Math 211-003: Solutions to Assignment 10

1. There are two tanks. The first tank initially has 10 gallons of pure water. The second tank initially has 8 gallons of a water/salt solution with 10 oz of ***salt***. Both tanks drain into the other at a rate of 2 gallons per minute. Find formulas to express the amount of salt in each tank.

   Solution: Let $\vec{x} = \begin{pmatrix} x & y \end{pmatrix}^t$ where x,y represent the amount of salt in the first and second tanks respectively. $x_{in} = y_{out} = 2\frac{y}{8} = \frac{y}{4}$ and $x_{out} = y_{in} = 2\frac{x}{10} = \frac{x}{5}$. So we reach the IVP system

   $$\vec{x}' = \begin{pmatrix} -\frac{1}{5} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{4} \end{pmatrix} \vec{x}, \text{ for } \vec{x}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

   The characteristic equation is $\lambda^2 + \frac{9}{20}\lambda = \lambda(\lambda + \frac{9}{20}) = 0$

   For $\lambda = 0$, one such eigenvector is $\vec{v} = \begin{pmatrix} 5 & 4 \end{pmatrix}^t$.

   $$\Rightarrow \vec{x_1}(t) = e^{0t}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

   For $\lambda = -\frac{9}{20}$, we have an eigenvector $\vec{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^t$

   $$\Rightarrow \vec{x_2}(t) = e^{-\frac{9}{20}t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-\frac{9}{20}t} \\ -e^{-\frac{9}{20}t} \end{pmatrix}$$

   So

   $$\vec{x}(t) = C_1\begin{pmatrix} 5 \\ 4 \end{pmatrix} + C_2\begin{pmatrix} e^{-\frac{9}{20}t} \\ -e^{-\frac{9}{20}t} \end{pmatrix}$$

   Solving $\vec{x}(0) = \begin{pmatrix} 0 & 10 \end{pmatrix}^t$ yields $C_1 = \frac{10}{9}$ and $C_2 = -\frac{50}{9}$

   $$\Rightarrow \vec{x}(t) = \frac{10}{9}\begin{pmatrix} 5 - 5e^{-\frac{9}{20}t} \\ 4 + 5e^{-\frac{9}{20}t} \end{pmatrix}$$

   □

2. Solve the following IVP:

$$\vec{x}' = \begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix} \vec{x} \text{ for } \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: The characteristic equation is $\lambda^2 - 9 = (\lambda - 3)(\lambda + 3) = 0$. For $\lambda = -3$, one such eigenvector is $\vec{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^t$. So we get

$$\vec{x_1}(t) = e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \end{pmatrix}$$

For $\lambda = 3$, one such eigenvector is $\vec{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^t$. So we get

$$\vec{x_2}(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix}$$

So

$$\vec{x}(t) = C_1 \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix}$$

Solving $\vec{x}(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^t$ yields $C_1 = \frac{2}{3}$ and $C_2 = -\frac{1}{3}$

$$\Rightarrow \vec{x}(t) = \frac{1}{3} \begin{pmatrix} 4e^{-3t} - e^{3t} \\ 2e^{-3t} + e^{3t} \end{pmatrix}$$

□

3. Find a general solution to the following system

$$\vec{x}' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \vec{x}$$

Solution: The characteristic equation is $\lambda^2 + 1 = 0$.
We have the complex root $\lambda = i = 0 + 1 \cdot i$ with eigenvector

$$\vec{v} = \vec{v_1} + i\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

So

$$\vec{x_1}(t) = e^{\alpha t}[\cos \beta t \vec{v_1} - \sin \beta t \vec{v_2}] = \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}$$

$$\vec{x_2}(t) = e^{\alpha t}[\cos \beta t \vec{v_2} + \sin \beta t \vec{v_1}] = \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

Our general solution is

$$\vec{x}(t) = C_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

□

4. Solve the following IVP:

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \vec{x} \text{ for } \vec{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Solution: The characteristic equation is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$. So we only will be able to find one independent eigenvector for $\lambda = 2$. One such vector is $\vec{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^t$. So

$$\vec{x_1}(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

Now we need to find $\vec{w}$ such that $A\vec{w} = 2\vec{w} + \vec{v}$. So we try any vector that's not a multiple of $\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow A\vec{u} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\vec{u} - \vec{v}$$

So we want $\vec{w} = -\vec{u}$. It can be verified that $A\vec{w} = 2\vec{w} + \vec{v}$. So

$$\vec{x_2}(t) = e^{\lambda t}(\vec{w} + t\vec{v}) = e^{2t} \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} e^{2t}(t-1) \\ te^2t \end{pmatrix}$$

So

$$\vec{x}(t) = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t}(t-1) \\ te^2t \end{pmatrix}$$

Solving for $\vec{x}(0) = \begin{pmatrix} 0 & 2 \end{pmatrix}^t$ yields $C_1 = C_2 = 2$

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} 2te^{2t} \\ 2(t+1)e^{2t} \end{pmatrix}$$

$\square$

5. Find a general solution to the following system

$$\vec{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: The characteristic equation is $\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$. For $\lambda = 1$, one such eigenvector is $\vec{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^t$. So we get

$$\vec{x_1}(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

For $\lambda = 4$, one such eigenvector is $\vec{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^t$. So we get

$$\vec{x_2}(t) = e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}$$

So now we need to find $\vec{x_p} = X \int X^{-1}\vec{b}dt$

$$X = [\vec{x_1}, \vec{x_2}] = \begin{pmatrix} e^t & 2e^{4t} \\ -e^t & e^{4t} \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X^{-1} = \frac{1}{3e^{5t}} \begin{pmatrix} e^{4t} & -2e^{4t} \\ e^t & e^t \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{-4t} & e^{-4t} \end{pmatrix}$$

$$\int X^{-1}\vec{b}dt = \int \frac{1}{3} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{-4t} & e^{-4t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt = \int \begin{pmatrix} -e^{-t} \\ 2e^{-4t} \end{pmatrix} dt = \frac{1}{3} \begin{pmatrix} e^{-t} \\ -\frac{1}{2}e^{-4t} \end{pmatrix}$$

So

$$\vec{x_p} = X \int X^{-1}\vec{b}dt = \begin{pmatrix} e^t & 2e^{4t} \\ -e^t & e^{4t} \end{pmatrix} \frac{1}{3} \begin{pmatrix} e^{-t} \\ -\frac{1}{2}e^{-4t} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 - 1 \\ -1 - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

So the general solution is

$$\vec{x}(t) = C_1 \begin{pmatrix} e^t \\ -e^t \end{pmatrix} + C_2 \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

□

6. Given an $n \times n$ matrix A, show that the set

$$\mathbf{E}_\lambda := \{\vec{v} | A\vec{v} = \lambda\vec{v}\}$$

is a vector subspace whenever $\mathbf{E}_\lambda \neq \{\vec{0}\}$.

Easy way: Since $A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0}$, $\mathbf{E}_\lambda = null(A - \lambda I)$. As we covered in class, a nullspace is a vector subspace, so $\mathbf{E}_\lambda$ is one as well. □

Brute force way: We need to verify the two properties of a vector space:

$$\vec{x}, \vec{y} \in \mathbf{E}_\lambda \text{ so } A\vec{x} = \lambda\vec{x} \text{ and } A\vec{y} = \lambda\vec{y}$$

So

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y} = \lambda(\vec{x} + \vec{y}) \Rightarrow (\vec{x} + \vec{y}) \in \mathbf{E}_\lambda$$

And

$$A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\lambda\vec{x} = \lambda(\alpha\vec{x}) \Rightarrow \alpha\vec{x} \in \mathbf{E}_\lambda$$

So $\mathbf{E}_\lambda$ is a vector subspace of $\mathbb{R}^n$ □

7. Find the values of α such that the system

$$\begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix} \vec{x} = \vec{b}$$

is guaranteed to have a solution for any choice of $\vec{b}$.

Solution: Guaranteed solution $\iff$ A is non-singular $\iff$ $detA \neq 0$.

$$\begin{vmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{vmatrix} = \alpha \begin{vmatrix} \alpha & 1 \\ 1 & \alpha \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & \alpha \end{vmatrix} = \alpha^3 - 2\alpha = \alpha(\alpha - \sqrt{2})(\alpha + \sqrt{2}) \neq 0$$

So our solution set is $(-\infty, -\sqrt{2}) \bigcup (-\sqrt{2}, 0) \bigcup (0, \sqrt{2}) \bigcup (\sqrt{2}, \infty)$ $\square$

8. Determine which of the following vectors are in V, where

$$V = span\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$$

$$\vec{v_1} = \begin{pmatrix}1\\-1\\2\end{pmatrix}, \vec{v_2} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \vec{v_3} = \begin{pmatrix}0\\0\\1\end{pmatrix}, \vec{v_4} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solution:

$$\vec{v_1} = 2\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow \vec{v_1} \in V$$

$$\vec{v_2} \in \mathbb{R}^4, \text{ so it can not be in } V \subseteq \mathbb{R}^3$$

$$\vec{v_3} \text{ will yield no solution } \Rightarrow \vec{v_3} \text{ is not in } V$$

$$\vec{v_4} = 0\begin{pmatrix}1\\0\\1\end{pmatrix} \Longrightarrow \vec{v_4} \in V$$

In fact $\vec{0}$ is always in any span (assuming it is of the same dimension). □

9. Determine which of the following vectors are in V, where

$$V = span \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} e^2 \\ \pi \\ \cos 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 5 \\ 4 \\ \frac{7}{2} \end{pmatrix}$$

Solution: Consider

$$A = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right]$$

The question now can be stated as finding a solution to $A\vec{c} = \vec{v_i}$ for $i = 1, 2, 3$ or 4. But $det(A) = -2 \neq 0 \Rightarrow A$ is non-singular. So a solution must exist for any choice of vector (yes even for $\vec{v_2}$). In fact

$$\vec{v_2} = \frac{\pi + e^2 - \cos 1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\cos 1 + e^2 - \pi}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{\pi + \cos 1 - e^2}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

□

10. Find a basis for

$$V = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ -4 \end{pmatrix} \right\}$$

Solution: We need to see if there is a solution to $A\vec{c} = \vec{0}$, where

$$A = \left[\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ -4 \end{pmatrix} \right]$$

Doing so would show that the vectors are not independent, allowing us to remove vectors to get a basis.

Consider the augmented system

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 3 & 1 & 4 & -2 & 0 \\ 4 & 1 & 6 & -4 & 0 \end{pmatrix}$$

By performing the following operations $R_2 - 2R_1 \to R_2$, $R_3 - 3R_1 \to R_3$, and $R_4 - 4R_1 \to R_4$, we get the system

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 2 & -4 & 0 \\ 0 & -2 & 4 & -8 & 0 \\ 0 & -3 & 6 & -12 & 0 \end{pmatrix}$$

Again by performing the operations $R_3 - 2R_2 \to R_3$ and $R_4 - 3R_2 \to R_4$, we get

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We get the solution set $(2t - 2s)\vec{v_1} + (2s - 4t)\vec{v_2} + s\vec{v_3} + t\vec{v_4} = \vec{0}$. By setting $s = 0$ and $t = 1$, we see that $2\vec{v_1} - 4\vec{v_2} + \vec{v_4} = \vec{0}$, so $\vec{v_4}$ may be removed, as it can be expressed in terms of $\vec{v_1}$ and $\vec{v_2}$. Likewise, using $s = 1$ and $t = 0$, we see that $-2\vec{v_1} + 2\vec{v_2} + \vec{v_3} = \vec{0}$ so $\vec{v_3}$ may be removed as well. So our basis will be following two vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

□

11.

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) What is the dimension of $null(A)$.
b) Find a basis for $null(A)$.
c) Find the general solution for

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution: a) There are two free columns $\Rightarrow$ two free variables $\Rightarrow dim(null A) = 2$.

b) Assigning $x_2 = s$ and $x_4 = t$,we have the following two equations: $x_3 - t = 0$ and $x_1 + x_3 + t = 0$.

$$\Rightarrow null(A) = \left\{ s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = span \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

These two vectors form a basis for $null(A)$.

c) Any solution to the equation must be of the form $\vec{x}(t) = \vec{p} + \vec{v}$, where

$$\vec{v} = s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

and $\vec{p}$ is any vector satisfying

$$A\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

One such vector is $\vec{p} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}^t$. So we get

$$\vec{x}(t) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

12. Solve the following IVP:

$$x' = x \cos t \text{ for } x(0) = 5$$

Solution: This is a separable 1st order IVP. So

$$\frac{dx}{dt} = x \cos t \Rightarrow \frac{dx}{x} = \cos t \; dt \text{ assuming } x \neq 0$$

By integrating we get

$$\ln |x| = \sin t + C \Rightarrow x(t) = Ae^{\sin t}$$

$$x(0) = Ae^{\sin 0} = A = 5$$

So we get the solution

$$x(t) = 5e^{\sin t}$$

$\square$

13. Find a general solution for

$$x'' - 3x' + 2x = \sin t$$

Solution: We can either solve this ODE using the undetermined coefficients method or by the Laplace transform (using arbitrary initial conditions). We will do the former.

Consider the characteristic equation $\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$. So we have two roots $\lambda_1 = 1, \lambda_2 = 2$. This gives us our two independent homogeneous solutions,

$$x_1(t) = e^t \text{ and } x_2(t) = e^{2t}$$

Now we find $x_p(t)$. Our trial solution will be $x_p(t) = A\cos t + B\sin t$.

$x_p = A\cos t + B\sin t,\ x_p' = -A\sin t + B\cos t,$ and $x_p'' = -A\cos t + -B\sin t$

Plugging these functions into our ODE yields

$$(A - 3B)\cos t + (3A + B)\sin t = \sin t \Rightarrow A = \frac{3}{10}, B = \frac{1}{10}$$

So our general solution is

$$x(t) = C_1 e^t + C_2 e^{2t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t$$

□

14. Suppose $(\lambda, \vec{v})$ is an eigenpair for an $n \times n$ matrix A. Suppose also that $(\mu, \vec{v})$ is an eigenpair for $n \times n$ matrix B. Show that $(e^{\lambda+\mu}, \vec{v})$ is an eigenpair for e^{A+B}

Solution: To find this solution, we first show that $(\lambda+\mu, \vec{v})$ is an eigenpair for $A + B$. So we calculate

$$(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda + \mu)\vec{v}$$

Now we use the example from class to conclude that, because $(\lambda + \mu, \vec{v})$ is an eigenpair for $A + B$, $(e^{\lambda+\mu}, \vec{v})$ is an eigenpair for e^{A+B}. $\square$