Math 451 HW # 7, due Friday 10/21/22 **NAME:**

This should be relatively straightforward once you get past the notation.

1. Lee Exercise 11.17 (page 280, SECOND)
   Given polar (r, θ) and rectangular $(x := r\cos\theta, y := r\sin\theta)$ coordinates on $\mathbb{R}^2$ we have that the coordinate vector fields transform, using Equation (11.4) on page 275, by
   $$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}, \end{aligned}$$
   for arbitrary coordinate transformations in any finite dimension. Using this fact, consider $f(x, y) = x^2$ on $\mathbb{R}^2$ and let X be the vector field
   $$X = \text{ grad } f = 2x\frac{\partial}{\partial x}$$
   Compute the coordinate expression of X in polar coordinates (on some open subset on which they are defined) using Equation (11.4) on page 275, and show that it is *not* equal to
   $$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}.$$
   Takeaway: The partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field. However, they can be interpreted as the components of a covector field. This is the most important application of covector fields.

2. (Linear Algebra Warm Up 1)
   Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear map. Show that the dual map $A^* : W^* \to V^*$ is given in coordinates as follows. Let $\{e_i\}$ and $\{f_j\}$ be bases for V and W, and let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases for V^* and W^*. If $Ae_i = A_i^j f_j$ then $A^* f^j = A_i^j e^i$.

3. (Linear Algebra Warm Up 2)
   Let V be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V. The inner product determines an isomorphism $\phi : V \to V^*$.
   (a) Show that the isomorphism ϕ is given in coordinates as follows. Let $\{e_i\}$ be a basis for V, let $\{e^i\}$ be the dual basis, and write $g_{ij} = \langle e_i, e_j \rangle$. Then $\phi(e_i) = g_{ij}e^j$.
   (b) The inner product, together with the isomorphism ϕ, define an inner product on V^*. Write this in coordinates as $g^{ij} = \langle e^i, e^j \rangle$. Show that the matrix (g^{ij}) is the inverse of the matrix (g_{ij}).

4. Let M be a smooth manifold with a Riemannian metric $g : TM \otimes TM \to \mathbb{R}$. If $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of f with respect to g is the vector field ∇f defined by
   $$df = g(\nabla f, \cdot).$$
   (a) In local coordinates $\{x^i\}$, if $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij}$, explain how to compute ∇f in terms of g_{ij} and $\partial f/\partial x^i$. *Hint: This should fall out of the preceding two linear algebra warm ups.*
   (b) Let $f : M \to \mathbb{R}$ and let $p \in M$. Show that if $V \in T_pM$ satisfies $df_p(V) > 0$, then there exists a Riemannian metric g on M with $\nabla f(p) = V$.

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?