Math 4081 HW#7, due Wednesday 4/11/18 **NAME:**

1. (6 points) Define a 1-form α on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by

$$\alpha = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

(a) Calculate $\int_C \alpha$ for any circle C of radius r around the origin.

(b) Prove that in the half plane $\{x > 0\}$, α is the differential of a function. Hint: try $\arctan(y/x)$ as a random possibility.

(c) Let $A \subset \mathbb{R}^2 \setminus \{(0,0)\}$ denote the positive x-axis, and let $\gamma : [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a loop which is transverse to A. Show that the intersection number $A \cdot \gamma \in \mathbb{Z}$ satisfies

$$\frac{1}{2\pi} \int_\gamma \alpha = A \cdot \gamma.$$

2. (8 points) Lee 14.6 SECOND
Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

(a) Compute ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).$$

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota_{S^2}^* \omega$ to S^2, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota_{S^2}^* \omega$ is nowhere zero.

3. (4 points) Lee 14.7 b, c SECOND
In each of the following cases, M and N are smooth manifolds, ω is a smooth differential form on N, and $F : M \to N$ is a smooth map. **In each case, compute $d\omega$ and $F^*\omega$, and verify by direct computation that $F^*(d\omega) = d(F^*\omega)$.**

(a) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$, $\quad \omega = y dz \wedge dx$,

$$F(\theta, \varphi) = \left((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi\right).$$

(b) $M = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$ and $N = \mathbb{R}^3 \setminus \{0\}$, $\quad \omega = \dfrac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$,

$$F(u,v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right).$$

4. How difficult was this assignment?