Math 4081, Midterm **Name (print):** ______________________

Prof. Jo Nelson

Due: March 9, 2018 at 5pm **UNI:** ______________________

This exam contains 6 pages (including this cover page) and 5 problems.

Put your name on the top of every page, in case the pages become separated.

You may *not* discuss this exam with anyone other than me.

Answer all questions, writing in complete sentences as appropriate. When using lemmas, propositions, or theorems from Lee please give a reference. The following rules apply:

- **Turning in the exam**. You may either slide your exam under my office door Math 624 or upload the midterm to gradescope on Friday by 5pm 3/9/18.
- **Allowable materials**. This is an open book exam. In particular, you are allowed to use your notes, wikipedia, the course textbooks, and other textbooks. You can use electronic versions of textbooks. You may NOT google the questions.
- **Office hours**. I will have office hours 12-1pm and 2.30-3.30pm on Monday 3/5/18 and Wednesday 3/7/18. You may come to discuss the midterm with me during my office hours.
- Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) The *zero section* of the tangent bundle TM is the set of zero tangent vectors,

$$Z = \{(p, 0)\} \subset TM = \{(p, V) \mid p \in M, V \in T_pM\}.$$

Show that if $(p, 0) \in Z$, then there is a canonical (not depending on a choice of coordinates) isomorphism

$$T_{(p,0)}TM = T_pM \oplus T_pM.$$

Hint: There is a nice classification result for finite dimensional vector spaces that will be useful.

2. (10 points) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

3. (10 points) For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$ compute the Lie bracket $[X, Y]$.

   a)

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial y} \\ Y_1 &= e^{xyz}\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \end{aligned}$$

   b)

$$\begin{aligned} X_2 &= \sin x\frac{\partial}{\partial y} + yz^2\frac{\partial}{\partial z} \\ Y_2 &= x^3\, y\frac{\partial}{\partial x} + \cos(yz)\frac{\partial}{\partial z} \end{aligned}$$

4. (10 points) Let $\beta = \{v_1, .., v_k\}$ be an ordered basis for V. Show that:
a) Replacing one v_i by a nonzero multiple cv_i yields an equivalently oriented ordered basis if $c > 0$ and an oppositely ordered basis if $c < 0$.
b) Transposing two elements (e.g. interchanging the places of v_i and v_j for $i \neq j$ yields an oppositely oriented ordered basis.
c) Subtracting from one v_i a linear combination of the others yields an equivalently oriented ordered basis.

5. (10 points) If $\dim X + \dim Z = \dim Y$ and X is transverse to Z prove that

$$X \cap Z = (-1)^{(\text{codim} X)(\text{codim } Z)} Z \cap X$$