# JO NELSON: AN INTEGRAL LIFT OF CYLINDRICAL CONTACT HOMOLOGY

## 1. Ingredients of "classical" contact homology $CH^{\mathbb{Q}}$

Consider $(Y^{2n-1}, \ker\lambda = \xi)$ a nondegenerate contact structure, cooriented for orientation reasons. $R$ is the Reeb vector field, $c_1(\xi) = 0$, and $J$ is an almost complex structure on $(\mathbb{R}_\tau \times M, d(e^\tau\lambda))$, $J\partial_\tau = R$. Assume $J$ $d(e^\tau\lambda)$-compatible.

Let $\gamma\pm$ be periodic Reeb orbits, and set $\mathcal{M}^J(\gamma_+, \gamma_-)$ to be the moduli space of $u\colon (\mathbb{R}\times S^1, j_0) \to (\mathbb{R}\times Y, J)$ such that $u$ is $J$-holomorphic, and the limits of $\pi_Y u$ as $s \to \pm\infty$ to be reparametrizations of $\gamma_\pm$, and the limits of $\pi_{\mathbb{R}} u$ to be $\pm\infty$.

(CW's question: is the $c_1 = 0$ hypothesis only for obtaining an integral grading via the Conley-Zehnder index? JN's answer: yes.)

Grading is given by $CZ$ index, and the dimension of $\mathcal{M}^J$ is $|\gamma_+| - |\gamma_-|$, where we set $|\gamma| = CZ(\gamma) + n - 3$; when $\dim Y = 3$, this becomes $CZ(\gamma) - 1$.

The chain complex $CC_*^{\mathbb{Q}}$ is defined to be the $\mathbb{Q}$-vector space generated by all the non-bad Reeb orbits. The CZ index of a Reeb orbit can either have the same parity or flip between odd and even. Bad Reeb orbits are the even covers of flippy CZ guys. In dimension 3, they are even covers of negative hyperbolic orbits; negative hyperbolic orbits are characterized by the linearized return map having negative real eigenvalues. (NB's question: why are these "bad"? JN's answer: because even when you have transversality, no way to prove invariance; they also mess up orientations.)

The differential maps $\partial\colon CC_* \to CC_{*-1}$, and it's defined by

$$\langle \partial^{\mathbb{Q}} x, y\rangle = \sum_{u\in\mathcal{M}(x,y)/\mathbb{R}, |x|-|y|=1} \tfrac{m(x)}{m(u)}\epsilon(u).$$

(JO: On the next line is written another differential, with $m(y)$ in the numerator. If we have all the transversality in the world then these two differentials yield isomorphic homologies over $\mathbb{Q}$-coefficients.)

## 2. Issues

We are working with multiply-covered curves, hence there are compactness issues. The following "conjeorem" was stated in EGH's 2000 propaganda paper:

**Theorem 2.1** ("conjeorem")**.** *Let $(Y^{2n-1}, \lambda)$ be a nondegenerate contact manifold, and let $J$ be generic. Assume there are no contractible Reeb orbits of index $-1, 0, 1$. Then $\partial^{\mathbb{Q}}$ is well-defined, $(\partial^{\mathbb{Q}})^2 = 0$, and the homology of $CC_*$ is an invariant of $\xi$ (e.g. independent of $\lambda$ and $J$).*

BEHWZ in 2003 showed that there's no bubbling (just as in Hamiltonian Floer), but there is breaking of curves into "buildings". There is a maximum principle in $\pi_{\mathbb{R}} u$, but you can grow minima, so you can grow new punctures at the negative ends. (The picture drawn is a cylinder, with a minimum growing in the $\mathbb{R}$-direction, hence can break into a pair of pants, a cylinder, and a disk.) This makes us sad, because we only want once-broken cylinders to occur in the boundary. This cylinder-degeneration is exactly what leads us to the "no bad orbits" hypothesis.

(NB's question: why is the cap [finite energy plane] that comes off a "break" and not a "bubble"? JF's answer: because of the Hamiltonian, there's no $S^1$-symmetry, hence there's only one real gluing parameter.)

In dimension 3 you can get automatic transversality and index calculations to work favorably; the hypotheses are called "dynamically separated" provided $3 \leq CZ(\gamma) \leq 5$ for $\gamma$ simple and contractible and $CZ(\gamma^{k+1}) = CZ(\gamma^k) + 4$. When $\gamma$ is noncontractible we have an analogous definition, but we have to keep track of free homotopy classes of the iterates of $\gamma$, which is annoying to explain in a 1 hour talk. (By the way, the "dynamically separated" hypothesis is usually true up to large action $\int_\gamma \lambda$.)

**Theorem 2.2** (JN, 2013)**.** *If $(Y^3, \lambda)$ is nondegenerate and dynamically separated and $J$ is generic, then $CH_*^{\mathbb{Q}}$ is defined (perhaps up to some large action/index) and invariant under choices of $J$ and dynamically separated $\lambda$.*

**Definition 2.3.** $(Y, \lambda)$ is "dynamically convex" if $c_1(\xi)|_{\pi_2(Y)} = 0$ and $CZ(\gamma) \geq 3$ for all contractible Reeb orbits $\gamma$.

*Example* 2.4. Any strictly convex hypersurface in $(\mathbb{R}^4, \omega_0)$ transverse to the radial vector field.

**Lemma 2.5** (Hutchings–Nelson, 2013)**.** *Say $(Y^3, \lambda)$ is dynamically convex and $J$ is generic. Assume also that $CZ(\gamma) > 3$ for $\gamma$ nonsimple and contractible. Then the only buildings of index 2 in $\mathbb{R} \times M$ are unbroken cylinders between $x, z$ with $|x| - |z| = 2$, a once-broken cylinder between three orbits with each pair of index difference 1, or the pair of pants with a cap and cylinder that was mentioned before. Furthermore, no index-2 cylinders can limit to the third configuration.*

**Corollary 2.6.** *$\partial^{\mathbb{Q}}$ is well-defined and squares to zero.*

## 3. Invariance?

The normal approach looks bad, so let's try to define a nonequivariant version of $CH$ and get invariance. This will look like the positive part of symplectic homology, while the equivariant version looks like the positive part of symplectic homology.

Now, instead of looking at a single $J$ for all time, let's take a domain-dependent $J_t$ for $t \in S^1$.

*GOOD NEWS*: for a generic $S^1$-family $(J_t)$, $\mathcal{M}^{(J_t)}(\gamma_+, \gamma_-)$ is a manifold of dimension $CZ(\gamma_+) - CZ(\gamma_-) + 1$.

Now, to define the differential, we need to throw in some point constraints. So, define $ev_\pm \colon \mathcal{M}^{(J_t)} \to \operatorname{im}(\gamma_\pm)$, defined as limits as $s \to \pm\infty$. Take the *base point* $p_{\overline{\gamma}}$ on the underlying embedded Reeb orbit $\overline{\gamma}$ associated to any $\gamma \in \mathcal{P}(\lambda)$. In the language of evaluation maps we write

$$\begin{array}{rrcl} e_+ : & \mathcal{M}^{(J_t)}(\gamma_+, \gamma_-) & \to & \overline{\gamma_+} \\ & u & \mapsto & \lim\limits_{s \to +\infty} \pi_Y u(s, 0) \\ \\ e_- : & \mathcal{M}^{(J_t)}(\gamma_+, \gamma_-) & \to & \overline{\gamma_-} \\ & u & \mapsto & \lim\limits_{s \to -\infty} \pi_Y u(s, 0) \end{array} \tag{1}$$

by specifying $e_+(u) = p_{\overline{\gamma_-}}$ or $e_-(u) = p_{\overline{\gamma_+}}$.

For the chain complex, no longer have to throw out the bad ones, but will have twice as many orbits as before. Can think of this as a Morse–Bott situation, where the Morse–Bott manifolds are the Reeb orbits, and we use the height function as our Morse function on $S^1 \sim \gamma$, which gives rise to two generators for each $\gamma$. This is the motivation for the following definition:

$$NCC_* := \bigoplus_{\widehat{\gamma}, \check{\gamma}} \mathbb{Z}\langle \check{\gamma}, \widehat{\gamma} \rangle,$$

where we set $|\check{\gamma}| := |\gamma| = CZ(\gamma) + n - 3$, $|\widehat{\gamma}| := |\gamma| + 1 = CZ(\widehat{\gamma}) + n + -2$. The hats and checks are coming from the fact that we're thinking Morse–Bott, so we've put the height Morse function on all the Reeb orbits, which has two critical points and corresponding Morse index.

We will define the differential in terms of the following cascade Morse-Bott moduli spaces.

We will use these base points to define the *cascade Morse-Bott moduli spaces*, whose elements are counted by the cascade differential; they are denoted by $\mathcal{M}^{(J_t)}(\check{\alpha};\check{\beta})$, $\mathcal{M}^{(J_t)}(\widehat{\alpha};\check{\beta})$, $\mathcal{M}^{(J_t)}(\check{\alpha};\widehat{\beta})$, and $\mathcal{M}^{(J_t)}(\widehat{\alpha};\widehat{\beta})$. In these definitions we require $\alpha$ and $\beta$ to be distinct Reeb orbits; definitions for when they are the same Reeb orbit will follow presently.

As a set, each of these spaces is a disjoint union of subsets $\mathcal{M}^{\mathcal{J}}(\cdot,\cdot)_\ell$ indexed by *level* $\ell$. Higher levels consist of tuples $(u_1, .., u_\ell)$ of broken cylinders subject to certain conditions. Before explaining these conditions we define the base level, $\mathcal{M}^{\mathcal{J}}(\cdot,\cdot)_1$.

$$
\begin{aligned}
\mathcal{M}^{(J_t)}\left(\check{\alpha};\check{\beta}\right)_1 &:= \left\{u \in \mathcal{M}^{(J_t)}(\alpha;\beta) \mid e_+(u) = p_{\overline{\alpha}}\right\}\\
\mathcal{M}^{(J_t)}\left(\widehat{\alpha};\check{\beta}\right)_1 &:= \mathcal{M}^{(J_t)}(\alpha;\beta)\\
\mathcal{M}^{(J_t)}\left(\check{\alpha};\widehat{\beta}\right)_1 &:= \left\{u \in \mathcal{M}^{(J_t)}(\alpha;\beta) \mid e_+(u) = p_{\overline{\alpha}},\ e_-(u) = p_{\overline{\beta}}\right\}\\
\mathcal{M}^{(J_t)}\left(\widehat{\alpha};\widehat{\beta}\right)_1 &:= \left\{u \in \mathcal{M}^{(J_t)}(\alpha;\beta) \mid e_-(u) = p_{\overline{\beta}}\right\}
\end{aligned}
\tag{2}
$$

**Definition 3.1.** Moving upwards, we define the *higher levels* $\mathcal{M}^{(J_t)}_{|\alpha|-|\beta|}\left(\widetilde{\alpha},\widetilde{\beta}\right)_\ell$ assuming $\alpha = \gamma_0$, $\gamma_1$, ... , $\gamma_{\ell-1}$, $\gamma_\ell = \beta$ are all distinct Reeb orbits, where $\widetilde{}$ over a Reeb orbit indicates a decoration of either a $\widehat{}$ or $\check{}$. We define $\mathcal{M}^{(J_t)}\left(\widetilde{\alpha},\widetilde{\beta}\right)_\ell$ to be the set of tuples

$$
(u_1, .., u_\ell) \in \prod_{i=1}^{\ell} \mathcal{M}^{(J_t)}(\gamma_{i-1},\gamma_i)
$$

such that

If $\widetilde{\alpha} = \check{\alpha}$ then the positive end of $u_0$ has a point constraint, $e_+(u_0) = p_{\overline{\alpha}}$

If $\widetilde{\beta} = \widehat{\beta}$ then the negative end of $u_\ell$ has a point constraint, $e_-(u_\ell) = p_{\overline{\beta}}$

If $1 \leq i \leq \ell$ then the three points $p_{\overline{\gamma_i}}$, $e_-(u_{i-1})$, $e_+(u_i)$ are cyclically ordered around the image of the Reeb orbit $\gamma_i$, with respect to the orientation given by the Reeb orbit.

When $\alpha$ and $\beta$ are the same Reeb orbit we define

$$
\mathcal{M}^{(J_t)}\left(\check{\alpha};\check{\alpha}\right) = \mathcal{M}^{(J_t)}\left(\check{\alpha};\widehat{\alpha}\right) = \mathcal{M}^{(J_t)}\left(\widehat{\alpha};\widehat{\alpha}\right) = \emptyset
$$

and

$$
\mathcal{M}^{\mathcal{J}}\left(\widehat{\alpha};\check{\alpha}\right) := \left\{\begin{array}{cc} 2\{\mathrm{pt}\} & \text{if } \alpha \text{ is bad;} \\ \emptyset & \text{if } \alpha \text{ is good.}\end{array}\right\}
\tag{3}
$$

The differential is defined in block form by:

$$
\partial := \left(\begin{array}{cc} \check{\partial} & \partial^+ \\ \partial^- + \mathbf{obg} & \widehat{\partial}\end{array}\right),
$$

where **obg** is a correction term that accounts for a contribution to the differential counted via obstruction bundle gluing in the presence of certain contractibe Reeb orbits.

We define

$$\begin{array}{llll} \check{\partial} & : & \check{CC}_* \to \check{CC}_{*-1} \\ \check{\alpha} & \mapsto & \displaystyle\sum_{\substack{\check{\beta} \\ |\alpha|-|\beta|=1}} \sum_{u\in\mathcal{M}^{(J_t)}(\check{\alpha},\check{\beta})} \epsilon(u)\check{\beta} \end{array} \qquad \begin{array}{llll} \partial^+ & : & \widehat{CC}_* \to \check{CC}_* \\ \widehat{\alpha} & \mapsto & \displaystyle\sum_{\substack{\check{\beta} \\ |\alpha|-|\beta|=0}} \sum_{u\in\mathcal{M}^{(J_t)}(\widehat{\alpha},\check{\beta})} \epsilon(u)\check{\beta} \end{array}$$

$$\begin{array}{llll} \partial^- & : & \check{CC}_* \to \widehat{CC}_{*-2} \\ \check{\alpha} & \mapsto & \displaystyle\sum_{\substack{\widehat{\beta} \\ |\alpha|-|\beta|=2}} \sum_{u\in\mathcal{M}^{(J_t)}(\check{\alpha},\widehat{\beta})} \epsilon(u)\widehat{\beta} \end{array} \qquad \begin{array}{llll} \widehat{\partial} & : & \widehat{CC}_* \to \widehat{CC}_{*-1} \\ \widehat{\alpha} & \mapsto & \displaystyle\sum_{\substack{\widehat{\beta} \\ |\alpha|-|\beta|=1}} \sum_{u\in\mathcal{M}^{(J_t)}(\widehat{\alpha},\widehat{\beta})} \epsilon(u)\widehat{\beta} \end{array}$$

(Answer to NB's question: can use $\mathbb{Z}$ coefficients because the time-dependent $J$ allows us to rule out isotropy. Indeed, that time-dependence implies the existence of somewhere-injective points.)

**Theorem 3.2.** *For $(Y^3,\lambda)$ dynamically convex and $J$ generic, $NCC_*$ is a chain complex and $H_*$ is independent of $(J_t)$ and dynamically convex $\lambda$.*

Let's relate this back to ordinary cylindrical contact homology. When there is sufficient automatic transversality (as needed to define $\partial^{\mathbb{Q}}$) then we can use this $J$ to look at $NCH_*$. In this case, $\check{\partial} = \partial^{\mathbb{Q}}$, and $\widehat{\partial}$ is the other $\partial^{\mathbb{Q}}$ when counting cylinders between good orbits. The counts between bad orbits can be expressed in terms of $\check{\partial}$ and $\widehat{\partial}$ with additional simplification.

**Theorem 3.3** (Hutchings–Nelson)**.** *If $(Y,\lambda)$ is nondegenerate and dynamically convex, and $J$ is generic, then for*

$$\partial_0 := \begin{pmatrix} \check{\partial} & \partial^+ \\ \partial^- + \mathbf{obg} & \widehat{\partial} \end{pmatrix}$$

*and*

$$\partial_1 := \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}$$

$(NCC_* \otimes \mathbb{Z}[u], \partial^{\mathbb{Z}} = \partial_0 + \partial_1 u^{-1}, J)$ *is a chain complex. ($u$ is a formal variable of degree 2.)*