(\gamma_+;\gamma_-)$ can have **nonpositive** virtual dimension...
- Compactness issues are severe



Desired compactification when $CZ(x) - CZ(z) = 2$.



Adding to 2 becomes hard

Early results

- Automatic transversality results of Wendl, Hutchings, and Taubes in **dimension 4.**
- Understand Riemann-Hurwitz and the Conley-Zehnder index
- Realize your original thesis project contained a useful geometric perturbation

Theorem (N. 2013)

Assume some strong things about contact forms associated to (M^3, ξ)*. Then* $\left(\partial^{\mathbb{Q}}\right)^2 = 0$ *and* $CH_*^{\mathbb{Q}}$ *is a contact invariant, which we can actually compute.*

Some computations

$$CH_*^{\mathbb{Q}}(S^3, \xi_{std}) = \begin{cases} \mathbb{Q} & * = 2k, k \geq 1, \\ 0 & \text{else.} \end{cases}$$

Theorem (Abbrescia - Huq-Kuruvilla - N - Sultani '15)

$$CH_*^{\mathbb{Q}}(L(n+1, n), \xi_{std}) = \begin{cases} \mathbb{Q}^n & * = 0, \\ \mathbb{Q}^{n+1} & * = 2k, k \geq 1, \\ 0 & \textit{else.} \end{cases}$$

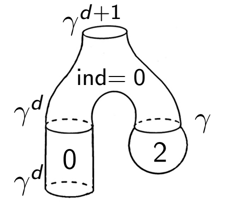
The simple singularities arise from $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SL}(2;\mathbb{C})$. The link L_Γ of this singularity is contactomorphic to S^3/Γ.

Conjecture (McKay correspondence)

Rank$(CH_*^{\mathbb{Q}}(L_\Gamma)) = \#$ *conjugacy classes of* $\Gamma \subset \mathrm{SL}(2;\mathbb{C})$.

The return of regularity

- Do more index calculations
- Learn some intersection theory
- Team up with Hutchings
- Obstructions to $\left(\partial^{\mathbb{Q}}\right)^2 = 0$ can be excluded!



Definition

A nondegenerate $(M^3, \xi = \ker \alpha)$ is **dynamically convex** whenever

- there are no contractible Reeb orbits, or
- $c_1(\xi)|_{\pi_2(M)} = 0$ and $* = CZ(\text{contractible } \gamma) - 1 \geq 2$.

Any convex hypersurface transverse to the radial vector field Y in $(\mathbb{R}^4, \omega_0)$ admits a dynamically convex contact form $\alpha := \omega_0(Y, \cdot)$.

Theorem (Hutchings-N. 2014)

If (M^3, α) is dynamically convex and every contractible Reeb orbit γ has $\mu_{CZ}(\gamma) = 3$ only if γ is embedded then $\left(\partial^{\mathbb{Q}}\right)^2 = 0$.

Technical considerations

- S^1-independent J work in $\mathbb{R} \times M$
- But not in cobordisms, so no chain maps.
- Invariance of $CH_*^{\mathbb{Q}}(M, \alpha, J)$ requires S^1-dependent J.

- Breaking S^1 symmetry invalidates $\left(\partial^{\mathbb{Q}}\right)^2 = 0$.

- Can define a Morse-Bott non-equivariant chain complex.
- Compactness issues require obstruction bundle gluing, producing a correction term.

- $\rightsquigarrow NCH_*$, but what about $CH_*^{\mathbb{Q}}$??

Full circle

- We S^1-equivariantize the nonequivariant theory NCH_* algebraically, yielding an integral lift of contact homology,

$$CH_*^{\mathbb{Z}} = H_*(\mathbb{Z}\langle \check{\gamma}, \hat{\gamma} \rangle \otimes \mathbb{Z}[[u]], \partial^{\mathbb{Z}}), \ \deg(u) = 2.$$

- $CH_*^{\mathbb{Z}}$ rescues the bad orbits, which contribute torsion
- Expect isomorphisms with flavors of symplectic homology

Theorem (Hutchings-N; pending orientations and edits)

For dynamically convex 3-manifolds, NCH_* and $CH_*^{\mathbb{Z}}$ are defined with coefficients in $\mathbb{Z}$ and are contact invariants. Moreover,

$$CH_*^{\mathbb{Z}}(M, \alpha, J) \otimes \mathbb{Q} \cong CH_*^{\mathbb{Q}}(M, \alpha, J),$$

thus $CH_*^{\mathbb{Q}}$ is also a contact invariant.

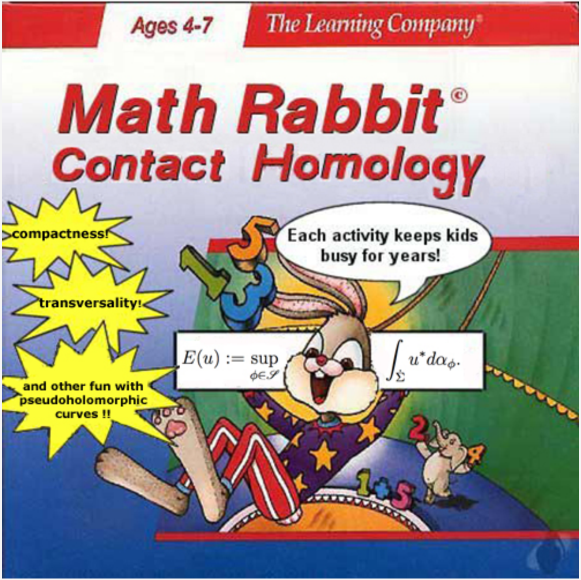
Final thoughts

- If we want to count curves directly we can't appeal to polyfolds or Kuranishi structures.
- These methods abstractly perturb the nonlinear Cauchy-Riemann equation.

- The "obvious" curve counts might need to be corrected by obstruction bundle gluing terms.

- Curve counts reveal info about dynamics, singularity theory, loop spaces, symplectic embeddings, and relationships to other Floer theories.
- And might further low energy space travel!

Thanks!



Ages 4-7
The Learning Company®
Math Rabbit©
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for years!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\Sigma} u^* d\alpha_\phi.$