RESEARCH STATEMENT

JON CHAIKA

1. Overview

My research has focused on:

- Generic and exceptional properties of IETs.
- Diophantine and shrinking target properties.

Highlights of my results for IETs include: The first examples of minimal IETs that have an ergodic measure with Hausdorff dimension less than 1 (Theorem 4), the first example of a topologically mixing IET (Theorem 5) and showing that a generic pair of IETs are disjoint (Theorem 1).

Highlights in the second area include introducing a diophantine property for sequences related to inhomogeneous approximation and establishing a necessary and sufficient condition for it (see Section 5.2).

Highlights in the overlap include a fairly complete treatment of the shrinking target property for IETs with monotone targets (see Section 5.1).

2. Notation and terminology

Definition 1. *Given* $L = (l_1, l_2, ..., l_d)$ *where* $l_i \geq 0$*,* $l_1 + ... + l_d = 1$*, we obtain* d *subintervals of* $[0, 1)$*,* $I_1 = [0, l_1), I_2 = [l_1, l_1 + l_2), ..., I_d = [l_1 + ... + l_{d-1}, 1)$*. Given a permutation* π *on* $\{1, 2, ..., d\}$*, we obtain a* d-Interval Exchange Transformation *(IET)* $T: [0, 1) \to [0, 1)$ *which exchanges the intervals* I_i *according to* π*. That is, if* $x \in I_j$ *then*

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

An IET corresponding to a permutation on $\{1, 2, ..., d\}$ is naturally parametrized by the standard simplex in $\mathbb{R}^d$, $\Delta_d = \{(l_1, l_2, ..., l_d) : l_i > 0, \sum l_i = 1\}$. This set carries a natural Lebesgue measure that we denote $\mathbf{m}_{\mathfrak{R}}$. We are particularly concerned with so called irreducible permutations π that contain some IETs with dense orbits. (That is, $\pi(\{1, ..., k\}) \neq \{1, ..., k\}$ for $k < d$ [16, Section 3].)The term almost every IET refers to Lebesgue measure on the disjoint union of simplices corresponding to irreducible permutations.

Throughout λ denotes Lebesgue measure on [0,1). R_α denotes rotation by α. The term transformation means a measure preserving transformation of Lebesgue space.

3. Disjointness for IETs

This project began with a simple question: Are two interval exchange transformations almost surely measure theoretically different? That is, for almost every pair of IETs (T, S) does there exist a measure preserving bijection (with measure preserving inverse) π such that $S \circ \pi = \pi \circ T$? Partial results had previously been known for some 3-IETs [3], however nothing was known in much generality.

Unsurprisingly, the answer is yes, a pair of IETs is almost surely very different.

Theorem 1. *Let T be an ergodic transformation. Almost every IET is disjoint from T.*

The above statement can not be strengthened to say that every pair of IETs is disjoint because every IET is isomorphic to uncountably many IETs.

Theorem 1 is interesting in its own right, providing a strong answer to the classification problem for IETs. Not only are almost every pair of IETs different, but they share no factors (these are consequences of the technical condition of ergodic transformations being disjoint). Moreover, any ergodic transformation shares no factors with almost every IET. Recall that S is a factor of T if there exists a measure preserving map π such that $S \circ \pi = \pi \circ T$. Also, we obtain the following corollary from the fact that almost every IET is uniquely ergodic [26] and [21] (or [5] for a later proof by elementary methods).

Corollary 1. *Let T be a uniquely ergodic IET. For almost every IET S, $T \times S$ is uniquely ergodic.*

Perhaps the most interesting consequence of this corollary is that for almost every pair of IETs (T, S) and any rectangle $R \subset [0, 1)^2$, the induced map of $T \times S$ on R is an exchange of a finite number of rectangles. Before this result it was unknown if almost every pair (T, S) had every point of $[0, 1)^2$ recurrent under $T \times S$ or had $T \times S$ minimal.

In proving this result a key step was the following strengthening of the fact that almost every IET is rigid [27, Theorem 1.4].

Theorem 2. *Let A be a sequence of natural numbers with density 1. Almost every IET has a rigidity sequence contained in A.*

Question 1. Can the above theorem be weakened to positive density?

Recall that a rigidity sequence for an IET T, is a sequence $n_1, n_2, ...$ such that $\lim\limits_{i\to\infty} \int_0^1 |T^n x - x| dx = 0$ and an IET is called rigid if it has a rigidity sequence.

This theorem and a straightforward strengthening have some interesting applications. Notably, it shows that almost every IET has a rigidity sequence that is not a rigidity sequence for almost every IET. This implies that IETs are almost surely distinguished by their rigidity sequences.

Additionally the methods of this paper established a lower bound for the growth of the primitive IETs introduced in [18] and studied in [27, Part II] (these are the IETs, with rational length data such that the orbit of the point zero hits every rational with denominator dividing the least common denominator of the length data). This bound is that the number of primitive d-IETs with least common denominator less than k is at least $c_d k^d$ (improving the bound in [27, Part II]). The number of primitive d-IETs with least common denominator less than k is trivially less than $C_d k^d$.

I have also proved the analogous result to Theorem 1 for flows on translation surfaces. However the following question remains open:

Question 2. Does every (or almost every) translation surface have the property that every ergodic flow T is disjoint from the flow in almost every direction?

As a side note, in a previous version of this project an interesting criterion for disjointedness was observed.

Theorem 3. *Let T, S be two transformations of Lebesgue space. If there exists a sequence $n_1, \ldots$ such that $T^{n_i} \to Id$ in the Strong Operator Topology on L^2 while $S^{n_i} \to 0$ in Weak Operator Topology on $L^2/\{constant\ functions\}$ then T and S are disjoint.*

Similar results had previously been used to distinguish rigid or partially rigid systems from mixing systems. This result is useful for distinguishing rigid weak mixing systems from each other.

4. Exotic Keane type examples

Michael Keane introduced a family of minimal but not uniquely ergodic 4-IETs [17]. These IETs have 2 ergodic measures: λ_2 and λ_3. Each such IET gives rise to a family of topologically identical IETs parametrized by [0,1] (see [25, Section 1] for further, more general discussion). The extremal points correspond to when one ergodic measure is Lebesgue and the other is singular with respect to Lebesgue measure. For a Keane type IET T we define $H_{dim}[T](\lambda_2, d_{\lambda_3})$ to be the Hausdorff dimension of λ_2 when λ_3 is Lebesgue measure, and $H_{dim}[T](\lambda_3, d_{\lambda_2})$ denotes the Hausdorff dimension of λ_3 when λ_2 is Lebesgue measure. Let $\hat{v}(T)$ denote $(H_{dim}[T](\lambda_2, d_{\lambda_3}), H_{dim}[T](\lambda_3, d_{\lambda_2})) \in [0,1]^2$. As a side note, in the intermediate situation both measures are absolutely continuous with respect to Lebesgue measure and therefore have Hausdorff dimension 1.

Theorem 4. *There exists an IET T such that $H_{dim}[T](\lambda_2, d_{\lambda_3}) = 0$.*

In further work [13], I have been able to show that for any $c \in [0,1]$ there is an IET T_1, with $H_{dim}[T_1](\lambda_2, d_{\lambda_3}) = c$. Likewise there is an IET T_2 with $H_{dim}[T_2](\lambda_3, d_{\lambda_2}) = c$. Additionally, there exists an IET T such that $\hat{v}(T) = (0,1)$. Likewise $\hat{v}(S)$ can be $(0,0), (1,1)$ or $(1,0)$. I found these results fairly surprising, because informally it said that the measures did not notice how small they looked to the other measure. Lastly, I have been able to show that there is a Keane type IET T such that the complement of the λ_3 generic points have Hausdorff dimension 0 [13].

Question 3. Can $\hat{v}(T)$ take any value in $[0,1]^2$?

Additionally, using Keane's example I constructed the first example of a topologically mixing IET, establishing that IETs can be topologically mixing. Recall that $T : X \to X$ is called topologically mixing if for any non-empty open set U and V there exists $N_{U,V} := N$ such that $T^n(U) \cap V \neq \emptyset$ for all $n > N$.

Theorem 5. *There exists a topologically mixing 4-IET.*

This is the smallest number of intervals that could possibly be topologically mixing because no 3-IET is topologically mixing [7].

Question 4. Is almost every 4-IET topologically mixing? Is a residual set of 4-IETs topologically mixing?

5. Diophantine Properties

Given $T\colon [0,1) \to [0,1)$ satisfying some mild assumptions (e.g. ergodicity or minimality), then $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, \epsilon)$ has full measure for almost every x and any $\epsilon > 0$. By replacing ϵ with a decreasing sequence $\epsilon_{\mathbf{n}}$, we address the question: *How quickly do points approach each other?* This question motivates restricting our attention to non-increasing sequences $a_1 \geq a_2 \geq ...$ The Borel-Cantelli Theorem motivates restricting our attention to when $\sum_{i=1}^{\infty} a_i = \infty$. In light of this, we say a sequence $\mathbf{a}$ is *standard* if it is non-increasing, with $\lim_{i \to \infty} a_i = 0$ and $\sum_{i=1}^{\infty} a_i = \infty$. Let $S_{\mathbf{a}}(T)=\{(x,y) : y \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, a_i)\}$, the set of all pairs (x,y) such that $d(T^i x, y) < a_i$ for infinitely many i. This set is not symmetric; if $T(x) = 2x \bmod 1$, and $\mathbf{a}$ is standard then $\lambda(S_{\mathbf{a}}(T) \cap [0,1) \times \{y\}) = 1$ for any y, but $\lambda(S_{\mathbf{a}}(T) \cap \{0\} \times [0,1)) = 0$. We think of $S_{\mathbf{a}}(T)$ having full measure as meaning that generically points approach each other with speed at least $\mathbf{a}$.

5.1. **IETs.** I have recently (partly jointly with Michael Boshernitzan) provided a fairly complete answer to the strictly shrinking target problem for IETs.

Theorem 6. *If T is ergodic with respect to μ then $\lambda(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, \frac{\epsilon}{i})) = 1$ for any $\epsilon > 0$ and μ almost every x.*

The conditions of the theorem are proper. If $\lim_{i \to \infty} i a_i = 0$ then there exists an irrational rotation R, such that $S_{\mathbf{a}}(R)$ has $\lambda \times \lambda$ measure 0 [7]. Also, even though almost every IET is ergodic with respect to λ and λ is always an invariant measure, the measure can not be replaced by λ. That is, there exists a minimal, non-uniquely ergodic 4-IET T, such that $(\lambda \times \lambda)(S_{\frac{\mathbf{c}}{\mathbf{i}}}) < 1$ for any c [7]. This work can be thought of as describing the appropriate universal inhomogeneous diophantine condition for IETs. This paragraph was joint work with Boshernitzan.

In independent work I investigated what happens for generic IETs [12].

Theorem 7. *Given a standard sequence* $\mathbf{a}$ *then for almost every IET* $(\lambda \times \lambda)(S_{\mathbf{a}}(T)) = 1$.

Moreover, $\lambda(S_{\mathbf{a}}(T) \cap \{x\} \times [0,1)) = 1$ and $\lambda(S_{\mathbf{a}}(T) \cap [0,1) \times \{y\}) = 1$ for every x and every y [12, Theorem 6 and Proposition 5]. The quantifiers in the above statement are a little weird, the target comes first and then comes the full measure set; this is necessary. For almost every IET T there exists a standard sequence $\mathbf{a_T}$ such that $(\lambda \times \lambda)(S_{\mathbf{a_T}}(T)) = 0$ [12, Theorem 7]. (These two results were already known for rotations [20].) In the proof of this statement the $\mathbf{a_T}$ seems contrived. This is not a consequence of the proof, but of reality. If one restricts to standard sequences $\mathbf{a}$ that satisfy the additional condition that $i a_i$ is eventually monotone (we call these *Khinchin sequences*) then there is a full measure set of IETs such that $(\lambda \times \lambda)(S_{\mathbf{a}}(T)) = 1$ for all Khinchin sequences $\mathbf{a}$ [12, Theorem 8]. More is true, there exists a full measure set of IETs such that for any IET T in the full measure set $\lambda(S_{\mathbf{a}}(T) \cap \{x\} \times [0,1)) = 1$ and $\lambda(S_{\mathbf{a}}(T) \cap [0,1) \times \{y\}) = 1$ for any Khinchin sequence $\mathbf{a}$, every x and every y. The Khinchin condition is common, appearing in Khinchin's Theorem [19, Theorem 32] and is satisfied by any sequence satisfying some regular growth condition (belonging to a discrete Hardy field is a much stronger condition and is satisfied by many natural sequences [4]).

5.2. **More General.** Some natural questions came up while investigating shrinking targets for IETs. As a consequence of our methods, showing $S_{\frac{\mathbf{c}}{\mathbf{i}}}(T)$ had full measure implied that $S_{\frac{\mathbf{c}}{\mathbf{2i}}}(T)$ had full measure. This is a consequence of a more general result that we found in joint work with Boshernitzan [6, Theorem 4].

Theorem 8. *Given any sequence* $\mathbf{x}$ *in* $[0,1)$ *and any sequence of real numbers* $\mathbf{s}$*, going to infinity, we have* $\lambda(\{y : \liminf_{n\to\infty} s_n|x_n - y| \in (0,\infty)\}) = 0$.

That is, for almost every y point $\liminf_{n\to\infty} s_n|x_n - y|$ takes values in the two point set $\{0,\infty\}$. In the context of dynamical systems, $x_i = T^i x$ and $s_i = (a_i)^{-1}$ and we see that $\lambda(S_{\mathbf{a}}(T)) = \lambda(S_{\mathbf{b}}(T))$ where $b_i = \frac{1}{2}a_i$.

In joint work with Boshernitzan, we further investigate shrinking target properties of abstract sequences [6]. A sequence $\mathbf{x}$ in $[0,1)$ is called *Borel-Cantelli* if for any standard sequence $\mathbf{a}$ we have $\lambda(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(x_i, a_i)) = 1$. We identify that many natural sequences are Borel-Cantelli: the Farey sequence, almost every sequence given by independent λ distributed random variables, the orbits of pseudo-Anosov IETs and so forth. In a number of cases, results of this ilk were known for any particular standard sequence; by establishing that these sequences are Borel-Cantelli we showed that $\lambda(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, a_i)) = 1$ for all standard sequences $\mathbf{a}$ simultaneously. Moreover, we identify a necessary and sufficient condition for a sequence to be Borel-Cantelli. From this condition, we see that l^1 perturbations of Borel-Cantelli sequences are Borel-Cantelli. This work also generalizes to Ahlfors regular spaces. It also generalizes to weaker conditions (conditions stronger than standard on $\mathbf{a}$).

Lastly, throughout this work, we have proved full measure statements. This is necessary, no λ measure preserving transformation has $S_{\frac{\mathbf{c}}{\mathbf{i}}}(T) = [0,1)^2$. This same statement is false if one removes the condition of measurability. It is also false for sequences. In fact, for every standard $\mathbf{a}$ there exists a uniformly distributed sequence $\mathbf{x}$ such that $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(x_i, a_i) = [0,1)$.

6. Schrödinger operators for IETs

In joint work with David Damanik and Helge Krüger we considered the dynamical Schrödinger operators given by IETs [9]. The set up of this problem is: Let T be an IET, $f : [0,1] \to \mathbb{R}$ continuous, and $x \in [0,1)$. Consider the Schrödinger operator $H\colon l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ by

$$[H_x u](n) = u(n-1) + u(n+1) + f(T^n(x))u(n).$$

We considered the spectrum associated with these operators. Using well established techniques (Kotani Theory) we developed a criterion for empty absolutely continuous spectrum.

Proposition 1. *Suppose* T *is an interval exchange transformation that satisfies the Keane condition. If* f *is a continuous sampling function such that* $f \circ T^n$ *is discontinuous for some* $n \geq 1$*, then* H_x *has empty absolutely continuous spectrum for every* $x \in [0,1)$.

This proposition is the motivation for the results that follow.

The following result essentially says that for many permutations all relevant Schrödinger operators have empty ac spectrum. To state this result we recall the definition of type W permutations.

Definition 2. *Suppose π is an irreducible permutation of r symbols. Define inductively a sequence $\{a_k\}_{k=0,\dots,s}$ as follows. Set $a_0 = 1$. If $a_k \in \{\pi^{-1}(1), r+1\}$, then set $s = k$ and stop. Otherwise let $a_{k+1} = \pi^{-1}(\pi(a_k) - 1) + 1$. The permutation π is of* Type W *if $a_s = \pi^{-1}(1)$.*

Theorem 9. *If T is a type W IET satisfying the Keane condition and f is a continuous function such that $f \circ T^n$ is continuous for all n then f is constant.*

By Proposition 1 this implies that the associated Schrödinger operator has empty absolutely continuous spectrum. Theorem 9 also implies that type W IETs satisfying the Keane condition are topologically prime and therefore topologically weakly mixing. There are non type W IETs that satisfy the Keane condition and are not topologically weak mixing [15].

Theorem 10. *If T is a weakly mixing IET and f is a Lipschitz function such that $f \circ T^n$ is continuous for all n then f is constant.*

This result is of particular interest because it uses mixing phenomena itself to establish empty absolutely continuous spectrum (via Proposition 1). The previous results used characteristics that also provided mixing phenomena.

Question 5. Does there exist a weak mixing IET T and a non-constant continuous function f such that $f \circ T^n$ is continuous for all n?

We also considered the analogue of the almost-Mathieu operator in this setting.

Theorem 11. *Let $f(x) = \lambda \cos(2\pi x)$, T be a d-IET, $d > 2$ satisfying the Keane condition. The dynamical Schrödinger operator H_x has empty absolutely continuous spectrum for every $x \in [0, 1)$.*

It should be noted that if $\lambda < 2$ then for any 2-IET the associated Schrödinger operator has non-empty absolutely continuous spectrum [2]. This shows that increasing complexity from a rotation dispels the rich almost-Mathieu phenomena.

7. Miscellaneous

7.1. **Cylinder flow over rotation.** In joint work with D. Ralston we consider the following cylinder flow over a rotation. Let

$$T_\alpha : [0, 1) \times \mathbb{Z} \to [0, 1) \times \mathbb{Z} \text{ by } T_\alpha(x, n) = (x + \alpha, n - 1 + 2\chi_{[0,.5)}(x)).$$

The first coordinate is rotation by α. The second coordinate moves up a level if the first coordinate is in the first half interval and down a level if the first coordinate is in the second half interval. This system was first introduced by K. Schmidt for the golden mean and studied in more generality by Aaronson, Conze, Keane and Oren. $\mathbb{R}$ actions which have this transformation as their discretization have recently been studied by P. Hubert and Ba. Weiss.

Let $V_{\alpha,x} = \{n > 0 : T^n_\alpha(x, 0) \in [0, 1) \times 0\}$. That is, the set of times that $(x, 0)$ returns to "level" 0. For any irrational α, T_α is ergodic with respect to Lebesgue measure (on $[0, 1) \times \mathbb{Z}$) and therefore $V_{\alpha,x}$ has density zero [14]. However, it is still large.

Theorem 12. *For almost every pair* (α, x), $\sum_{n \in V_{\alpha,x}} \frac{1}{n} = \infty$.

For every α there exists x_α such that $V_{\alpha,x_\alpha} = \emptyset$ [22]. More surprisingly,

Theorem 13. *There are uncountably many* α *such that* $\sum_{n \in V_{\alpha,x}} \frac{1}{n} < \infty$.

7.2. **Topological dynamics.** I considered the problem of given a sequence $\bar{x}$ in a metric space X and a metric space Y can $C(X, Y)$ detect the minimality or non-minimality of $\bar{x}$? That is, for each $\bar{x}$ is non-minimal is there $f \in C(X, Y)$ such that $(f(x_1), f(x_2), ...)$ is non-minimal. The answer is no. This stands in contrast to an earlier result of Michael Boshernitzan who showed that for any compact metric space X, if $\bar{x}$ is a non-minimal sequence in X then there exists a continuous function $f: X \to [0, 1]^2$ such that $(f(x_1), f(x_2), ...)$ is non-minimal.

8. Future projects

I find that mathematics often takes me in different directions than I expect, but there are a few projects that I am interested in pursuing in the future. I am interested in finding specific examples of behavior involved in weak mixing systems. I would like to find an explicit weak mixing IET T and points (x, y) such that (x, y) is $T \times T$ generic for Lebesgue measure in $[0, 1) \times [0, 1)$. In particular, if one examines the symbolic dynamical system related to an IET (where the action is continuous) each discontinuity splits into two points, a left side and a right side. Can the pair (left side, right side) be a generic point? I would like to find mixing sequences for some IETs. This would be of interest in its own right, but if I can find an IET with a rigidity sequence in this mixing sequence then they would be disjoint. If they both are uniquely ergodic then their product would be uniquely ergodic. One could then study the finite rectangle exchanges given by the induced map of this transformation on a rectangle.

I would like to address Veech's question of whether almost every non-rotation IET is *prime* [28]. This is probably hard, but there are partial steps along the way that are interesting in their own right. Does the generic IET commute with a root of the identity? What does the commutator of an IET look like? I have some partial results for linearly recurrent IETs. Do IETs satisfy property S introduced in [28]?

I would like to investigate which increasing sequences of natural numbers $n_1, n_2, ...$ have the property that $\langle\langle \mathbf{n}\alpha \rangle\rangle$ is Borel-Cantelli [6] for almost every α. Note that for any increasing sequence of natural numbers $\langle\langle \mathbf{n}\alpha \rangle\rangle$ is not Borel-Cantelli for a residual set of α.

Lastly, I would like to have some explicit examples of two non-isomorphic IETs that share non-trivial joinings.

References

[1] J. Aaronson and M. Keane, *The Visits to Zero of Some Deterministic Random Walks*, Proc. London Math. Soc., **44:3** (1982), 535-553.

[2] Avila, A: The absolutely continuous spectrum of the almost Mathieu operator. arXiv:0810.2965.

[3] Berthe, V; Chekhova, N; Ferenczi, S. Covering numbers: arithmetics and dynamics for rotations and interval exchanges. *J. D'Analyse Math.* **79** 1 (1999) 1-31

[4] Boshernitzan, M: Discrete "orders of infinity". *Amer. J. Math.* **106** (1984), no. 5, 1147–1198.

[5] Boshernitzan, M: A condition for minimal interval exchange maps to be uniquely ergodic. *Duke Math. J.* **52** (1985), no. 3, 723–752.
[6] Boshernitzan, M; Chaika, J: Borel-Cantelli sequences. preprint
[7] Boshernitzan, M; Chaika, J: Some Diophantine properties of IETs. In progress
[8] Chaika J: Hausdorff dimension for ergodic measures of interval exchange transformations. *J. Mod. Dynam.*
[9] Chaika, J; Damanik, D; Krüger, H: Schrödinger operators defined by interval exchange transformations. *J. Mod. Dynam.*
[10] Chaika, J: There exists a topologically mixing IET. preprint
[11] Chaika, J: Every transformation is disjoint from almost every IET. preprint
[12] Chaika, J: Shrinking targets for IETs: Extending a theorem of Kurzweil. preprint
[13] Chaika, J: Hausdorff dimension for ergodic measures of interval exchange transformations II. in progress
[14] J. P. Conze and M. Keane, *Ergodicite d'un Flot Cylindrique*, Fasc. I Proba., Rennes (1976).
[15] H. Hmili, Echanges d'intervalles non topologiquement faiblement mélangeants, Preprint (arXiv:math/0609192).
[16] Keane, M: Interval exchange transformations, *Math. Z.* **14l**, 25-3l (1975).
[17] Keane, M: Keane, M: Non-ergodic interval exchange transformations, *Israel J. Math.* **26** (2) (1977) 188-196.
[18] M. Keane, G. Rauzy, Unique ergodicité des éxchanges d'intervalles, *Math. Z.* **174** (1980), 203–212.
[19] A. Ya. Khinchin, "Continued Fractions," Dover Publications, Inc. (1997).
[20] Kurzweil, J: On the metric theory of inhomogeneous diophantine approximation, *Studia. Math.* **15** (1955) 84-112.
[21] Masur, H: Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982) 168-200.
[22] D. Ralston, *Heaviness in Circle Rotations*, preprint.
[23] I. Oren, *Ergodicity of Cylinder Flows Arising from Irregularities of Distribution*, Israel Journal of Mathematics, **44:2** (1983), 127-138.
[24] K. Schmidt, *A Cylinder Flow Arising from Irregularity of Distribution*, Compositio Math., **36:3** (1978), 225-232.
[25] Veech, W: Interval exchange transformations. *J. D'Analyse Math.* **33** (1978) 222-272.
[26] Veech, W: Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)* 115 (1982) 201-242.
[27] Veech W: The Metric Theory of interval exchange transformations, *American Journal of Mathematics* **106** (6) (1984) 1331-1422.
[28] Veech, W: A criterion for a process to be prime. *Monatsh. Math.* **94** (1982), no. 4, 335–341.

Department of Mathematics, Rice University, Houston, TX 77005, USA
E-mail address: Jonathan.M.Chaika@rice.edu

Department of Mathematics, Rice University, Houston, TX 77005, USA