Math 451 HW # 11, due Friday 12/2/22 **NAME:**

Pick 4 of the 6 problems to upload to gradescope.
(1, 3, 5 and 6 are require little to no understanding of cohomology)

1. Lee 16-10 [Second Edition]
   Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(r-2)^2+z^2=1$ around the z-axis (example 5.17), with its induced Riemannian metric and with the orientation determined by the outward unit normal.

   (a) Compute the surface area of D

   (b) Compute the integral over D of the 2-form $\omega = z dx \wedge dy$.

Plague Hint Guided Jones 11-10,  d'oh

Suppose $0 \le a \le b$. Find the surface area of the torus obtained by revolving the circle $(x-b)^2+z^2 = a^2$ in the xz-plane about the z-axis.
Suggestion: Show that the torus admits the parametrization $0 \le \varphi, \theta \le 2\pi$ by

$$\begin{aligned} x &= (b + a\cos\varphi)\cos\theta \\ y &= (b + a\cos\varphi)\sin\theta \\ z &= a\sin\varphi \end{aligned}$$

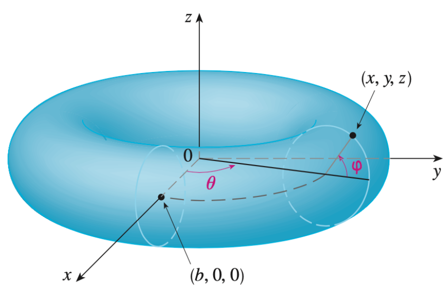

Figure 1: The hollow blue donut

2. A symplectic manifold is a smooth manifold M equipped with a nondegenerate closed 2-form ω. A closed nondegenerate 2-form is said to be a symplectic form.

   (a) Show that if there exists a symplectic form on a smooth manifold M, then $\dim M = 2n$.

   (b) Show that the only sphere S^n which admits a symplectic form is S^2.
   Hint: Use Stokes' theorem and the computation of the de Rham cohomology of S^n.

3. Lee 16-9 [Second Edition]
   Let ω be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$

   (a) Show that $\iota^*_{S^{n-1}}\omega$ is the Riemannian volume form of S^{n-1} with respect to the round metric and the standard orientation.

   (b) Show that ω is closed but not exact on $\mathbb{R}^n \setminus \{0\}$.

4. For each $n \ge 1$, compute the de Rham cohomology groups of $\mathbb{R}^n \setminus \{e_1, -e_1\}$ and for each nonzero cohomology group, give specific differential forms whose cohomology classes form a basis.

5. Note: You may find HW # 10, problem 3 helpful, as it concerns the Hodge star operator, which is the homomorphism $*: \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ satisfying

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_g \, dV_g.$$

In c) and d) take $\mathbb{R}^n$ to be a Riemannian manifold equipped with the Euclidean metric (e.g. inner product) and the standard orientation.

 (a) Show that $*: \Lambda^0 T^*M \to \Lambda^n T^*M$ is given by $*f = f dV_g$
 (b) Show that $** \omega = (-1)^{k(n-k)} \omega$ if $\omega \in \Omega^k(M)$.
 (c) Calculate $*dx^i$ for $i = 1, ..., n$
 (d) Calculate $*(dx^i \wedge dx^j)$ in the case when $n = 4$.

6. Lee 17-1 second
Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ be closed forms. Show that the deRham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map

$$\cup : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M),$$

called the **cup product** given by $[\omega] \cup [\eta] = [\omega \wedge \eta]$.

* What were your favorite topics this semester?